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ANALYTICITY ON FAMILIES OF CIRCLES

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ABSTRACT It is known that if f is a continuous function on the complex plane which extends holomorphically from each circle surrounding the origin then f is not necessarily holomorphic. In the paper we prove that if, in addition, f extends holomorphically from each circle belonging to an open family of circles which do not surround the origin then f is holomorphic.

1. Introduction and the main result

Write $\Delta(a,\rho) = \{\zeta \in \mathbb{C}\colon\ |\zeta - a| < \rho\}$ and $\Delta = \Delta(0,1)$. If $0 < r_1 < r_2 < \infty$ write $A(a, r_1, r_2) = \{\zeta \in \mathbb{C}\colon\ r_1 \le |\zeta - a| \le r_2\}$. We say that a continuous function on $b\Delta(a,\rho)$ extends holomorphically from $b\Delta(a,\rho)$ if it has a continuous extension to $\overline{\Delta}(a,\rho)$ which is holomorphic on $\Delta(a,\rho)$.

A family $\mathcal{C}$ of circles is called a *test family for holomorphy* (on $\mathbb{C}$) if every continuous function on $\mathbb{C}$ that extends holomorphically from each circle in $\mathcal{C}$ is holomorphic on $\mathbb{C}$. We will consider open families of circles, that is, families of the form $\{b\Delta(a,\rho)\colon\ (a,\rho) \in \mathcal{P}\}$ where $\mathcal{P}$ is an open subset of $\mathbb{C} \times (0,\infty)$.

There are large families of circles that are not test families for holomorphy. For instance, the function

$$f(z) = \begin{cases} z^2/\overline{z} & (z \in \mathbb{C} \setminus \{0\}) \\ 0 & (z = 0) \end{cases}$$

is continuous on $\mathbb{C}$ and extends holomorphically from each circle that surrounds the origin, yet f is not holomorphic. This shows that the family of all circles that surround the origin is not a test family for holomorphy. In the present paper we prove that the family of all circles that surround the origin is a maximal open family that is not a test family for holomorphy:

Theorem 1.1 *Let f be a continuous function on $\mathbb{C} \setminus \{0\}$ which extends holomorphically from each circle that surrounds the origin. Suppose that, in addition, f extends holomorphically from each circle belonging to a nonempty open family of circles that do not surround the origin. Then f is an entire function, that is, f is a holomorphic function on $\mathbb{C} \setminus \{0\}$ which has a removable singularity at* 0.

We prove Theorem 1.1 in the first part of the paper. To do this, we first follow [G2] to formulate the problem in $\mathbb{C}^2$. Then we prove the theorem by using some tools from several complex variables, in particular, the edge of the wedge theorem and the continuity principle. In the second part we look at special cases of nonholomorphic continuous functions on $\mathbb{C} \setminus \{0\}$ which extend holomorphically from every circle surrounding the origin. In particular, we consider functions constant on lines passing through the origin and functions constant on rays passing through the origin.

2. Varieties $\Lambda_{a,\rho}$ and domains $\Omega(A)$

Given $a \in \mathbb{C}$ and $\rho > 0$ let $\Lambda_{a,\rho} = \{(z,w) \in \mathbb{C}^2\colon\ (z-a)(w-\overline{a}) = \rho^2,\ 0 <$

$|z-a| < \rho\}$. This is a closed complex submanifold of $\mathbb{C}^2 \setminus \Sigma$ attached to the real two plane $\Sigma = \{(z,\overline{z})\colon\ z \in \mathbb{C}\}$ along the circle $b\Lambda_{a,\rho} = \{(z,\overline{z})\colon\ z \in b\Delta(a,\rho)\}$. We call $\Lambda_{a,\rho}$ *the variety associated with* $b\Delta(a,\rho)$. The varieties $\Lambda_{a,\rho}$ are important in our context because of the following property: a continuous function f on $b\Delta(a,\rho)$ extends holomorphically from $b\Delta(a,\rho)$ if and only if the function $(z,\overline{z}) \mapsto F(z,\overline{z}) = f(z)$ defined on $b\Lambda_{a,\rho}$ has a bounded continuous extension to $\Lambda_{a,\rho} \cup b\Lambda_{a,\rho}$ which is holomorphic on $\Lambda_{a,\rho}$ [G2].

Let $0 < r_1 < r_2 < \infty$ and let $a \in \mathbb{C}$. Denote by $\Omega(A(a,r_1,r_2))$ the union of all $\Lambda_{b,\rho}$ such that $b\Delta(b,\rho) \subset \mathrm{Int}A(a,r_1,r_2)$ surrounds a. The set $\Omega(A(a,r_1,r_2))$ is an unbounded open connected subset of $\mathbb{C}^2 \setminus \Sigma$ which is attached to Σ along $\{(z,\overline{z})\colon\ z \in \mathrm{Int}A(a,r_1,r_2)$. We shall need the following

Theorem 2.1 [G2] *Let f be a continuous function on $A(a,r_1,r_2)$. The following are equivalent*

(i) f extends holomorphically from each circle $b\Delta(b,\rho) \subset A(a,r_1,r_2)$ which surrounds the point a

(ii) the function $F(z,\overline{z}) = f(z)$ defined on $\{(z,\overline{z})\colon\ z \in A(a,r_1,r_2)\}$ extends to a bounded continuous function on $\Omega(A(a,r_1,r_2)) \cup b\Omega(A(a,r_1,r_2))$ which is holomorphic on $\Omega(A(a,r_1,r_2))$.

We list some simple properties of $\Lambda_{a,\rho}$ and $\Omega(A)$. The proofs are elementary. They can be found in [G2].

