

UNIVERSITY OF LJUBLJANA
INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS
DEPARTMENT OF MATHEMATICS
JADRANSKA 19, 1111 LJUBLJANA, SLOVENIA

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**STRONGLY PSEUDOCONVEX
HANDLEBODIES**

Franc Forstnerič Jernej Kozak

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Franc Forstnerič and Jernej Kozak

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1 Introduction

Let \mathbb{C}^n be the complex n -dimensional Euclidean space with coordinates $z = (z_1, \dots, z_n)$, $z_j = x_j + iy_j$. Let J denote the standard almost complex structure operator on $T\mathbb{C}^n$: $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$, $J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}$. A \mathcal{C}^1 submanifold $M \subset \mathbb{C}^n$ is *totally real* at $p \in M$ if $T_p M \cap JT_p M = \{0\}$, that is, the tangent space $T_p M \subset T_p \mathbb{C}^n$ contains no complex line. A \mathcal{C}^2 function $\rho: U \subset \mathbb{C}^n \rightarrow \mathbb{R}$ is *strongly plurisubharmonic* on U if

$$\mathcal{L}_\rho(z; v) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) v_j \bar{v}_k > 0 \quad (z \in U, v \in \mathbb{C}^n \setminus \{0\}).$$

$\mathcal{L}_\rho(z; v)$ is called the *Levi form* of ρ at z in the direction of the vector v .

Assume that $D \subset \mathbb{C}^n$ is a closed, smoothly bounded, strongly pseudoconvex domain. Thus $D = \{\rho \leq 0\}$ where ρ is a strongly plurisubharmonic function in an open set $U \supset D$, with $d\rho \neq 0$ on $bD = \{\rho = 0\}$. Let $M \subset \mathbb{C}^n$ be a smooth totally real submanifold with boundary $bM = S \cup S'$, where each of the sets S, S' is a union of connected components of bM (S' may be empty). Assume furthermore that

$$M \cap D = S \subset bD, \quad T_p(S) \subset T_p^{\mathbb{C}}(bD) := T_p(bD) \cap JT_p(bD) \quad (p \in S).$$

Such M will be called a *totally real handle* attached to D along the *Legendrian* (*complex tangential*) submanifold $S \subset bD$. (Some authors reserve the word ‘handle’ for the case when M is diffeomorphic to the closed ball in some \mathbb{R}^k and $bM = S^{k-1}$.) We consider the following problem.

The handlebody problem. Given a (small) open set $U \supset M$, find a closed strongly pseudoconvex domain $K \subset \mathbb{C}^n$ satisfying $D \cup M \subset K \subset D \cup U$ which admits a strong deformation retraction onto $E := D \cup M$.

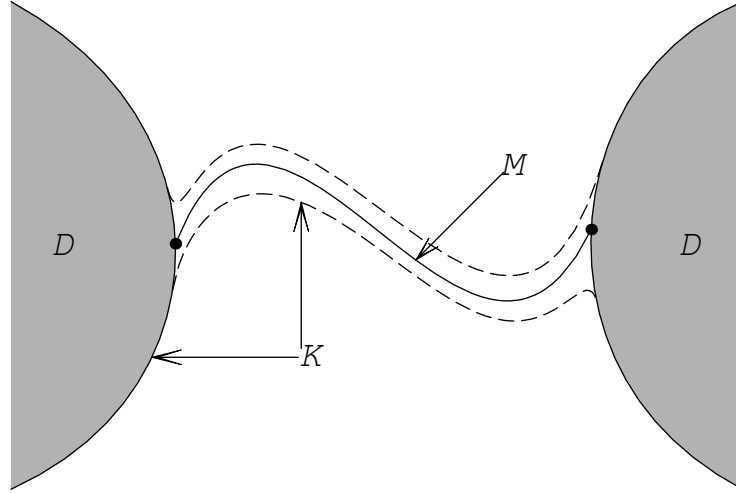


Figure 1: A handlebody K with center $E = D \cup M$

Such K will be called a *strongly pseudoconvex handlebody with center E* (Figure 1). The existence of a strong deformation retraction of K onto E implies that K is *homotopically equivalent* to E .

Recall that any totally real submanifold M in \mathbb{C}^n (or in any complex manifold) has a basis of strongly pseudoconvex tubular neighborhoods. (If $M \subset \mathbb{C}^n$ is compact and of class \mathcal{C}^2 , we may take neighborhoods defined by the Euclidean distance to M .) Hence the above problem is nontrivial only along the attaching submanifold $S = D \cap M \subset bD$. If S fails to be Legendrian in bD at some point $p \in S$ then $D \cup M$ has a nontrivial local envelope of holomorphy at p containing small analytic discs with boundaries in $bD \cup M$ (this follows from the results in [AH]; see also the Remark on p. 534 of [R]). Hence there exist no small pseudoconvex neighborhoods at such points. Local envelope may also appear at points $p \in M$ for which $T_p M$ contains a nontrivial complex subspace [Bi]. This justifies our hypotheses on M and S .

The simpler problem concerning the existence of a basis of (strongly) pseudoconvex neighborhoods of $E = D \cup M$ (but without insisting on the deformation retraction onto E) been considered by several authors; see e.g. Stolzenberg [S], Hörmander and Wermer [HoW], Fornæss and Stout [FS1], [FS2], Chirka and Smirnov [SC], and Rosay [R].

However, in many problems one actually needs strongly pseudoconvex handlebodies which have ‘the same shape’ as $D \cup M$. An example is the construction of Stein manifolds with prescribed homotopy type in [E] and [Go]. Another is the *bumping method construction* (initiated by Grauert) of holomorphic objects on Stein manifolds: we fix a strongly plurisubharmonic exhaustion function ρ on the manifold and extend the object stepwise (by approximation) from one sublevel set of ρ to another by attaching small ‘bumps’. (Such constructions have been used for instance in [Gr], [HL], [FP].) Usually it is easy to proceed through the noncritical levels of ρ where we can use contractible bumps, but when trying to cross a critical point of ρ the handlebody problem appears as follows. The change of topology of the sublevel sets at a Morse critical point p of ρ can be described by attaching to a subcritical sublevel set $D = \{\rho \leq c\}$ a totally real handle M passing through p , with $\dim M$ equal to the index of ρ at p . Suppose that we can extend the object to a small neighborhood $U \supset D \cup M$ (here we often need a topological condition). We then wish to find a handlebody $K \subset U$ with center $D \cup M$ such that a certain supercritical sublevel set of ρ (containing the critical point p) is a noncritical strongly pseudoconvex extension of K ; this allows us to proceed as in the noncritical case. This approach was recently used in the construction of holomorphic submersions of Stein manifolds to complex Euclidean spaces [F]. We present a small modification of the relevant lemma from [F] in Sect. 4 below.

It seems that the only general constructions of strongly pseudoconvex handlebodies in the existing literature can be found in the paper of Eliashberg [E] and in the Ph. D. dissertation of B. Boonstra [Bo] (1995, unpublished). Eliashberg used handlebodies to find Stein manifolds of dimension ≥ 3 with prescribed homotopy type (see Gompf [Go] for dimension 2 and Chapter 11 in [GS]).

One of the purposes of this paper is to close a gap in Eliashberg’s construction which we shall describe below. (Perhaps the results in [Bo] would also suffice, but the constructions there are less explicit and the work is unpublished.) To describe the situation, write the coordinates on \mathbb{C}^n in the form $z = x + iy$, where $x, y \in \mathbb{R}^n$. Set $|x|^2 = x_1^2 + \dots + x_n^2$, $|y|^2 = y_1^2 + \dots + y_n^2$. Let

$$D_\lambda = \{x + iy \in \mathbb{C}^n : |y|^2 \geq 1 + \lambda|x|^2\}, \quad M = \{iy : |y| \leq 1\}.$$

Thus M is the unit ball in the Lagrangian subspace $i\mathbb{R}^n \subset \mathbb{C}^n$, attached to the quadric domain D_λ along the $(n - 1)$ -sphere $S = bM = \{iy : |y| = 1\} \subset bD_\lambda$

which is Legendrian in bD_λ . Note that D_λ is strongly pseudoconvex precisely when $\lambda > 1$.

