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Abstract. Using the relation between the set of embeddings of tori into Euclidean spaces modulo ambient isotopies and the homotopy groups of Stiefel manifolds, we prove new results on embeddings of tori into Euclidean spaces.

1. Introduction

We shall denote by CAT the category PL of piecewise-linear manifolds or the category $DIFF$ of smooth manifolds (all assumed to be finite-dimensional).

Let $\mathrm{Emb}^m_{CAT}(N^n)$ be the set of all CAT embeddings $N^n \to \mathbb{R}^m$ of a closed (i.e. compact, connected and without boundary) n-dimensional CAT manifold N^n into the m-dimensional Euclidean space $\mathbb{R}^m$, modulo an ambient CAT isotopy. Then the standard inclusion $i : \mathbb{R}^m \hookrightarrow \mathbb{R}^{m+k}$ induces the map

$$i_* : \mathrm{Emb}^m_{CAT}(N^n) \to \mathrm{Emb}^{m+k}_{CAT}(N^n).$$

The study of this map is a classical problem in topology of manifolds (see [1], [2], [3], [7], [8] [12], and [14]). The case when $N^{p+q} = S^p \times S^q$ is known to be of particular interest because it sheds light on the general phenomena. The following summarizes the main known results on this problem:

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Theorem 1.1 *Let CAT denote either the PL or the DIFF category. Then the inclusion-induced mapping:*

$$i = i_{p,q}^{m,m+k} : \mathrm{Emb}^m_{CAT}(S^p \times S^q) \to \mathrm{Emb}^{m+k}_{CAT}(S^p \times S^q)$$

(1) *is trivial for $q > p$, $m = 2q$ and $m+k = 2q+p+1$* (cf. [12]; Theorem 4)*;*
(2) *is trivial for $m = p + 2q$, $m + k = p + 2q + 1$, $2 \le p \le q$ and q even ≥ 4* (cf. [14])*;*
(3) *is non-trivial for $q \in \{1, 3, 7\}$, $p = q - 1$, $m = 2q + 1$ and $m + k = 3q$* (cf. [1])*;*
(4) *is non-trivial for $p \le \rho(q)-1$, $m = 2q+1$ and $m+k = 2q+p+1$, where $\rho(q) = 2^c + 8d$ for $q + 1 = (2a + 1) \cdot 2^{4d+c}$, $c \in \{0, 1, 2, 3\}$* (cf. [12])*; and*
(5) *is surjective for $m = p+2q$, $m+k = p+2q+1$, $2 \le p \le q$ and q odd ≥ 3* (cf. [14]).

The purpose of this note is to obtain a new result on embeddings of tori in Euclidean spaces (stated below), more precisely – we obtain new information on the map

$$i_* : \mathrm{Emb}^m_{CAT}(N^n) \to \mathrm{Emb}^{m+k}_{CAT}(N^n).$$

for the case of knotted tori, by applying recent results from [9], which relate the set of embeddings of a torus to a Euclidean space, modulo ambient isotopies, with the homotopy groups of Stiefel manifolds. We also give a new short proof of a result from [14] (cf. Theorem 3.1).

Theorem 1.2 *Let CAT denote either the PL or the DIFF category. If DIFF=PL then assume $q \ge 26$. If CAT=DIFF then assume $q \ge 40$. Then the inclusion-induced maps:*

$$i_{7,q}^{2q-4,2q-3} : \mathrm{Emb}^{2q-4}_{CAT}(S^7 \times S^q) \longrightarrow \mathrm{Emb}^{2q-3}_{CAT}(S^7 \times S^q)$$

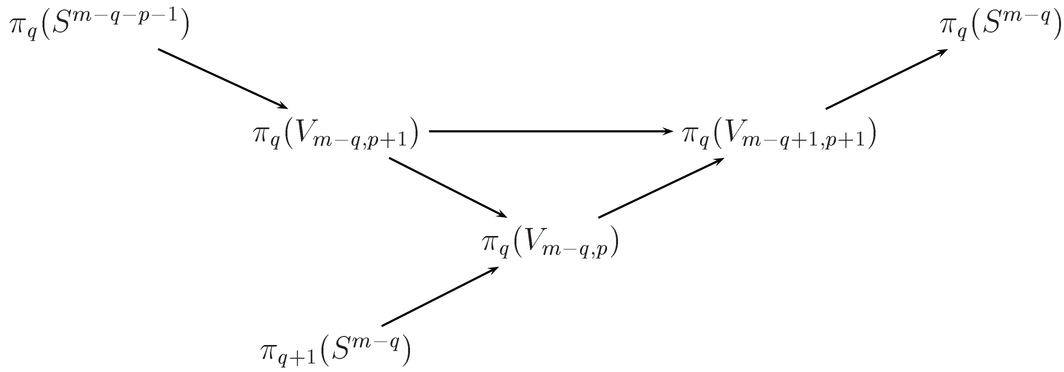
and

$$i_{8,q}^{2q-3,2q-2} : \mathrm{Emb}^{2q-3}_{CAT}(S^8 \times S^q) \longrightarrow \mathrm{Emb}^{2q-2}_{CAT}(S^8 \times S^q)$$

are monomorphisms.