Proposition 2.1 *Let $(z,w) \in \mathbb{C}^2 \setminus \Sigma$. Then $(z,w) \in \Lambda_{a,R}$ if and only if there is a $t > 0$ such that $a = z + t(z-\overline{w})$ and $R = \sqrt{t(t+1)}|z-\overline{w}|$. In fact, given $R > 0$ we have*

$$a = z + 2^{-1}[\sqrt{1 + 4R^2/|z-\overline{w}|^2} - 1](z-\overline{w}). \tag{2.1}$$

Note that the two dimensional subspace perpendicular to the Lagrangian two-plane Σ is $i\Sigma = \{(z,-\overline{z})\colon\ z \in \mathbb{C}\}$. Our next lemma tells how a variety $\Lambda_{a,\rho}$ intersects the two dimensional planes perpendicular to Σ:

Proposition 2.2 *Let $z \in \mathbb{C}$, $t > 0$ and $\varphi \in \mathbb{R}$. Then $(z,\overline{z}) + (te^{i\varphi}, -te^{-i\varphi}) \in \Lambda_{a,R}$ if and only if $a = z + \sqrt{t^2+R^2}e^{i\varphi}$.*

Proposition 2.3 *Let $A = A(a,r_1,r_2)$. Then $\Omega(A)$ is an unbounded open connected subset of $\mathbb{C}^2 \setminus \Sigma$ attached to Σ along $\{(z,\overline{z})\colon\ z \in IntA\}$. If $\gamma = (r_1+r_2)/2$ then $b\Omega(A)$ consists of $\tilde{A} = \{(z,\overline{z})\colon\ z \in A\}$ together with all $\Lambda_{b,\rho}$ associated with those $b\Delta(b,\gamma) \subset A$ which are tangent to both $b\Delta(a,r_1)$ and $b\Delta(a,r_2)$. Further, $\Omega(A)$ is a disjoint union of $\Lambda_{b,\gamma}$ such that $b\Delta(b,\gamma) \subset IntA$.*

Let $z_0 \in \mathrm{Int}A$. Let $\Gamma(z_0) \subset \mathrm{Int}A$ be the circle of radius $\gamma = (r_1+r_2)/2$. which passes through z_0 and whose center $b(z_0)$ lies on the line through a and z_0. For each $\varphi \in \mathbb{R}$ define $T_\varphi(z) = z_0 + e^{i\varphi}(z-z_0)$. T_φ is the rotation with center z_0 for the angle φ. There is a $\delta(z_0)$, $0 < \delta(z_0) < \pi/2$, such that $T_\varphi(\Gamma(z_0)) \subset \mathrm{Int}A$ $(-\delta(z_0) < \varphi < \delta(z_0))$ and such that both $T_{\delta(z_0)}(\Gamma(z_0))$ and $T_{-\delta(z_0)}(\Gamma(z_0))$ meet bA (in fact, they meet both circles that bound A). Fix φ, $-\delta(z_0) < \varphi < \delta(z_0)$. There is a $\tau(z_0,\varphi) > 0$ such that

$$T_\varphi(\Gamma(z_0)) + t\frac{b(z_0)-z_0}{|b(z_0)-z_0|}e^{i\varphi} \subset \mathrm{Int}A \quad (0 \le t < \tau(z_0,\varphi))$$

while $T_\varphi(\Gamma(z_0))+\tau(z_0,\varphi)((b(z_0)-z_0)/|b(z_0)-z_0|)e^{i\varphi}$ meets bA (in fact, it meets both circles that bound A). For each φ, $-\delta(z_0)<\varphi<\delta(z_0)$, let $\eta(z_0,\varphi)=\sqrt{\tau(z_0,\varphi)^2+2\tau(z_0,\varphi)\gamma}$. By Proposition 2.2 we have

$$(z_0,\overline{z_0})+\left(t\frac{b(z_0)-z_0}{|b(z_0)-z_0|}e^{i\varphi},-t\frac{\overline{b(z_0)}-\overline{z_0}}{|b(z_0)-z_0|}e^{-i\varphi}\right)\in\Omega(A)$$

provided that $0<t<\eta(z_0,\varphi)$. Let

$$D(z_0)=\{t\frac{b(z_0)-z_0}{|b(z_0)-z_0|}e^{i\varphi}\colon\ 0<t<\eta(z_0,\varphi),\ -\delta(z_0)<\varphi<\delta(z_0)\}.$$

It is easy to see that the function δ is continuous on Int$A$ and that η is a continuous function of z_0 and φ where it is defined. For each $z_0\in$ Int$A$ the set

$$(z_0,\overline{z_0})+\{(\zeta,-\overline{\zeta})\colon\ \zeta\in D(z_0)\}$$

is contained in $\Omega(A)$. This proves

Proposition 2.4 *Let $z_0\in$ Int$A$. There are a neighbourhood $U\subset\Sigma$ of $(z_0,\overline{z_0})$, an open convex cone $\mathcal{C}\subset\mathbb{C}$ with vertex at the origin, containing $\{t(a-z_0)\colon\ t>0\}$, and an $r>0$ such that if*

$$P=\{(\zeta,-\overline{\zeta})\colon\ \zeta\in\mathcal{C},\ |\zeta|<r\}$$

then $U+P\subset\Omega(A)$.