The main purpose of Section 3 in [E] is to find for each open set $U \supset M$ a strongly pseudoconvex handlebody $K \subset D_\lambda \cup U$ with center $E_\lambda = D_\lambda \cup M$. Eliashberg seeks handlebodies of the form $K = \{|y| \geq \varphi(|x|)\}$ for a suitable function φ . Unfortunately the proof of Lemma 3.4.3. in [E] is not complete. Moreover, Remark 3.4 in Sect. 3 below shows that it is impossible to complete that proof unless one first proves Lemma 3.4.2. in [E] with a different choice of some of the constants. Since Lemma 3.4.3. is a key technical lemma in [E] which is also used in many subsequent works (including Gompf's paper [Go] and the monograph [GS] by Gompf and Stipsicz), a clarification of this point would be welcome. (We are not aware of anything in print at the time of this writing.)

Here we give a complete self-contained construction of strongly pseudoconvex handlebodies satisfying Lemma 3.4.3. in [E] (Proposition 3.1 below). Our construction is based on a pair of differential inequalities (6) for a real function f of a real variable which together characterize strong pseudoconvexity of $D_+ = \{x + iy \in \mathbb{C}^n : |y| \geq f(|x|)\}$ (resp. of $D_- = \{|y| \leq f(|x|)\}$) along the hypersurface $\Sigma = \{|y| = f(|x|)\}$ (Proposition 2.1 and Corollary 2.2 in Sect. 2). In Sect. 3 we construct an explicit family of solutions, following [E] as much as possible.

We believe that one can obtain the desired handlebodies also by analyzing solutions of the second order differential equation (7) which is obtained by changing the second order differential inequality in (6) to equality. Each solution f of this equation in the region $t > 0$, $f > 0$ gives a weakly pseudoconvex hypersurface $\Sigma = \{|y| = f(|x|)\} \subset \mathbb{C}^n$; which side of Σ happens to be pseudoconvex is determined by the sign of the first inequality in (6). We looked at solutions that have a first order contact at some point with a quadratic model $|y| = \sqrt{\lambda|x|^2 + a}$ for some $\lambda > 1$, $a > 0$. Numerical results confirmed the expected behavior of these solutions and indicated that they give the desired handlebodies. However, the problem is to show that these solutions remain in $t > 0$, $f > 0$. The analysis of the equation is rather delicate at the boundary points $t = 0$, $f = 0$ where the coefficients of the equation have first resp. second order poles. Since we have not been able to complete this analysis on time for this publication, we decided to postpone it to another occasion.

In Sect. 4 we give a simpler construction of monotone families of strongly pseudoconvex handlebodies whose center is a sublevel set of a general quadratic strongly plurisubharmonic function $\rho: \mathbb{C}^n \rightarrow \mathbb{R}$ with an attached Lagrangian disc M (describing the change of topology of the sublevel set $\{\rho < c\}$ at the critical point $0 \in \mathbb{C}^n$). Although these handlebodies are not thin everywhere around M , they can be made arbitrarily thin in a neighborhood of the origin.

This already suffices in the ‘bumping method’ as is shown in Sect. 5 of [F].

It is possible to adapt the construction of handlebodies to more general situations by using standard tools such as the (almost) biholomorphic changes of coordinates, the bumping and patching techniques for strongly plurisubharmonic functions, etc. A particularly simple case is when the handle M is real analytic and satisfies $J(T_p M) \subset T_p bD$ for each $p \in M \cap D$ (such a handle can be locally flattened and the resulting domain can be osculated along $M \cap D$ by a quadratic model of the type considered here). One may also find handlebodies corresponding to handles of lower dimension. However, lacking a specific application, we leave these extensions to another occasion.

2 Pseudoconvexity of spherical domains

Let $z = (z_1, \dots, z_n) = x + iy \in \mathbb{C}^n$, with $z_k = x_k + iy_k$ for $k = 1, \dots, n$. Set $|x|^2 = x_1^2 + \dots + x_n^2$, $|y|^2 = y_1^2 + \dots + y_n^2$. Let U be a nonempty open set in \mathbb{R}^n which is invariant under the action of the orthogonal group $O(n)$ (i.e., $x \in U$ and $|x'| = |x|$ implies $x' \in U$). Set $I = \{|x|^2 : x \in U\} \subset \mathbb{R}_+$. Assume that $\theta : I \rightarrow (0, +\infty)$ is a positive function of class \mathcal{C}^2 .

2.1 Proposition. *Let $n > 1$. The domain*

$$D_- = \{x + iy \in \mathbb{C}^n : x \in U, |y|^2 < \theta(|x|^2)\} \quad (1)$$

is strongly pseudoconvex along the hypersurface $\Sigma = \{|y|^2 = \theta(|x|^2)\}$ if and only if θ satisfies the following differential inequalities on I :

$$\theta' < 1, \quad 2|x|^2\theta\theta'' < (1 - \theta')(|x|^2\theta'^2 + \theta). \quad (2)$$

(θ and its derivatives are calculated at $|x|^2$). The domain

$$D_+ = \{x + iy \in \mathbb{C}^n : x \in U, |y|^2 > \theta(|x|^2)\} \quad (3)$$

is strongly pseudoconvex along Σ if and only if the reverse inequalities hold in (2). If θ solves the differential equation

$$2|x|^2\theta\theta'' = (1 - \theta') \cdot (|x|^2\theta'^2 + \theta) \quad (4)$$

then $\theta' < 1$ implies that D_- is weakly pseudoconvex along Σ while $\theta' > 1$ implies that D_+ is weakly pseudoconvex along Σ .

Proof. Set $\rho(x + iy) = |y|^2 - \theta(|x|^2)$. A calculation gives for $1 \leq j \neq k \leq n$

$$\begin{aligned} -\rho_{z_k} &= x_k\theta' + iy_k, \\ -2\rho_{z_k\bar{z}_k} &= 2x_k^2\theta'' + \theta' - 1, \\ -2\rho_{z_j\bar{z}_k} &= 2x_jx_k\theta'', \end{aligned}$$

where θ and its derivatives are evaluated at $|x|^2$. The calculation of the Levi form of $\Sigma = \{\rho = 0\}$ can be simplified by observing that ρ is invariant under the action of the real orthogonal group $O(n)$ on \mathbb{C}^n by $A(x + iy) = Ax + iAy$ for $A \in O(n)$. Fix a point $p = r + is \in \Sigma$ ($r, s \in \mathbb{R}^n$). After an orthogonal rotation we may assume that $p = (x_1 + iy_1, i\tilde{y}_2, \dots, i\tilde{y}_n)$ with $x_1 = |r| \geq 0$. Applying another orthogonal map which is the identity on $\mathbb{C} \times \{0\}^{n-1}$ we may further assume that $p = (x_1 + iy_1, iy_2, 0, \dots, 0)$ where $y_1^2 + y_2^2 = |s|^2 = \theta(x_1^2)$. At this point we have

$$\rho_{z_1}(p) = -x_1\theta' - iy_1, \quad \rho_{z_2}(p) = -iy_2, \quad \rho_{z_k}(p) = 0 \text{ for } k = 3, \dots, n.$$