2. Proof of Theorem 1.2

The inclusion $i : V_{m-q,p+1} \hookrightarrow V_{m-q+1,p+1}$ is homotopic to the map which preserves the first p vectors of the $(p+1)$-frame and maps the last vector of the frame to $(0, \ldots, 0, 1) \in \mathbb{R}^{m-q+1}$. Therefore one obtains the following commutative diagram:



Here the NW-SE sequence and the SW-NE sequence are parts of the respective fibration sequences.

Let $m = 2q-4$. In this case the homotopy group in the NW corner of the diagram is $\pi_q(S^{q-12})$, whereas the homotopy group at the bottom of the diagram is $\pi_{q+1}(S^{q-4})$. These groups are in the stable range and are known to be trivial (cf. [13]).

Therefore the homomorphism $i_* : \pi_q(V_{m-q,p+1}) \to \pi_q(V_{m+1,p+1})$ is a monomorphism. Since in this case $m \geq 3q/2 + p + 2$, as well as $m \geq 2p + q + 2$, the inclusion-induced homomorphism (cf. [9]):

$$\mathrm{Emb}^m_{CAT}(S^p \times S^q) \longrightarrow \mathrm{Emb}^{m+1}_{CAT}(S^p \times S^q)$$

is also a monomorphism, hence the assertion is proved.

If $m = 2q-3$ then the homotopy groups in question are $\pi_q(S^{q-12}) = 0$ and $\pi_{q+1}(S^{q-3}) = 0$ and we prove the assertion in a similar way. $\square$

3. On the Vrabec triviality theorem

We give a new short proof of the following result from [14] (note that in [14] a more general situation than that of knotted tori was considered):

Theorem 3.1 *Let CAT denote either the PL or the DIFF category. Then the homomorphism*

$$i = i_{p,q}^{p+2q,p+2q+1} : \mathrm{Emb}_{CAT}^{p+2q}(S^p \times S^q) \to \mathrm{Emb}_{CAT}^{p+2q+1}(S^p \times S^q)$$

is trivial for $2 \leq p \leq q$ *and* q *even* ≥ 4.

Proof. For a manifold N let $\tilde{N} = (N \times N) \setminus \Delta$, where Δ is the diagonal of the product. Then there is a natural $\mathbb{Z}_2$ action on $\tilde{N}$. Let $\pi_{eq}^m(\tilde{N})$ denote the set of all equivariant homotopy classes of the equivariant maps $\tilde{N} \to S^m$.

Assume that $p \leq q$, $m \geq 2p + q + 2$ and $m \geq 3q/2 + p + 2$ or $m \geq 3(q+p)/2 + 2$ in the PL or DIFF category, respectively. Then there are one-to-one correspondences (cf. [6], [9], [10]):

$$\mathrm{Emb}_{CAT}^m(S^p \times S^q) \xrightarrow{\cong} \pi_{eq}^{m-1}(\widetilde{S^p \times S^q}) \xrightarrow{\cong} \pi_q(V_{m-q,p+1}),$$

where $V_{k,l}$ denotes the Stiefel (k,l)-manifold (cf. [4]). Moreover, the first two of the sets above have group structures such that these one-to-one correspondences are isomorphisms (cf. [11]).

The isomorphisms above (with the conditions on m, p, q) are natural with respect to the inclusions $\mathbb{R}^m \hookrightarrow \mathbb{R}^{m+k}$ in the sense that they give the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Emb}_{CAT}^m(S^p \times S^q) & \longrightarrow & \mathrm{Emb}_{CAT}^{m+k}(S^p \times S^q) \\ \cong\downarrow & & \downarrow\cong \\ \pi_q(V_{m-q,p+1}) & \longrightarrow & \pi_q(V_{m-q+k,p+1}) \end{array}$$

where the lower horizontal map is induced by the natural inclusion between the two Stiefel manifolds. In particular, if $q = 2k$, $2 \leq p \leq 2k-2$ and $m = 2q + p = 4k + p$ the group $\pi_q(V_{m-q,p+1})$ is known to be torsion (cf. [5]), while $\pi_q(V_{m-q+1,p+1})$ is free on one generator. □

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