Fix $R>0$. From (2.1) we get that if $(z,w)\in\Lambda_{a,R}$ then $|a|\le|z|+2R^2/|z-\overline{w}|$ which, by Proposition 2.3, implies that given $r_1,r_2,\ 0<r_1<r_2<\infty$

$$\begin{aligned}&\text{there are } \delta>0 \text{ and } M<\infty \text{ such that}\\ &\{(z,w)\colon\ |z|\le\delta,\ |w|\ge M\}\subset\Omega(A(0,r_1,r_2)).\end{aligned}\tag{2.2}$$

3. Functions that extend holomorphically from every circle which surrounds the origin

Suppose that f is a continuous function on $\mathbb{C}\setminus\{0\}$ which extends holomorphically from each circle that surrounds the origin. Define F on $\Sigma\setminus\{(0,0)\}$ by

$$F(z,\overline{z})=f(z).$$

Theorem 2.1 implies that for each $n\in\mathbb{N},\ n\ge2$ the function F extends from $\tilde{A}(0,1/n,n)=\{(z,\overline{z})\colon\ z\in A(0,1/n,n)\}$ to a bounded continuous function on $\Omega(A(0,1/n,n))\cup b\Omega(A(0,1/n,n)$ which is holomorphic on $\Omega(A(0,1/n,n))$. This defines a holomorphic function $F(z,w)$ on

$$\Omega=\cup_{n=2}^{\infty}\Omega(A(0,1/n,n))$$

whose restriction to each $\Omega(A(0,1/n,n))$ has a bounded continuous extension to $\Omega(A(0, 1/n,n)) \cup b\Omega(A(0,1/n,n)$ which is holomorphic on $\Omega(A(0,1/n,n))$ and on $\tilde{A}(0,1/n,n) \subset b\Omega(A(0,\ 1/n,n)$ coincides with $F(z,\overline{z})$.

We now show that $\Omega = \{(z,w)\colon |w| > |z|\}$. One way to see this is by using Proposition 2.2. We show this by using Proposition 2.1.

Suppose that $(z,w) \in \mathbb{C}^2 \setminus \Sigma$, that is, $w \neq \overline{z}$. By Proposition 2.1 we have $(z,w) \in \Lambda_{a,\rho}$ if and only if

$$a = z + t(z - \overline{w}), \quad \rho = \sqrt{t(t+1)}|z - \overline{w}| \tag{3.1}$$

for some $t > 0$. Let L be the line through $(z+\overline{w})/2$ which is perpendicular to the line through z and $\overline{w}$ and let $\Pi \subset \mathbb{C}$ be the open halfplane bounded by L which contains z. It is easy to see that Π is the union of all $\Delta(a,\rho)$ such that a and ρ satisfy (3.1) for some $t > 0$. It follows that $(z.w) \in \Lambda_{a,\rho}$ for some $b\Delta(a,\rho)$ that surrounds the origin if and only if $0 \in \Pi$, that is, if and only if $|w| > |z|$.

For each $z \neq 0$ we describe $[(z,\overline{z}) + i\Sigma] \cap \Omega$. Recall that $i\Sigma = \{(\zeta, -\overline{\zeta})\colon\ \zeta \in \mathbb{C}\}$. Write $z = |z|e^{i\alpha}$. Then $(z,\overline{z}) + (\zeta, -\overline{\zeta}) \in \Omega$ if and only if $|z+\zeta| < |\overline{z} - \overline{\zeta}|$, that is, if and only if $\mathrm{Re}(\overline{z}\zeta) < 0$. This happens if and only if $\mathrm{Re}(e^{-i\alpha}\zeta) < 0$, that is, if and only if $\zeta \in e^{i\alpha}\{z\colon\ \mathrm{Re} z < 0\}$. Let $L(z)$ be the line through the origin which is perpendicular to the line through 0 and z and let $P(z)$ be the halfplane bounded by $L(z)$ which does not contain z. Then

$$[(z,\overline{z}) + i\Sigma] \cap \Omega = (z,\overline{z}) + \{(\zeta, -\overline{\zeta})\colon\ \zeta \in P(z)\}.$$

Obviously $i\Sigma \cap \Omega = \emptyset$. Thus, Ω can be written as a disjoint union of halpfplanes

$$\Omega = \cup_{z \in \mathbb{C}\setminus\{0\}} [(z,\overline{z}) + \{(\zeta, -\overline{\zeta})\colon\ \zeta \in P(z)\}].$$

Further,

$$b\Omega = (i\Sigma) \cup (\cup_{z \in \mathbb{C}\setminus\{0\}} \{(\zeta, -\overline{\zeta})\colon\ \zeta \in L(z)\}).$$

Note that we cannot conclude in general that the function F extends continuously to $\Sigma\setminus\{(0,0)\}$. If $(z,\overline{z}) \in \Sigma\setminus\{(0,0)\}$ and if $(z_n,w_n) \in \Omega,\ (z_n,w_n) \to (z,\overline{z})$ then $\lim_{n\to\infty} F(z_n, w_n) = F(z,\ \overline{z}) = f(z)$ provided that there is an $m \in \mathbb{N}$ such that $(z_n,w_n) \in \Omega(A(0,1/m, m))$ for all n.

However, if $F(z,\overline{z}) = f(z)$ has a holomorphic extension Φ into a small ball $B \subset \mathbb{C}^2 \setminus \{(0,0)\}$ centered at $(z_0,\overline{z_0}) \in \Sigma \setminus \{(0,0)\}$ then $\Phi \equiv F$ on $B \cap \Omega$. This follows from the fact that there are a neighbourhood $U \subset \Sigma$ of $(z_0,\overline{z_0})$, an open convex cone $K \subset i\Sigma$ with vertex at the origin and an $\eta > 0$ such that if $K_\eta = \{w \in K,\ |w| < \eta\}$ then $U + K_\eta \subset \Omega(A(0,1/m,m) \cap B$ for some $m \in \mathbb{N}$ and hence $F(z,\overline{z})$ has a continuous extension from U to $U \cup [U + K_\eta]$ which is holomorphic on $U + K_\eta$. However, such extension is unique and since $\Phi|[U \cup [U + K_\eta]]$ is such an extension we must have $\Phi \equiv F$ on $U + K_\eta$. Since $(U + K_\eta) \cap B$ is an open subset of $\Omega \cap B$ and since $\Omega \cap B$ is connected it follows that $\Phi \equiv F$ on $\Omega \cap B$.