Hence the complex tangent space $T_p^{\mathbb{C}}\Sigma = \{v \in \mathbb{C}^n : \sum_{k=1}^n \frac{\partial \rho}{\partial z_k}(p)v_k = 0\}$ consists of all $v \in \mathbb{C}^n$ satisfying $v_1 = -\lambda iy_2$, $v_2 = \lambda(x_1\theta' + iy_1)$ for arbitrary choices of $\lambda \in \mathbb{C}$ and $v'' = (v_3, \dots, v_n) \in \mathbb{C}^{n-2}$. We also have

$$\begin{aligned} 2\rho_{z_1\bar{z}_1}(p) &= 1 - \theta' - 2x_1^2\theta'', \\ 2\rho_{z_k\bar{z}_k}(p) &= 1 - \theta' \quad (k = 2, \dots, n), \\ 2\rho_{z_j\bar{z}_k}(p) &= 0 \quad (1 \leq j \neq k \leq n). \end{aligned}$$

For $v \in T_p^{\mathbb{C}}\Sigma$ we thus get (noting that $y_1^2 + y_2^2 = \theta(x_1^2)$)

$$\begin{aligned} 2\mathcal{L}_\rho(p; v) &= (1 - \theta' - 2x_1^2\theta'')|\lambda|^2 y_2^2 + (1 - \theta')|\lambda|^2 (x_1^2\theta'^2 + y_1^2) + (1 - \theta')|v''|^2 \\ &= |\lambda|^2 \left(-2x_1^2 y_2^2 \theta'' + (1 - \theta')(x_1^2\theta'^2 + \theta) \right) + (1 - \theta')|v''|^2 \end{aligned} \quad (5)$$

where θ and its derivatives are evaluated at $x_1^2 = |r|^2$. Thus $\mathcal{L}_\rho(p; v) > 0$ for all choices of $\lambda \in \mathbb{C}$ and $v'' \in \mathbb{C}^{n-2}$ with $|\lambda|^2 + |v''|^2 > 0$ if and only if

$$\theta' < 1, \quad 2x_1^2 y_2^2 \theta'' < (1 - \theta')(x_1^2\theta'^2 + \theta).$$

Observe that $0 \leq y_2^2 \leq |s|^2 = \theta(x_1^2)$, and y_2^2 assumes both extreme values 0 and $\theta(x_1^2)$ when (y_1, y_2) traces the circle $y_1^2 + y_2^2 = \theta(x_1^2)$. Thus the second inequality above holds at all points of this circle precisely when it holds at the point $y_1 = 0$, $y_2 = \sqrt{\theta(x_1^2)}$. This gives the conditions

$$\theta' < 1, \quad 2x_1^2\theta\theta'' < (1 - \theta')(x_1^2\theta'^2 + \theta)$$

characterizing strong pseudoconvexity of D_- along the mentioned circle in Σ . Since $x_1 = |r|$, the above is equivalent to the pair of inequalities (2) at $p = r + is$. Similarly we see that negativity of $\mathcal{L}_\rho(p; v)$ for all choices of λ and v'' (which characterizes strong pseudoconvexity of D_+) is equivalent to the reverse inequalities in (2).

Assume now that θ satisfies (4). As before we reduce to the case $p = x + iy = (x_1 + iy_1, iy_2, 0, \dots, 0)$. From (5) we obtain

$$\begin{aligned} 2\mathcal{L}_\rho(p; v) &= |\lambda|^2 (-2x_1^2 y_2^2 \theta'' + 2x_1^2 \theta \theta'') + (1 - \theta')|v''|^2 \\ &= 2|\lambda|^2 x_1^2 y_1^2 \theta'' + (1 - \theta')|v''|^2. \end{aligned}$$

(We used $\theta(x_1^2) - y_1^2 = y_2^2$.) From (4) we see that θ'' is of the same sign as $1 - \theta'$. Thus $\theta' < 1$ implies $\mathcal{L}_\rho(p; v) \geq 0$, with equality precisely when $v'' = 0$ and $0 = x_1 y_1 = x \cdot y$. In this case $D_- = \{\rho < 0\}$ is weakly pseudoconvex along $\Sigma = \{\rho = 0\}$, strongly pseudoconvex on $\{x + iy \in \Sigma: x \cdot y \neq 0\}$, and has one zero eigenvalue of the Levi form at each point of $\{x + iy \in \Sigma: x \cdot y = 0\}$. When $\theta' > 1$ the analogous conclusions hold for $D_+ = \{\rho > 0\}$. If $\theta' = 1$ holds identically then $\rho = |y|^2 - |x|^2 + c = -\Re(\sum_{j=1}^n z_j^2) + c$ is pluriharmonic. ♠

The second inequality in (2) simplifies further in the variables $|x|, |y|$:

2.2 Corollary. *Let $U \subset \mathbb{R}^n \setminus \{0\}$ be an $O(n)$ -invariant open set and $f: I \rightarrow (0, +\infty)$ a \mathcal{C}^2 function on $I = \{|x|: x \in U\}$. The domain $D_- = \{x + iy \in \mathbb{C}^n: x \in U, |y| < f(|x|)\}$ is strongly pseudoconvex along the hypersurface $\Sigma = \{|y| = f(|x|)\}$ if and only if*

$$\frac{f f'}{|x|} < 1, \quad f \cdot \left(f'' + \frac{f'^3}{|x|} \right) < 1 \quad (6)$$

for all $x \in U$, where f and its derivatives are evaluated at $|x|$. The domain $D_+ = \{x + iy: x \in U, |y| > f(|x|)\}$ is strongly pseudoconvex along Σ when the reverse inequalities hold in (6). If f satisfies the differential equation

$$f \cdot \left(f'' + \frac{f'^3}{|x|} \right) = 1 \quad (7)$$

then $f f'/|x| < 1$ implies that D_- is weakly pseudoconvex along Σ while $f f'/|x| > 1$ implies that D_+ is weakly pseudoconvex along Σ .

Proof. Set $t = |x| > 0$ for $x \in U$. The functions f and θ are related by $f(t)^2 = \theta(t^2)$. Differentiation gives $f(t)f'(t) = t\theta'(t^2)$ whence $\theta' < 1$ is equivalent to $f f'/t < 1$. Another differentiation of $f(t)f'(t) = t\theta'(t^2)$ gives

$$f f'' + f'^2 = 2t^2 \theta'' + \theta' = 2|x|^2 \theta'' + f f'/t.$$

Hence $2|x|^2 \theta'' = f f'' + f'^2 - f f'/t$. Multiplying by $\theta = f^2$ we obtain the first line in the following display. In the second line we used $|x|^2 \theta'^2 = (t\theta')^2 = f^2 f'^2$:

$$\begin{aligned} 2|x|^2 \theta \theta'' &= f^2 (f f'' + f'^2 - \frac{f f'}{t}), \\ (1 - \theta') (|x|^2 \theta'^2 + \theta) &= (1 - \frac{f f'}{t}) (f^2 f'^2 + f^2) = f^2 (1 - \frac{f f'^3}{t} + f'^2 - \frac{f f'}{t}). \end{aligned}$$

Comparing the two sides, dividing by $f^2 > 0$ and cancelling the common terms $f'^2 - f f'/t$ we see that the second inequality in (2) is equivalent to $f(f'' + f'^3/t) < 1$. Similarly one treats the other cases. ♠

2.3 Remarks. (A) The differential inequalities (2) and (6) are invariant up to the sign with respect to taking the inverses. More precisely, assume

$\theta'(|x_0|^2) \neq 0$ for $x_0 \in U$ and denote by τ the local inverse of θ . At points where $\theta' > 0$ the inequalities (2) transform into the reverse inequalities for τ :

$$\tau' > 1, \quad 2|y|^2\tau\tau'' > (1 - \tau') (|y|^2\tau'^2 + \tau).$$

On the other hand, near points where $\theta' < 0$ the inequalities (2) transform into the same inequalities for $\tau = \theta^{-1}$. This can be explained geometrically as follows. If $\theta'(|x_0|^2) > 0$ then for x near x_0 we have $|y|^2 < \theta(|x|^2)$ if and only if $|x|^2 > \tau(|y|^2)$, and strong pseudoconvexity of the latter region is equivalent to the above inequality for τ according to Proposition 2.1. If $\theta'(|x_0|^2) < 0$ then for x near x_0 we have $|y|^2 < \theta(|x|^2)$ if and only if $|x|^2 < \tau(|y|^2)$, and pseudoconvexity is now characterized by (2). Similarly the equations (4) and (7) are invariant with respect to taking the inverses.