4. Intersecting varieties $V_{a,\rho}$ with Ω

Given $a \in \mathbb{C}$ and $\rho > 0$ let

$$V_{a,\rho} = \{(z,w)\colon\ (z-a)(w-\overline{a}) = \rho^2\}.$$

Thus, $\Lambda_{a,\rho} = \{(z,w) \in V_{a,\rho} :\ 0 < |z-a| < \rho\}$. We compute $V_{a,\rho} \cap b\Omega$. The equation of $V_{a,\rho}$ is $w = \overline{a} + \rho^2/(z-a)$, so we compute the intersection of $V_{a,\rho}$ with $b\Omega = \{(z,w)\colon\ |z| = |w|\}$ by solving $|\overline{a} + \rho^2/(z-a)| = |z|$. We get $|\overline{a}(z-a) + \rho^2| = |z|.|z-a|$ so

$$[\overline{a}(z-a) + \rho^2].[a(\overline{z} - \overline{a}) + \rho^2] - \rho^2 z\overline{z} = z\overline{z} = z\overline{z}[(z-a)(\overline{z} - \overline{a}) - \rho^2].$$

The left hand side equals

$$\begin{aligned} & a\overline{a}[(z-a) + \rho^2/\overline{a}].[(\overline{z} - \overline{a}) + \rho^2/a] - \rho^2[(z-a)(\overline{z} - \overline{a}) + a\overline{z} + \overline{a}z - a\overline{a}] = \\ & = a\overline{a}(z-a)(\overline{z} - \overline{a}) + \rho^4 - \rho^2(z-a)(\overline{z} - \overline{a}) + a\overline{a}\rho^2 \\ & = (a\overline{a} - \rho^2)[(z-a)(\overline{z} - \overline{a}) - \rho^2] \end{aligned}$$

and the equation becomes

$$[z\overline{z} - (a\overline{a} - \rho^2].[(z-a)(\overline{z} - \overline{a}) - \rho^2] = 0.$$

If the circle $b\Delta(a,\rho)$ surrounds the origin, that is, if $|a| < \rho$ then the set of solutions is $b\Delta(a,\rho)$. The case of interest to us will be the case when $|a| > \rho$, that is, when $b\Delta(a,\rho)$ does not surround the origin. In this case the set of solutions is $b\Delta(a,\rho) \cup b\Delta(0, \sqrt{|a|^2 - \rho^2})$. Note that these two circles intersect at right angle.

Since

$$|z-a|^2[|\overline{a} + \rho^2/(z-a)|^2 - |z|^2] = [(a\overline{a} - \rho^2 - z\overline{z}].[(z-a)(\overline{z} - \overline{a}) - \rho^2] \tag{4.1}$$

the point $(z, \overline{a} + \rho^2/(z-a))$ belongs to Ω if and only if the expression on the left in (4.1) is positive, that is, if and only if the expression on the right in (4.1) is positive.

5. Proof of Theorem 1.1, Part 1

Suppose that f extends holomorphically from each circle that surrounds the origin. We know that there is a holomorphic function F on Ω which, for each $n \geq 2$ has a bounded continuous extension to $\Omega(A(0,1/n,n)) \cup b\Omega(A(0,1/n,n))$ which coincides with $F(z,\overline{z}) = f(z)$ on $A(0,1/n,n)$.

Suppose that $a_0 \in \mathbb{C},\ 0 < r_1 < r_2 < |a_0|$ and suppose that f extends holomorphically from each circle $b\Delta(b,\rho) \subset A(a_0,r_1,r_2)$ which surrounds a_0. Notice that no such $b\Delta(a,\rho)$ surrounds the origin. Without loss of generality assume that a_0 is real and $a_0 > 0$. By Theorem 2.1 the function $F(z,\overline{z}) = f(z)$ has a bounded continuous extension F_1 from $\tilde{A}((a_0,r_1,r_2)$ to $\Omega(A(a_0,r_1,r_2)) \cup b\Omega(A(a_0,r_1,r_2))$ which is holomorphic on $\Omega(A(a_0,r_1,r_2))$.

Let $r = (r_1 + r_2)/2$ and consider the circle $b\Delta(a_0,r)$ and associated varieties $\Lambda_{a_0,r}$ and $V_{a_0,r}$. Note that $\Omega(A(a,r_1,r_2))$ is an open neighbourhood of $\Lambda_{a_0,r}$. It is the disjoint union of $\Lambda_{b,r}$ such that $b\Delta(b,r) \subset \mathrm{Int}A(a_0,r_1,r_2)$ surrounds a.

Write $D_1 = \Delta(a_0,r) \cap \Delta(0, \sqrt{|a_0|^2 - r^2})$, $D_2 = \mathbb{C} \setminus [\overline{\Delta}(a_0,r) \cup \overline{\Delta}(0, \sqrt{|a_0|^2 - r^2})$, $D_3 = \Delta(0, \sqrt{|a_0|^2 - r^2}) \setminus \overline{\Delta}(a_0,r)$, $D_4 = \Delta(a_0,r) \setminus \overline{\Delta}(0, \sqrt{|a_0|^2 - r^2})$, and for each j, $1 \leq j \leq 3$, let $V_j = \{(z, \overline{a_0} + r^2/(z-a_0))\colon\ z \in D_j\}$, and let $V_4 = \{(z, \overline{a_0} + r^2/(z-a_0))\colon\ z \in D_4 \setminus \{a_0\}\}$. Note that V_j, $1 \leq j \leq 4$, are components of $V_{a_0,r} \setminus b\Omega$.

Write

$$\lambda = \{(z, \overline{a_0} + r^2/(z - a_0))\colon\ z \in b\Delta(a_0, r) \cap \overline{\Delta}(0, \sqrt{|a_0|^2 - r^2})\}.$$

Note that λ is an arc which is a part of $b\Lambda_{a_0,r}$. Proposition 2.4 implies that for each $(z_0, \overline{z_0}) \in \lambda$ there are a neighbourhood $U \subset \Sigma$ of $(z_0, \overline{z_0})$, an open convex cone $K \subset i\Sigma$ with vertex at the origin and an $\eta > 0$ such that if $K_\eta = \{w \in K,\ |w| < \eta\}$ then $U + K_\eta \subset \Omega(A(0, 1/m, m))$ for some $m \in \mathbb{N}$ and $U - K_\eta \subset \Omega(A(a_0, r_1, r_2))$. By the edge of the wedge theorem it follows that there is an open connected neighbourhood $\mathcal{V}$ of λ in $\mathbb{C}^2$ such that $F(z, \overline{z}) = f(z)$ has a holomorphic extension Φ to $\mathcal{V}$. With no loss of generality assume that $\mathcal{V} \cap \Omega$ and $\mathcal{V} \cap \tilde{A}(a_0, r_1, r_2)$ are connected. By the discussion at the end of Section 3 we have $\Phi \equiv F$ on $\Omega \cap \mathcal{V}$ and $\Phi \equiv F_1$ on $\Omega(A(a_0, r_1, r_2)) \cap \mathcal{V}$.

Parts V_3 and V_4 of $V_{a_0,r} \setminus b\Omega$ are contained in Ω so F is well defined and holomorphic on V_3 and V_4. The function F_1 is well defined on parts V_1 and V_4 which are contained in $\Omega(A(a_0, r_1, r_2))$. Together with $\{(z, \overline{a_0} + r^2/(z - a_0))\colon\ z \in b\Delta(0, \sqrt{|a_0|^2 - r^2} \cap \Delta(a_0, r)$, these parts form $\Lambda_{a_0,r}$. We first show that on V_4 where both F and F_1 are defined, these two functions coincide. To see this, notice that by Proposition 2.4 for each $z_0 \in b\Delta(a_0, r) \setminus \overline{\Delta}(0, \sqrt{|a_0|^2 - r^2})$ there are an open neighbourhood $U \subset \Sigma$ of $(z_0, \overline{z_0})$, an open convex cone $K \subset i\Sigma$ with vertex at the origin and an $\eta > 0$ such that if $K_\eta = \{w \in K,\ |w| < \eta\}$ then $U + K_\eta \subset \Omega(A(a_0, r_1, r_2)) \cap \Omega(A(0, 1/m, m))$ for some $m \in \mathbb{N}$ and such that V_4 meets $U + K_\eta$. This implies that $F = F_1$ on $U + K_\eta$ (since the boundary values $F(z, \overline{z}) = f(z)$, are the same). Since V_4 meets $U + K_\eta$ it follows that $F = F_1$ on V_4.

We also show that the function $F_1|V_1$ is the analytic continuation of $F|V_3$ across $\mathrm{Int}\lambda$. Note that λ is the intersection of the boundaries of V_1 and V_3 in $V_{a_0,\rho}$. Recall that there are an open neighbourhood $\mathcal{V} \subset \mathbb{C}^2$ of λ and a holomorphic function Φ on $\mathcal{V}$ such that $\Phi \equiv F$ on $\mathcal{V} \cap \Omega$ and $\Phi \equiv F_1$ on $\mathcal{V} \cap \Omega(A(a_0, r_1, r_2))$, so there is a single holomorphic function Φ on $V_{a_0,r} \cap \mathcal{V}$ such that $\Phi \equiv F$ on $V_3 \cap \mathcal{V}$ and $\Phi \equiv F_1$ on $V_1 \cap \mathcal{V}$.

6. Proof of Theorem 1.1, Part 2

We have $\lambda \subset \Sigma \subset b\Omega$. Notice that $(tz, w) \in \Omega$ whenever $(z, w) \in \overline{\Omega} \setminus \{(0, 0)\}$ and $0 \le t < 1$. There are a neighbourhood $\mathcal{W}$ of λ in $b\Omega$ and an $\varepsilon > 0$ such that $(tz, w) \in \mathcal{V}$ whenever $(z, w) \in \mathcal{W}$ and $1 - \varepsilon < t < 1 + \varepsilon$. Note that, by transversality $\{(tz, w)\colon\ (z, w) \in \mathcal{W},\ 1 - \varepsilon < t < 1 + \varepsilon\}$ is a neighbourhood of λ in $\mathbb{C}^2$. Thus, shrinking $\mathcal{V}$ to $\{(tz, w)\colon\ (z, w) \in \mathcal{W},\ 1 - \varepsilon < t < 1 + \varepsilon\}$ we may, with no loss of generality, assume that

$$(tz, w) \in \Omega \cup \mathcal{V} \text{ whenever } (z, w) \in \mathcal{V} \text{ and } 0 \le t \le 1. \tag{6.1}$$

Choose $\varepsilon > 0$ very small and consider the union W of all varieties $\Lambda_{a,r}$ where $|a - a_0| < \varepsilon$. This is a tiny open neighbourhood of $\Lambda_{a_0,r}$ which is foliated by $\Lambda_{a,r}$. Repeating the reasoning above for each $\Lambda_{a,r}$ we see that the function

$$\Psi(z, w) = \begin{cases} \Phi(z, w) & (z, w) \in \mathcal{V} \\ F_1(z, w) & (z, w) \in W \end{cases}$$

is well defined on $\mathcal{V} \cup W$ and thus holomorphic on $\mathcal{V} \cup W$, and on $[\mathcal{V} \cup W] \cap \Omega$ coincides with F.