(B) If $f(t)$ ($t \in \mathbb{R}$) is a function of class \mathcal{C}^1 and piecewise \mathcal{C}^2 , we adopt the convention that f satisfies the second inequality in (6) at a point of discontinuity t_0 of the second derivative f'' when both the left and the right limit of f'' at t_0 satisfies it. (At endpoints we consider only the one sided limit.) A similar convention is adopted for (2). ♠

2.4 Example. We illustrate the above by looking at models domains defined by the quadratic function

$$\rho_\lambda(z) = \rho_\lambda(x + iy) = \lambda|x|^2 - |y|^2 \quad (\lambda \in \mathbb{R}, z \in \mathbb{C}^n)$$

which will be used in the following section. Setting $g_{\lambda,a}(t) = +\sqrt{\lambda t^2 + a}$ we have $\{\rho_\lambda < -a\} = \{x + iy \in \mathbb{C}^n : |y| > g_{\lambda,a}(|x|)\}$. From $(\frac{\partial^2 \rho_\lambda}{\partial z_j \partial \bar{z}_k}) = \frac{(\lambda-1)}{2}I$ we see that ρ_λ is strongly plurisubharmonic when $\lambda > 1$, strongly plurisuperharmonic when $\lambda < 1$, and $\rho_1(x+iy) = |x|^2 - |y|^2 = \Re(\sum_{j=1}^n z_j^2)$ is pluriharmonic. It is easily verified directly that $g_{\lambda,a}$ satisfies (6) on $\{t \in \mathbb{R}_+ : \lambda t^2 + a \geq 0\}$ if $\lambda < 1$, and it satisfies the reverse inequalities in (6) if $\lambda > 1$. If $\lambda \neq 0$ then $g = g_{\lambda,a}$ satisfies the differential equation $g(g'' + \frac{g'^3}{\lambda t}) = \lambda$.

2.5 Remark on [E]. We wish to compare the condition (6) above with Lemma 3.3.1. in [E, p. 38]. That lemma asserts that for any positive \mathcal{C}^2 function $\varphi(r)$, defined on an interval $r \in I \subset \mathbb{R}$ and satisfying

$$\varphi' > \frac{r}{\min \varphi} \quad (*), \quad \min \left(\varphi'', \frac{\varphi'}{r} \right) + \frac{\varphi'^3}{r} - \frac{1}{\varphi}(1 + \varphi'^2) > 0, \quad (**)$$

the domain $\{|y| > \varphi(|x|)\}$ is strongly pseudoconvex along $\{|y| = \varphi(|x|)\}$. It is immediate that (*) implies $\varphi\varphi'/r > 1$ which is the reverse of the first inequality in (6). If this holds then (**) implies the reverse of the second inequality in (6) (but not vice versa). To see this, delete the term $-\varphi'^2/\varphi < 0$ on the left hand side of (**) (the result remains > 0) and multiply the result by $\varphi > 0$. At points

where $\varphi'' \leq \varphi'/r$ we obtain the reverse of the second inequality in (6) above. At points where $\varphi'' > \varphi'/r$ the inequality (**) becomes $\varphi\varphi'/r + \varphi\varphi'^3/r - 1 > 0$ which trivially holds since $\varphi\varphi'/r - 1 > 0$ and the middle term is positive.

The conclusion is that *Eliashberg's conditions (*, **) are sufficient (but not necessary) for strong pseudoconvexity of $\{|y| > \varphi(|x|)\}$.*

3 Strongly pseudoconvex handlebodies

In this section we find functions $f: I \rightarrow (0, +\infty)$ on intervals $I \subset \mathbb{R}_+ = [0, +\infty)$ which satisfy one of the following pairs of differential inequalities:

$$f \cdot \left(f'' + \frac{f'^3}{t} \right) < 1 \quad \text{and} \quad \frac{ff'}{t} < 1, \quad (8)$$

$$f \cdot \left(f'' + \frac{f'^3}{t} \right) > 1 \quad \text{and} \quad \frac{ff'}{t} > 1. \quad (9)$$

If f is of class \mathcal{C}^1 and piecewise \mathcal{C}^2 then at a point of discontinuity of f'' it should be understood that f satisfies the first inequality in (8) resp. in (9) if the one-sided limits of f'' at that point satisfy it. By Corollary 2.2 the condition (8) characterizes strong pseudoconvexity of the domain $\{x + iy \in \mathbb{C}^n: |y| < f(|x|)\}$ along $\Sigma = \{|y| = f(|x|)\}$ while (9) does the same for $\{x + iy \in \mathbb{C}^n: |y| > f(|x|)\}$.

The following Proposition is essentially due to Eliashberg (Lemma 3.4.3. in [E]), but see Remark 3.4 below.

3.1 Proposition. *Let $\lambda > 1, a > 0$ and $g(t) = +\sqrt{\lambda t^2 + a}$. For every sufficiently small $\epsilon > 0$ there exists a number $\sigma = \sigma(\epsilon) \in (0, \epsilon)$ and a continuous, positive, strictly increasing function $f = f_\epsilon: [\sigma, +\infty) \rightarrow [f(\sigma), +\infty)$ which is \mathcal{C}^∞ on $(\sigma, +\infty)$, satisfies (9) and also the following:*

- (i) $f(t) = g(t)$ for $t \geq \epsilon$,
- (ii) $f(t) < g(t)$ for $\sigma \leq t < \epsilon$,
- (iii) $f'(\sigma+) = \lim_{t \downarrow \sigma} f'(t) = +\infty$, and
- (iv) the inverse function $f^{-1}: \mathbb{R}_+ \rightarrow [\sigma, +\infty)$ is of class \mathcal{C}^∞ and satisfies (8) provided that we set $f^{-1}(u) = \sigma$ for $0 \leq u \leq f(\sigma)$.

3.2 Corollary. *Let $\lambda > 1, a > 0$, $D = \{x + iy \in \mathbb{C}^n: |y|^2 \geq \lambda|x|^2 + a\}$ and $M = \{iy: y \in \mathbb{R}^n, |y| \leq a\}$. If f satisfies Proposition 3.1 then $K = \{x + iy \in \mathbb{C}^n: |x| \leq f^{-1}(|y|)\}$ (Figure 2) is a smooth strongly pseudoconvex handlebody with center $E = D \cup M$, satisfying $D \cup \{|x| \leq \sigma\} \subset K \subset D \cup \{|x| < \epsilon\}$.*

Remark. Handlebodies on figures 2 and 4 are shown in the coordinate system $(|x|, |y|) \in \mathbb{R}_+^2$. The actual handlebody is the preimage under the map $x+iy \rightarrow (|x|, |y|)$.

Proof of Proposition 3.1. We use the ideas from [E] but adjust some of the functions and constants involved in the construction. Without loss of generality we may take $a = 1$ and $g(t) = \sqrt{\lambda t^2 + 1}$ (the general case follows by rescaling).