Choose $\tau > 0$ so small that if $D = \Delta(0, \sqrt{|a_0|^2 - r^2} + \tau) \cap \Delta(a_0, r - \tau)$ then $\{(z, \overline{a_0} + r^2/(z - a_0)) \colon z \in bD\} \subset \Omega \cup \mathcal{V}$.

Let $\mathcal{D} = \{(z, \overline{a_0} + r^2/(z - a_0)) \colon z \in \overline{D}\}$. Note that $\mathcal{D}$ is an analytic disc whose boundary $b\mathcal{D} = \{(z, \overline{a_0} + r^2/(z - a_0)) \colon z \in bD\}$ is contained in $\Omega \cup \mathcal{V}$. The disc $\mathcal{D}$ is contained in $\Lambda_{a_0, r} \subset W$ so Ψ is holomorphic in a neighbourhood of $\mathcal{D}$. A neighbourhood of $b\mathcal{D}$ is contained in $\Omega \cup \mathcal{V}$ and on this neighbourhood Ψ coincides with F.

We now apply the continuity principle. To do this, we first define $T_t(z, w) = (tz, w)$ $(0 \le t \le 1)$ and then define a continuous family of analytic discs by $\mathcal{D}_t = T_t(\mathcal{D})$ $(0 \le t \le 1)$. By (6.1) we have $b\mathcal{D}_t \subset \Omega \cup \mathcal{V}$ for all t, $0 \le t \le 1$. Moreover, $\mathcal{D}_1 = \mathcal{D}$ and $\mathcal{D}_0$ is contained in $w-$axis and has the origin in its interior. By the continuity principle it follows that F extends holomorphically into a neighbourhood of the origin.

To prove that f extends to an entire function one may use [G1]. Alternatively, one may use the Liouville theorem as follows: From (2.2) it follows that F is bounded on $\Delta(0, \delta) \times \mathbb{C}$ for some $\delta > 0$. By the Liouville theorem the function $\zeta \mapsto F(z, \zeta)$ is constant for each $z \in \Delta(0, \delta)$ so F does not depend on w on $\Delta(0, \delta) \times \mathbb{C}$. It follows that F is a function of z only so $f(z) = F(z, \overline{z})$ $(z \in \mathbb{C} \setminus \{0\})$ is a restriction of an entire function to $\mathbb{C} \setminus \{0\}$. This completes the proof.

7. Examples

By Theorem 1.1 the family of all circles that surround the origin is a maximal *open* family of circles that is not a test family for holomorphy. Even its closure, that is the family of all circles that either surround the origin or pass through the origin is not a maximal family that is not a test family for holomorphy. To see this, let $a \in \mathbb{C}$, $\rho > 0, |a| > \rho$, and let

$$g(z) = \begin{cases} 0 & \text{if } z = 0 \\ (z^2/\overline{z})[(z - a)(\overline{z} - \overline{a}) - \rho^2] & \text{if } z \ne 0. \end{cases}$$

The function g vanishes identically on $b\Delta(a, \rho)$ and hence extends holomorphically from $b\Delta(a, \rho)$. Since $(z^2/\overline{z})[(z - a)(\overline{z} - \overline{a}) - \rho^2] = z^3 - az^2 - \overline{a}(z^3/\overline{z}) + a\overline{a}(z^2/\overline{z}) - \rho^2(z^2/\overline{z})$ is a polynomial in z and $1/\overline{z}$ it follows that g extends holomorphically from every circle that surrounds the origin. Since g is continuous on $\mathbb{C}$ it extends holomorphically also from every circle that passes through the origin. This shows that if $|a_i| > \rho_i > 0$, $1 \le i \le n$, then the function

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ (z^2/z)^n \Pi_{j=1}^n [(z - a_j)(\overline{z} - \overline{a}_j) - \rho_j^2] & \text{if } z \ne 0 \end{cases}$$

is continuous on $\mathbb{C}$, extends holomorphically from all circles that surround the origin, from all circles that pass through the origin, and from all circles $b\Delta(a_i, \rho_i)$, $1 \le i \le n$ yet f is not holomorphic.

In our next example, let g be a function from the disc algebra and define

$$f(z) = g(z/\overline{z}) \quad (z \in \mathbb{C} \setminus \{0\}). \tag{7.1}$$

Suppose that $b\Delta(a, \rho)$ surrounds the origin. Then $|a| < \rho$ and for $|\zeta| = 1$ we have

$$f(a + \zeta\rho) = g\left(\frac{a + \zeta\rho}{\overline{a} + \overline{\zeta}\rho}\right) = g\left(\zeta \frac{\zeta + a/\rho}{1 + (\overline{a}/\rho)\zeta}\right)$$

which shows that the function $\zeta \mapsto f(a+\zeta\rho)$ $(\zeta \in b\Delta)$ extends to a function from the disc algebra. Thus, f extends holomorphically from every circle surrounding the origin. Since the boundary values of the functions from the disc algebra can be highly non-smooth this example shows that a highly nonsmooth function on $\mathbb{C}\setminus\{0\}$ can be holomorphically extendible from every circle surrounding the origin.

8. Analyticity on circles for functions constant on lines

In the second example in Section 7 the function f is constant on each line passing through the origin, that is,

$$f(tz) = f(z) \quad (z \in \mathbb{C}\setminus\{0\},\ t \in \mathbb{R}\setminus\{0\}). \tag{8.1}$$

In this section we look more closely at such functions.

Theorem 8.1 *Suppose that f is a continuous function on $\mathbb{C}\setminus\{0\}$ that is a constant on each line passing through the origin, that is, f satisfies (8.1). If f extends holomorphically from one circle surrounding the origin then it extends holomorphically from every circle surrounding the origin. This happens if and only if there is a function g from the disc algebra such that $f(z) = g(z/\overline{z})$ $(z \in \mathbb{C}\setminus\{0\})$.*

Proof. Suppose that f is a continuous function on $\mathbb{C}\setminus\{0\}$ that satisfies (8.1). Then there is a continuous function g on $b\Delta$ such that

$$f(z) = g(z/\overline{z}) \quad (z \in \mathbb{C}\setminus\{0\}. \tag{8.2}$$

Assume that f extends holomorphically from a circle $b\Delta(a,\rho)$ that surrounds the origin. By (8.1) we may assume that $\rho = 1$. If $a = 0$ then the function $\zeta \mapsto g(\zeta^2)$ $(\zeta \in b\Delta)$ extends to a function in the disc algebra which implies that g extends to a function from the disc algebra. Suppose that $a \neq 0$. Composing f with a rotation if necessary we may assume that $0 < a < 1$. By our assumption there is a function h from the disc algebra such that

$$h(\zeta) = g\left(\frac{a+\zeta}{a+\overline{\zeta}}\right) = g\left(\zeta\frac{a+\zeta}{1+\zeta a}\right) \quad (\zeta \in b\Delta).$$

Clearly $\xi \mapsto h\big((\xi-t)/(1-t\xi)\big)$ $(\xi \in b\Delta)$ extends to a function from the disc algebra for every t, $0 \le t < 1$. Put $t = (1-\sqrt{1-a^2})/a$. Then $0 < t < 1$ and

$$\frac{a+\frac{\xi-t}{1-\xi t}}{1+a\frac{\xi-t}{1-\xi t}} = \frac{\xi+t}{1+t\xi} \quad (\xi \in b\Delta)$$

which implies that

$$\xi \mapsto g\left(\frac{\xi-t}{1-\xi t}\frac{\xi+t}{1+\xi t}\right) = g\left(\frac{\xi^2-t^2}{1-t^2\xi^2}\right) \quad (\xi \in b\Delta)$$

extends to a function from the disc algebra which implies that $\xi \mapsto g((\xi-t^2)/(1-t^2\xi))$ $(\xi \in b\Delta)$ extends to a function from the disc algebra. Consequently g extends to a function in the disc algebra and so f is of the form (7.1). By the discussion following (7.1) it follows that the function f extends holomorphically from every circle that surrounds the origin.

9. More examples

Example 9.1 Let $0 < a < 1$ and let $\Phi(\zeta) = (a+\zeta)/|a+\zeta|$ $(\zeta \in b\Delta)$. Then Φ: $b\Delta \to b\Delta$ is diffeomorphism. Define

$$f(z) = \Phi^{-1}(z/|z|) \ \ (z \in \mathbb{C} \setminus \{0\}). \tag{9.1}$$

The function f is continuous on $\mathbb{C} \setminus \{0\}$ and is constant on each ray emanating from the origin, that is,

$$f(tz) = f(z) \ \ (z \in \mathbb{C} \setminus \{0\},\ t > 0). \tag{9.2}$$

If an f satisfying (9.2) extends holomorphically from a circle $b\Delta(a, \rho)$ that surrounds the origin then it extends holomorphically from $b\Delta(ta, t\rho)$ for every $t > 0$. So, when studying holomorphic extendibility from circles $b\Delta(a, \rho)$ we may, with no loss of generality, assume that $\rho = 1$.

Let f be as in (9.1). Since

$$f(a+\zeta) = \Phi^{-1}((a+\zeta)/|a+\zeta|) = \zeta \ \ (\zeta \in b\Delta)$$

it follows that f extends holomorphically from $b\Delta(a, 1)$. By (9.2)

$$\frac{a+\zeta}{|a+\zeta|} = \sqrt{\frac{(a+\zeta)^2}{(a+\zeta)(a+\overline{\zeta})}} = \sqrt{\frac{a+\zeta}{a+\overline{\zeta}}} = \sqrt{\zeta\frac{a+\zeta}{1+a\zeta}} \ \ (\zeta \in b\Delta)$$

it follows that

$$\zeta \mapsto f\left(\sqrt{\zeta\frac{a+\zeta}{1+a\zeta}}\right)$$

extends to a function from the disc algebra which happens if and only if

$$\zeta \mapsto f\left(\sqrt{M(\zeta)\frac{a+M(\zeta)}{1+aM(\zeta)}}\right)$$

extends to a function from the disc algebra for an automorphism M of Δ. In particular, if $M(\zeta) = (\zeta - a)/(1 - a\zeta)$ it follows that

$$\zeta \mapsto f\left(\sqrt{\zeta\frac{\zeta-a}{1-a\zeta}}\right) \ \ (z \in b\Delta)$$

extends to a function from the disc algebra which is equivalent to the fact that f extends holomorphically from $b\Delta(-a, 1)$. It will follow from Theorem 10.1 that these two circles are the only circles of radius 1 from which f extends holomorphically.

Example 9.2 Let g be a function in the disc algebra which is not an even function. Let $f(z)=g(z/|z|)$ $(z\in\mathbb{C}\setminus\{0\})$. Then f is continuous on $\mathbb{C}\setminus\{0\}$ and extends holomorphically from $b\Delta(0,1)$. It will follow from Theorem 10.1 below that $b\Delta(0,1)$ is the only circle of radius 1 from which f extends holomorphically.

10. Analyticity on circles for functions constant on rays

In both examples in Section 9 the function f satisfies (9.2), that is, f is constant on each ray emanating from the origin. In this section we look more closely at such functions.