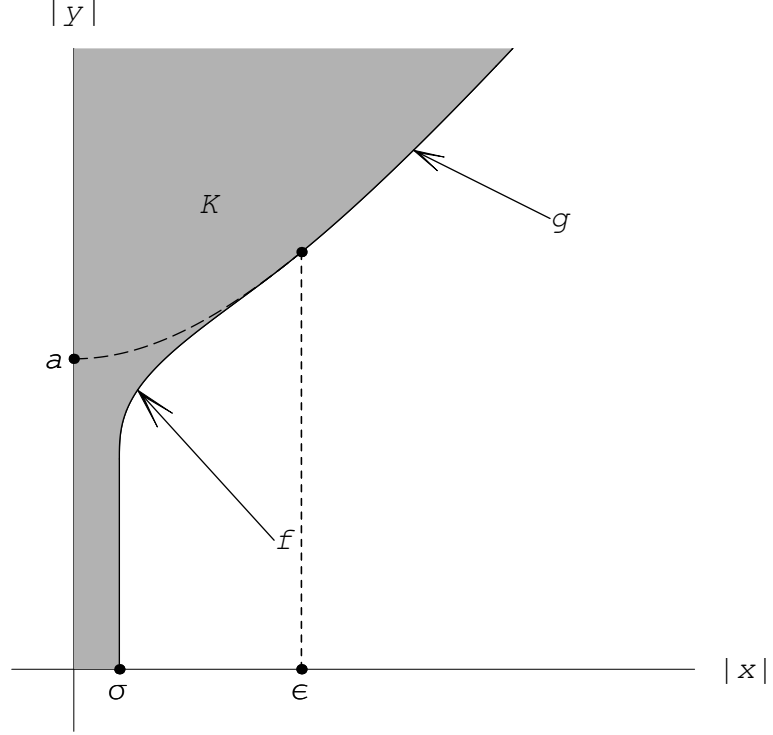


Figure 2: The handlebody K

A calculation gives for $t > 0$

$$g'(t) = \frac{\lambda t}{g(t)} > 0, \quad g''(t) = \frac{\lambda}{g(t)^3} > 0, \quad g'''(t) = -\frac{3\lambda^2 t}{g(t)^5} < 0$$

which shows that g is increasing, convex, and g' is concave. We also obtain $g'(t) - tg''(t) = \lambda^2 t^3 / g(t)^3 > 0$. Fix a small $\epsilon > 0$ and let $c := g'(\epsilon) - \epsilon g''(\epsilon) > 0$. Choose a number $0 < \eta < \min(\epsilon, c^3/3)$ and let $c_1 := c + \eta g''(\epsilon)$. Let $\sigma > 0$ be a number satisfying $2\sigma < \eta < \epsilon$ (its precise value will be determined later).

We shall first obtain a solution f of class \mathcal{C}^1 and piecewise \mathcal{C}^2 on $(\sigma, +\infty)$; the final solution will be obtained by smoothing. Let

$$f(t) = g(\epsilon) + \int_{\epsilon}^t f'(\tau) d\tau \quad (\sigma \leq t < +\infty)$$

where f' is a continuous and piecewise \mathcal{C}^1 function defined as follows:

$$f'(t) = \begin{cases} g'(t), & \text{if } \epsilon \leq t; \\ g'(\epsilon) + g''(\epsilon)(t - \epsilon), & \text{if } \eta \leq t < \epsilon; \\ c_1 + \eta \log(\eta/t), & \text{if } 2\sigma \leq t < \eta; \\ 2\sqrt{\sigma}/\sqrt{t - \sigma}, & \text{if } \sigma < t < 2\sigma. \end{cases}$$

The graph of f' is shown on Figure 3. (However, due to technical difficulties we show the case for large ϵ . For small $\epsilon > 0$ the derivative of the linear part of the graph should be close to $\lambda > 1$. The same remark applies to Figure 5.)

Note that f' is continuous at $t = \eta$, with $f'(\eta) = c_1$. To insure the continuity of f' at $t = 2\sigma$ we choose σ to be the solution of $c_1 + \eta \log(\eta/2\sigma) = 2$. Clearly $f(t) = g(t)$ for $t \geq \epsilon$. It is also clear that $f'(t) > g'(t)$ for $\sigma < t < \epsilon$: on $t \in [\eta, \epsilon]$ the graph of f' is the tangent line to the graph of g' at $(\epsilon, g'(\epsilon))$ which stays above g' due to the concavity of g' ; on $(\sigma, \eta]$ this is clear since g' is increasing while f' is decreasing. Hence f is strictly increasing and satisfies $f(t) < g(t)$ for $\sigma \leq t < \epsilon$. Also $f'(\sigma+) = +\infty$. It remains to show that $f(\sigma+) > 0$ and that f satisfies (9) on (σ, ϵ) .

Case 1: $\eta \leq t < \epsilon$. On this interval

$$f'(t) = g'(\epsilon) + g''(\epsilon)(t - \epsilon) = c + tg''(\epsilon) > tg''(\epsilon).$$

The graph of f' is the tangent line to the graph of g' at the point $(\epsilon, g'(\epsilon))$. Since g' is strongly concave, we conclude $f'(t) > g'(t)$ for all $t \in [\eta, \epsilon)$. We have

$$f(t) > g(\epsilon) - \int_0^{\epsilon} (g'(\epsilon) + g''(\epsilon)(\tau - \epsilon)) d\tau > g(\epsilon) - \epsilon g'(\epsilon) = 1/g(\epsilon).$$

Since $f'(t) > 0$ and $f''(t) = g''(\epsilon)$, we get $f(f'' + f'^3/t) > ff'' > g''(\epsilon)/g(\epsilon) = \lambda/g(\epsilon)^4$ which is > 1 if ϵ is small (since $\lambda > 1$ and $g(\epsilon) \approx g(0) = 1$). From $f(t) > 1/g(\epsilon)$ and $f'(t) > tg''(\epsilon)$ we also get $f(t)f'(t)/t > g''(\epsilon)/g(\epsilon) > 1$.

Case 2: $2\sigma \leq t < \eta$. Using $f(\eta) > 1/g(\epsilon)$, $f'(\eta) < g'(\epsilon)$ (Case 1) we get

$$f(t) > f(\eta) - \int_0^{\eta} (f'(\eta) - \eta \log(\tau/\eta)) d\tau > \frac{1}{g(\epsilon)} - \eta g'(\epsilon) - \eta^2 =: M.$$

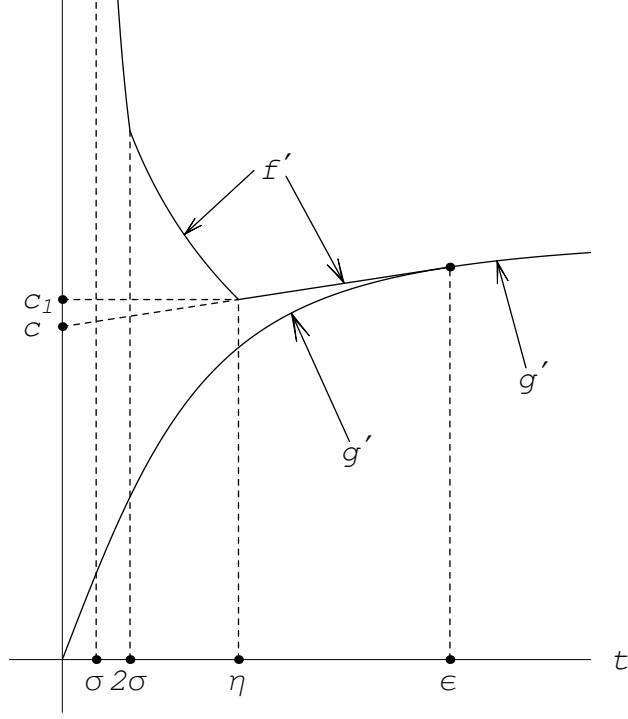


Figure 3: The graph of f'

Clearly $M > 1/2$ when $\epsilon > 0$ is small. From $f''(t) = -\eta/t$, $f'(t) = f'(\eta) + \eta \log(\eta/t) > f'(\eta) > c > 0$ and $0 < 3\eta < c^3$ we obtain

$$f\left(f'' + \frac{f'^3}{t}\right) - 1 > M\left(\frac{-\eta}{t} + \frac{c^3}{t}\right) - 1 > \frac{M}{t}(c^3 - \eta - 2t) > \frac{M}{\eta}(c^3 - 3\eta) > 0.$$

Also, $ff'/t > Mf'(\eta)/\eta > Mc/\eta > 3M/c^2 > 1$ (since $c > 0$ is small) which verifies the second inequality in (9).

Case 3: $\sigma < t < 2\sigma$. As before we easily obtain a lower bound $f(t) > 1/2$ provided that $\epsilon > 0$ is sufficiently small. We have $f'(t) = 2\sqrt{\sigma}/\sqrt{t-\sigma}$, $f''(t) = -\sqrt{\sigma}/\sqrt{t-\sigma}^3$, and hence

$$f\left(f'' + f'^3/t\right) > \frac{1}{2}\left(\frac{-\sqrt{\sigma}}{\sqrt{t-\sigma}^3} + \frac{8\sigma\sqrt{\sigma}}{t\sqrt{t-\sigma}^3}\right) > \frac{\sqrt{\sigma}}{2\sqrt{t-\sigma}^3}(-1+4) \geq \frac{3}{2\sigma} > 1.$$

The second inequality in (2) is trivial as in Case 2.

The function f constructed above is invertible and its inverse function $f^{-1}: [f(\sigma), +\infty) \rightarrow [\sigma, +\infty)$ is of class \mathcal{C}^1 , piecewise \mathcal{C}^2 (actually piecewise real-analytic), and it satisfies (8). We extend f^{-1} to $[0, +\infty)$ by taking $f^{-1}(u) = \sigma$

for $u \in [0, f(\sigma)]$; this extension satisfies the same properties also near the point $u = f(\sigma)$. The final solution is obtained by smoothing $h := f^{-1}$ in a small neighborhood of any point of discontinuity of its second derivative. (We interpolate smoothly between the left and the right limit of h'' at such a point and integrate back; this does not change h and h' very much and hence the inequality (8) is preserved.) This completes the proof. ♠

A small modification of the above construction gives strongly pseudoconvex handlebodies $L \subset \mathbb{C}^n$ with center

$$E = \{x + iy \in \mathbb{C}^n : |y|^2 \leq \lambda|x|^2 + 1\} \cup i\mathbb{R}^n \quad (\lambda < 1).$$

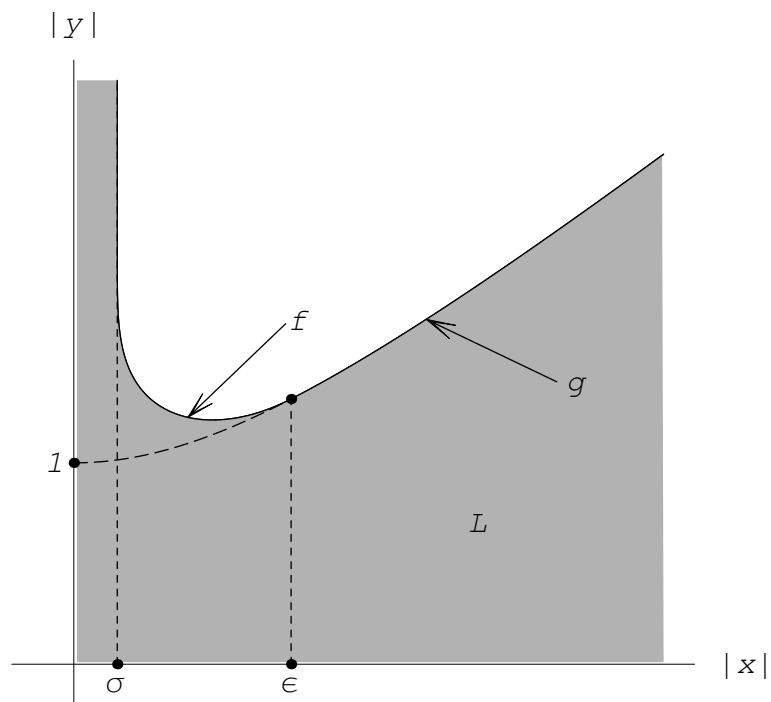


Figure 4: The handlebody L

A typical L is shown on Figure 4. Observe that $D = \{|y|^2 \leq \lambda|x|^2 + 1\}$ is strongly pseudoconvex precisely when $\lambda < 1$. It is an unbounded hyperboloid when $0 < \lambda < 1$, a tube when $\lambda = 0$ and a bounded ellipsoid when $\lambda < 0$. The Lagrangian plane $i\mathbb{R}^n$ is an (unbounded) handle attached to D along the sphere $\{iy : y \in \mathbb{R}^n, |y| = 1\}$. Boonstra [Bo] found explicit handlebodies for $\lambda < 0$ and gave an indirect ‘bumping and patching’ construction for $0 \leq \lambda < 1$. We give an explicit construction for all $\lambda < 1$. (Our example is easily modified to obtain handlebodies with center $D \cup M$ where $M \subset i\mathbb{R}^n$ is a compact domain

such that $D \cap i\mathbb{R}^n$ is contained in the relative interior of M .) Set

$$L = \{x + iy: |x| > \sigma, |y| \leq f(|x|)\} \cup \{x + iy: |x| \leq \sigma\}$$

where f is given by the following proposition. ♠

3.3 Proposition. *Let $\lambda < 1$ and $g(t) = +\sqrt{\lambda t^2 + 1}$. For every sufficiently small $\epsilon > 0$ there exists a number $\sigma = \sigma(\epsilon) \in (0, \epsilon)$ and a continuous function $f: [\sigma, +\infty) \rightarrow (0, +\infty)$, smooth on $(\sigma, +\infty)$, which satisfies the inequalities (8) and the following:*

- (i) $f(t) = g(t)$ for $t \geq \epsilon$,
- (ii) $f(t) > g(t)$ for $\sigma \leq t < \epsilon$,
- (iii) $f'(\sigma+) = \lim_{t \downarrow \sigma} f'(t) = -\infty$,
- (iv) *there exists a smooth inverse function f^{-1} near the point $u = f(\sigma)$, with $f^{-1}(u) = \sigma$ for $u \geq f(\sigma)$, satisfying the inequalities (8) on its domain.*

Proof. Choose numbers $0 < \sigma < 2\sigma < \eta < \epsilon$; additional conditions will be imposed later. We have $g'(\epsilon)/\epsilon = \lambda/g(\epsilon) < 1$. Choose a number k satisfying $g'(\epsilon)/\epsilon < k < 1$ and let $c := g'(\epsilon) - k\epsilon < 0$. Clearly $c > -1$ if ϵ is small. Choosing $\eta > 0$ sufficiently small we have $c_1 := g'(\epsilon) + k(\eta - \epsilon) = c + k\eta < 0$ and $\eta + c_1^3 < 0$. Let $\sigma \in (0, \eta/2)$ solve $c_1 - \eta \log(\eta/2\sigma) = -2$. With these choices we define f on $(\sigma, +\infty)$ by $f(t) = g(\epsilon) + \int_{\epsilon}^t f'(\tau) d\tau$ where

$$f'(t) = \begin{cases} g'(t), & \text{if } \epsilon \leq t; \\ g'(\epsilon) + k(t - \epsilon), & \text{if } \eta \leq t < \epsilon; \\ c_1 - \eta \log(\eta/t), & \text{if } 2\sigma \leq t < \eta; \\ -2\sqrt{\sigma}/\sqrt{t - \sigma}, & \text{if } \sigma < t < 2\sigma. \end{cases}$$

The graph of f' is shown on Figure 5. We verify that f satisfies (8). For $t \geq \epsilon$ this is clear since $f(t) = g(t)$. For $\eta \leq t < \epsilon$ we have

$$g(t) < f(t) \leq g(\epsilon) + \left| \int_t^{\epsilon} (g'(\epsilon) + k(\tau - \epsilon)) d\tau \right| < g(\epsilon) + \epsilon(1 + |g'(\epsilon)|).$$