Suppose that a continuous function f on $\mathbb{C}\setminus\{0\}$ satisfies (9.2). Assume that $0\leq d<1$ and that f extends holomorphically from $b\Delta(de^{i\alpha},1)$ for some $\alpha\in\mathbb{R}$. This means that $\zeta\mapsto f(e^{i\alpha}(d+\zeta))$ $(\zeta\in b\Delta)$ extends to a function in the disc algebra. Since

$$e^{i\alpha}(d+\zeta)/|d+\zeta|=e^{i\alpha}\sqrt{\zeta(d+\zeta)/(1+d\zeta)}\quad(\zeta\in b\Delta)$$

this happens if and only if

$$\left.\begin{array}{r}f\big(e^{i\alpha}\sqrt{\zeta(d+\zeta)/(1+d\zeta)}\big)=q(\zeta)\quad(\zeta\in b\Delta)\\ \text{where }\ q\text{ belongs to the disc algebra.}\end{array}\right\}\tag{10.1}$$

In the case when $d=0$ this implies that $\zeta\mapsto f(\zeta)$ $(\zeta\in b\Delta)$ extends to a function from the disc algebra. Suppose that $d\neq 0$. Put

$$\zeta=\frac{\xi-t}{1-t\xi}\quad\text{where}\quad t=\frac{1-\sqrt{1-d^2}}{d}$$

to get

$$\zeta\frac{d+\zeta}{1+d\zeta}=\frac{\xi^2-t^2}{1-t^2\xi^2}$$

so that (10.1) is equivalent to

$$\left.\begin{array}{r}f\left(e^{i\alpha}\sqrt{\dfrac{\xi^2-t^2}{1-t^2\xi^2}}\right)=G(\xi)\quad(\xi\in b\Delta)\\ \text{where }\ G\text{ belongs to the disc algebra.}\end{array}\right\}\tag{10.2}$$

In fact, $G(\xi)=q((\xi-t)/(1-t\xi))$ $(\xi\in b\Delta)$. Putting $Z=e^{i\alpha}\sqrt{(\xi^2-t^2)/(1-t^2\xi^2)}$ we get

$$\xi^2=\frac{(e^{-i\alpha}Z)^2+t^2}{1+t^2(e^{-i\alpha}Z)^2}$$

which implies that (10.2) is equivalent to

$$\left.\begin{array}{r}f(Z)=G\left(e^{-i\alpha}\sqrt{\dfrac{Z^2+e^{2i\alpha}t^2}{1+e^{-2i\alpha}t^2Z^2}}\right)\quad(Z\in b\Delta)\\ \text{where }\ G\text{ belongs to the disc algebra.}\end{array}\right\}\tag{10.3}$$

Theorem 10.1 *Let f be a continuous function on $\mathbb{C}\setminus\{0\}$ which satisfies (9.2), that is, f is constant on each ray emanating from the origin. Suppose that f extends holomorphically from $b\Delta(a,1)$ and $b\Delta(b,1)$ where $a,b\in\Delta,\ b\neq a, b\neq -a$. Then there is a function g in the disc algebra such that*

$$f(z)=g(z/|z|)\quad (z\in\mathbb{C}\setminus\{0\}). \tag{10.4}$$

Consequently, f satisfies (8.1), that is, f is constant on each line passing through the origin and extends holomorphically from each circle surrounding the origin.

Proof. Suppose that f extends holomorphically from $b\Delta(d_1e^{i\alpha_1},1)$ and $b\Delta(d_2e^{i\alpha_2},1)$ where $0\leq d_i<1$ $(i=1,2)$, $d_2e^{i\alpha_2}\neq d_1e^{i\alpha_1}$, $d_2e^{i\alpha_2}\neq -d_1e^{i\alpha_1}$. It is enough to prove that f is an even function for then the rest follows from Theorem 8.1.

Let $t_i=0$ if $d_i=0$ and $t_i=(1-\sqrt{1-d_i^2})/d_i$ if $d_i\neq 0$, $i=1,2$. Write $A_i=e^{2i\alpha_i}t_i^2$, $i=1,2$. By the discussion preceding Theorem 10.1 there are functions G_1,G_2 in the disc algebra such that

$$f(Z)=G_i\big(e^{-i\alpha_i}\sqrt{\frac{Z^2+A_i}{1+\overline{A_i}Z^2}}\big)\quad (Z\in b\Delta,\ i=1,2) \tag{10.5}$$

Write $W^2=(Z^2+A_1)/(1+\overline{A_1}Z^2)$ so that $Z^2=(W^2-A_1)/(1-\overline{A_1}W^2)$ $(W\in b\Delta)$ and $(Z^2+A_2)/(1+\overline{A_2}Z^2)=(W^2+C)/1+\overline{C}W^2)$ where $C=(A_2-A_1)/(1-A_1A_2)$. Now (10.5) implies that

$$G_1\left(e^{-i\alpha_1}\sqrt{\frac{Z^2+A_1}{1+\overline{A_1}Z^2}}\right)=G_2\left(e^{-i\alpha_2}\sqrt{\frac{Z^2+A_2}{1+\overline{A_2}Z^2}}\right)\quad (Z\in b\Delta)$$

which implies that

$$G_1(e^{-i\alpha_1}W)=G_2\big(e^{-i\alpha_2}\sqrt{\frac{W^2+C}{1+\overline{C}W^2}}\big)\quad (W\in b\Delta). \tag{10.6}$$

Since both G_1 and G_2 belong to the disc algebra it follows that the relation (10.6) continues holomorphically inside Δ, so (10.6) implies that either $C=0$ or G_2 is an even function. Assume that $C=0$. By (10.6) it follows that $A_1=A_2$ and $e^{i\alpha_1}=e^{i\alpha_2}$. It follows that $d_2e^{i\alpha_2}=\pm d_1e^{i\alpha_1}$ which is impossible by the assumption. Thus, G_2 is an even function and consequently, by (10.5), f is an even function. This completes the proof.

Remark Note that Theorem 10.1 implies that in Example 9.1 the circles $b\Delta(a,1)$ and $b\Delta(-a,1)$ are the only circles of radious one from which f extends holomorphically. Similarly, in Example 9.2, $b\Delta$ is the only circle of radius one from which f extends holomorphically.

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