By our choice of k the graph of f lies below the secant line through $(0, 0)$ and $(\epsilon, g'(\epsilon))$, and the secant is below g' due to concavity of g' . This gives $f'(t) < g'(\epsilon)t/\epsilon = \lambda t/g(\epsilon)$. Also, $f''(t) = k$. At points $t \in [\eta, \epsilon)$ where $f''(t) + f'(t)^3/t > 0$ we thus have

$$f(t)(f''(t) + f'(t)^3/t) \leq (g(\epsilon) + \epsilon(1 + |g'(\epsilon)|)) (k + \lambda^3 \epsilon^2/g(\epsilon)^3) < 1$$

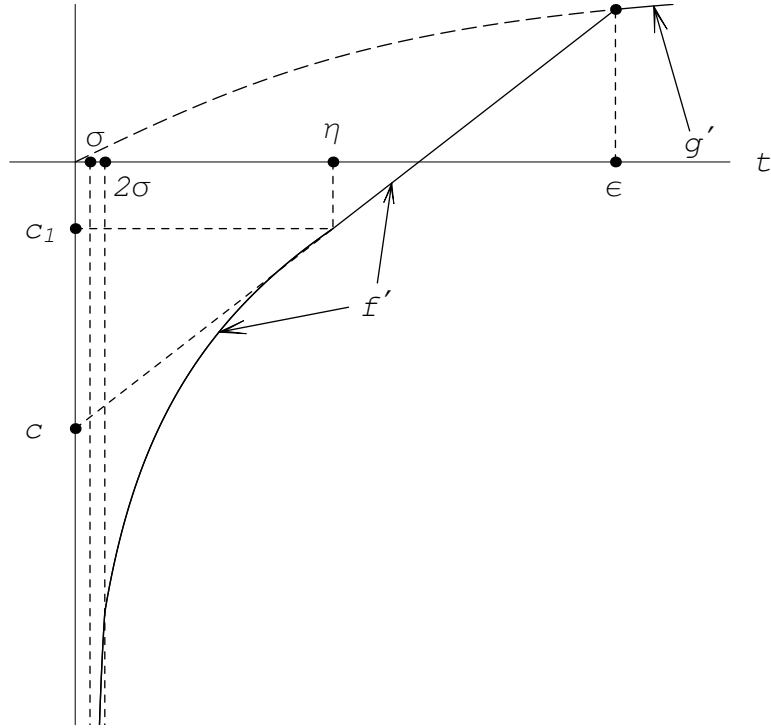


Figure 5: The graph of f'

provided that $\epsilon > 0$ is sufficiently small (since $k < 1$, $g(\epsilon) \approx 1$ and the other quantities are $O(\epsilon)$). At points where $f''(t) + f'(t)^3/t \leq 0$ the same estimate holds since $f(t) > 0$. Also, $f(t)f'(t)/t < (g(\epsilon) + O(\epsilon))\lambda/g(\epsilon) = \lambda + O(\epsilon) < 1$ if $\epsilon > 0$ is small.

For $t \in [2\sigma, \eta]$ the estimates (8) are almost trivial: from $f'(t) \leq f'(\eta) = c_1 < 0$ and $f''(t) = \eta/t$ we get $f''(t) + f'(t)^3/t \leq (\eta + c_1^3)/t < 0$ which implies the first estimate in (8) (since $f(t) > g(t) > 0$). Also $f(t)f'(t)/t < 0$ and hence (8) holds. Similarly we verify (8) on $(\sigma, 2\sigma]$. We complete the proof as in Proposition 3.1 by smoothing f^{-1} . ♠

3.4 A remark on [E]. Our Proposition 3.1 corresponds to Lemma 3.4.3. in [E] which is obtained by combining Lemmas 3.4.1. and 3.4.2. according to the sentence preceding Lemma 3.4.3. on p. 43. Lemma 3.4.2. in [E] provides a function φ on an interval $[\sigma, \epsilon^3/2]$ which satisfies the conditions (*, **) [E, p. 38] (implying strongy pseudoconvexity of $\{|y| > \varphi(|x|)\}$) and the boundary condition $\varphi'(\epsilon^3/2) = \epsilon$. A precise definition of φ is given on p. 43 in [E]. On the same page the author claims that Lemma 3.4.1. provides a (presumably \mathcal{C}^2) function $f: [\epsilon^3/2, \epsilon] \rightarrow (0, +\infty)$ satisfying (*, **) [E, p. 38], having a first order contact with φ at $t = \epsilon^3/2$ and with $g(t) = \sqrt{\lambda t^2 + 1}$ at $t = \epsilon$, and satisfying $f(t) \leq g(t)$ for all t . (In [E] our $\lambda > 1$ is written as λ^2 and our t is

denoted by r .) This should complete the proof of Lemma 3.4.3.

We show here that *there exists no function with these properties on $[\epsilon^3/2, \epsilon]$ if $\epsilon > 0$ is sufficiently small and $\lambda > 1$ is sufficiently close to 1.* If such f exists, it satisfies our condition (9) which is implied by (*, **) from [E] (Remark 2.5 above). Since $f(t) \leq g(\epsilon) = \sqrt{\lambda\epsilon^2 + 1}$ for all $t \in [\epsilon^3/2, \epsilon]$, (9) implies $f''(t) + f'(t)^3/t > 1/g(\epsilon)$ which is close to 1 if $\epsilon > 0$ is small. Write $h = f'$; thus $h \in \mathcal{C}^1([\epsilon^3/2, \epsilon])$ satisfies

$$h'(t) + h(t)^3/t > 1/g(\epsilon) \quad (t \in [\epsilon^3/2, \epsilon]); \quad (10)$$

$$h(\epsilon^3/2) = \epsilon, \quad h(\epsilon) = g'(\epsilon) = \lambda\epsilon/g(\epsilon) < \lambda\epsilon. \quad (11)$$

From (10), (11) we get $h'(\epsilon) > 1/g(\epsilon) - (\lambda\epsilon)^3/\epsilon = 1/g(\epsilon) - \lambda^3\epsilon^2$ which is close to 1 if ϵ is small. Consider first the case $h'(t) > 0$ for all $t \in [\epsilon^3/2, \epsilon]$. Then $h(2\epsilon^3) > h(\epsilon^3/2) = \epsilon$ and hence

$$\frac{h(\epsilon) - h(2\epsilon^3)}{\epsilon - 2\epsilon^3} < \frac{\lambda\epsilon - \epsilon}{\epsilon - 2\epsilon^3} = \frac{\lambda - 1}{1 - 2\epsilon^2} < 2(\lambda - 1)$$

if $\epsilon > 0$ is small. By Lagrange's mean value theorem the left hand side above equals $h'(\mu)$ for some $\mu \in (2\epsilon^3, \epsilon)$ and hence $h'(\mu) < 2(\lambda - 1)$. Since $h(\mu)^3/\mu < h(\epsilon)^3/\mu < (\lambda\epsilon)^3/2\epsilon^3 = \lambda^3/2$, we have

$$h'(\mu) + h(\mu)^3/\mu < 2(\lambda - 1) + \lambda^3/2$$

provided that $\epsilon > 0$ is sufficiently small. The right hand side is close to $1/2$ when λ is close to 1 which gives a contradiction to (10) at $t = \mu$ (since $1/g(\epsilon)$ is close to 1).

Suppose now that $h'(t) \leq 0$ for some $t \in [\epsilon^3/2, \epsilon]$. Since we have seen that $h'(\epsilon) > 0$, there exists a $\xi \in [\epsilon^3/2, \epsilon]$ such that $h'(\xi) = 0$ and $h'(t) > 0$ for $\xi < t \leq \epsilon$. Thus h is strictly increasing on $[\xi, \epsilon]$. From (10) we get $h(\xi)^3 > \xi/g(\epsilon)$ and hence $\xi < g(\epsilon)h(\xi)^3 < g(\epsilon)h(\epsilon)^3 < g(\epsilon)(\lambda\epsilon)^3$. Thus $4\xi < \epsilon$ provided that $\epsilon > 0$ is sufficiently small. By monotonicity we also have $h(4\xi) < h(\epsilon) < \lambda\epsilon$ and $h(4\xi) > h(\xi) > (\xi/g(\epsilon))^{1/3} > (\epsilon^3/2g(\epsilon))^{1/3} = \epsilon(2g(\epsilon))^{-1/3}$. Thus

$$\frac{h(\epsilon) - h(4\xi)}{\epsilon - 4\xi} < \frac{\lambda\epsilon - \epsilon(2g(\epsilon))^{-1/3}}{\epsilon - 4g(\epsilon)(\lambda\epsilon)^3} < \frac{\lambda - (2g(\epsilon))^{-1/3}}{1 - 4g(\epsilon)\lambda^3\epsilon^2} =: C(\lambda, \epsilon).$$

By Lagrange's theorem the left hand side above equals $h'(\mu)$ for some $\mu \in (4\xi, \epsilon)$. Since $\mu > 4\xi > 2\epsilon^3$, we also have $h(\mu)^3/\mu < (\lambda\epsilon)^3/2\epsilon^3 = \lambda^3/2$. Hence

$$h'(\mu) + h(\mu)^3/\mu < C(\lambda, \epsilon) + \lambda^3/2.$$

When $\epsilon \rightarrow 0$ the right hand side converges to $\lambda - 2^{-1/3} + \lambda^3/2$. If $\lambda > 1$ is close to one, this is close to $3/2 - 2^{-1/3} \approx 0.7 < 1$. For such values the inequality (10) is violated at $t = \mu$. This contradiction shows that no solution exists for $\lambda > 1$ close to 1 and $\epsilon > 0$ close to 0, nor does there exist a piecewise \mathcal{C}^2 solution in view of smoothing.

4 Handlebodies on general quadratic domains

In this section we consider handlebodies modeled on general quadratic strongly plurisubharmonic functions $\rho: \mathbb{C}^n \rightarrow \mathbb{R}$. Choose a $k \in \{0, 1, \dots, n\}$ and write the coordinates on \mathbb{C}^n in the form $\zeta = (z, w)$ with $z = x + iy \in \mathbb{C}^k$ and $w = u + iv \in \mathbb{C}^{n-k}$. Let A, B be positive definite real symmetric matrices of dimension k resp. $n - k$. Denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product on any \mathbb{R}^m . Given these choices let

$$\rho(z, w) = Q(y, w) - |x|^2, \quad Q(y, w) = \langle Ay, y \rangle + \langle Bv, v \rangle + |u|^2. \quad (12)$$

It is easily seen that ρ is strongly plurisubharmonic if and only if all eigenvalues of A are larger than 1. (Equivalently, the matrix $A - I$ must be positive definite which we denote by $A > I$.) Clearly ρ has a Morse critical point of index k at the origin and no other critical points. It is proved in [HaW] that every Morse critical point of a strongly plurisubharmonic function is of this form (modulo terms of order > 2) in some local holomorphic coordinates.

Assume now that $k \geq 1$. Let $\Lambda^k = \{(x + i0, 0) \in \mathbb{C}^n : x \in \mathbb{R}^k\}$. We identify $x \in \mathbb{R}^k$ with $(x + i0, 0) \in \Lambda^k \subset \mathbb{C}^n$ when appropriate.

4.1 Proposition. (Notation as above.) *Let ρ be given by (12) where $A > I$, $B > 0$. Given $r > 0, \epsilon > 0$ there exist constants $0 < r < c_0 < R$, $\delta > 0$ and a smooth, increasing, weakly convex function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\tau(z, w) = Q(y, w) - h(|x|^2)$ is a strongly plurisubharmonic function on \mathbb{C}^n , with a Morse critical point of index k at $0 \in \mathbb{C}^n$, satisfying the following:*

- (i) for $|x|^2 \leq r$ we have $\tau(z, w) = Q(y, w) - \delta|x|^2$,
- (ii) for $|x|^2 \geq R$ we have $\tau(z, w) = \rho(z, w) + c_0$, and
- (iii) $\{\rho \leq -c_0\} \cup \Lambda^k \subset \{\tau \leq 0\} \subset \{\rho < -r\} \cup \{Q < \epsilon\}$.

4.2 Corollary. *For every sufficiently small $c > 0$ the set $K_c = \{\tau \leq c\}$ is a strongly pseudoconvex handlebody with center*

$$E_{c-c_0} = \{\zeta \in \mathbb{C}^n : \rho(\zeta) \leq c - c_0\} \cup \Lambda^k$$

satisfying $E_{c-c_0} \subset K_c \subset \{\rho < -r\} \cup \{Q < \epsilon\}$.

Proof of Proposition 4.1. We modify slightly the construction in Sect. 5 of [F] (the function constructed there was not Morse). Let $t_0 = r + \epsilon$. Choose

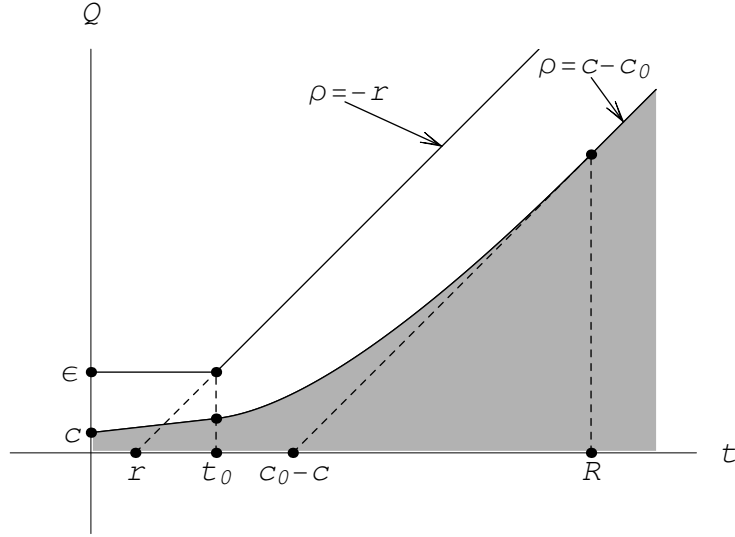


Figure 6: The handlebody $K_c = \{\tau \leq c\}$

$0 < \delta < 1, \mu > 1$ such that $\delta t_0 < \epsilon$ and $1 < \mu + \delta < \lambda_1$ where $\lambda_1 > 1$ denotes the smallest eigenvalue of A . Set $R = \mu^2 t_0 / (\mu + \delta - 1)^2$ and

$$h(t) = \begin{cases} \delta t, & \text{if } 0 \leq t \leq t_0; \\ \delta t + \mu(\sqrt{t} - \sqrt{t_0})^2, & \text{if } t_0 < t \leq R; \\ t - R + h(R), & \text{if } R < t \end{cases} .$$

It is easily verified that h is an increasing convex function of class \mathcal{C}^1 and piecewise \mathcal{C}^2 on \mathbb{R} which satisfies

$$2t\ddot{h} + \dot{h} = \mu + \delta < \lambda_1 \quad (t_0 \leq t \leq R)$$

and $\delta = \dot{h}(t_0) \leq \dot{h}(t) \leq 1 = \dot{h}(R)$ for all $t \in \mathbb{R}$. By smoothing h we obtain an increasing convex \mathcal{C}^∞ function, still denoted h , which equals δt for $0 \leq t \leq t_0$, it equals $t - R + h(R)$ for $t \geq R$, and satisfies

$$\dot{h}(t) < \lambda_1, \quad 2t\ddot{h}(t) + \dot{h}(t) < \lambda_1 \quad (t \in \mathbb{R}).$$

A simple calculation shows that, as a consequence of these inequalities, the associated function τ is strongly plurisubharmonic on \mathbb{C}^n and satisfies Proposition 4.1 with $c_0 = R - h(R)$. ♠

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Institute of Mathematics, Physics and Mechanics, University of Ljubljana,
Jadranska 19, 1000 Ljubljana, Slovenia