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SEIFERT CIRCLES, BRAID
INDEX AND THE ALGEBRAIC
CROSSING NUMBER

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Seifert circles, braid index and the algebraic crossing number

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ABSTRACT

We introduce new operations reducing the number of Seifert circles in link diagrams of a special type. The operations are similar to one described in [M–P]. We discuss a conjecture about the number of Seifert circles that can be canceled by applying the operation repeatedly. We translate the problem into one belonging to graph theory.

Keywords: braid index, cycle index, writhe

1. Introduction.

A classical conjecture in knot theory asserts that for every closed braid representing a given oriented link the writhe (or the algebraic crossing number) is uniquely determined, provided that the number of strings is minimal (equal to the braid index). In this note we study a certain idea of Murasugi and Przytycki that allows one to decrease the number of Seifert circles in certain diagrams. During the process the writhe is also changed — in a controlled manner. In our study we investigate the possibility of using the reduction method in different ways to obtain braids with different writhes. Below, we formulate the conjecture in a version that is slightly modified compared to the most popular one.

Conjecture 1.1. *Let D be an oriented diagram representing a given oriented link L . Assume that D has the minimum possible number of Seifert circles — $s(D)$ (among all diagrams representing L). Then $w(D)$ (the writhe of D) is uniquely determined. Moreover, for any diagram D' of L with $s(D) + k$ Seifert circles*

$$w(D) - k \leq w(D') \leq w(D) + k.$$

We formulated the conjecture for arbitrary diagrams and the number of Seifert circles rather than for closed braids and the number of strings. These formulations are equivalent, as it is known by Yamada [Y] that every diagram may be transformed into a closed braid with both the number of Seifert circles and the writhe unchanged (so the minimum number of Seifert circles is the same as the braid index of the considered link).

In this note we investigate a certain class of diagrams for which the number of Seifert circles may be easily decreased by means of an operation (applied repeatedly) introduced by Murasugi and Przytycki in [M–P]. We discuss the possibility of arriving at diagrams with the same number of Seifert circles but different writhes. Also we introduce some new

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operations that have an easily controlled effect on the corresponding Seifert graph. Finally, we translate the problem into one of graph theory.

2. MP-reducing operation.

We begin by describing the reducing operation of Murasugi and Przytycki (called an MP-operation in the sequel). The operation reduces the number of Seifert circles of the diagram by one. First, let us recall that a diagram of an oriented link may be seen as a configuration of simple oriented closed curves (the Seifert circles) with a number of crossings added — each crossing being a crossing between two of the Seifert circles. It may also be convenient to group together a number of crossings arranged like in a braid. Then we can perceive the diagram as a configuration of Seifert circles, with a number of boxes containing braids (braid-boxes) superimposed. This is illustrated in figure 1: there are 14 Seifert circles, two boxes supposed to contain four-string braids and four boxes supposed to contain five-string braids. Of course it is arbitrary to a large degree, how we group crossings in braid-boxes.

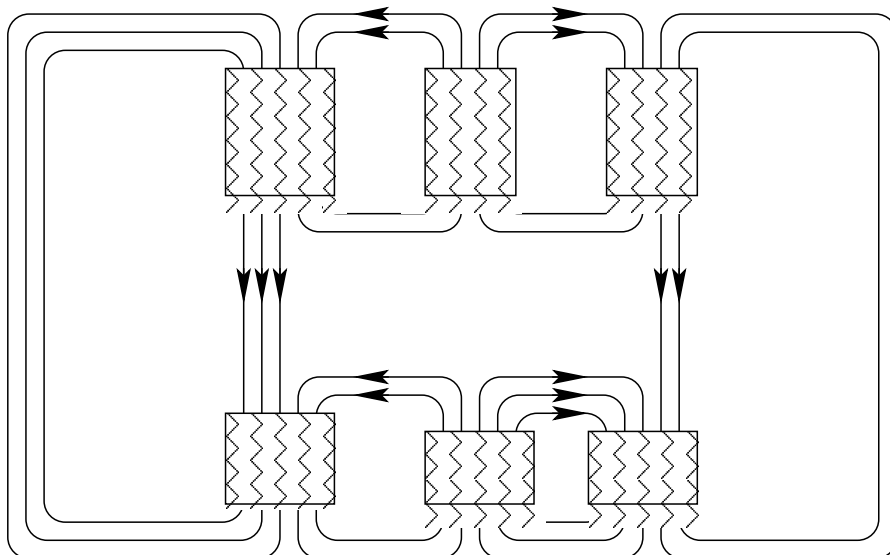


figure 1

For an MP-reducing operation we must find a pair of Seifert circles of the diagram such that there is precisely one crossing between the two Seifert circles. We will refer to such a configuration as a *reduction site* and we will call such a crossing a *special crossing*. Given a reduction site, the MP-operation may be performed in four ways depending on which of the two Seifert circles is chosen to be the *basic reduction circle* and which part of the crossing (the undercrossing or the overcrossing) will be modified. Once the choices are made, we proceed as follows. We take the chosen part of the crossing (the undercrossing or the overcrossing) and replace it with another arc (a tunnel or a bridge), this time a long

one. This is illustrated in figure 2.

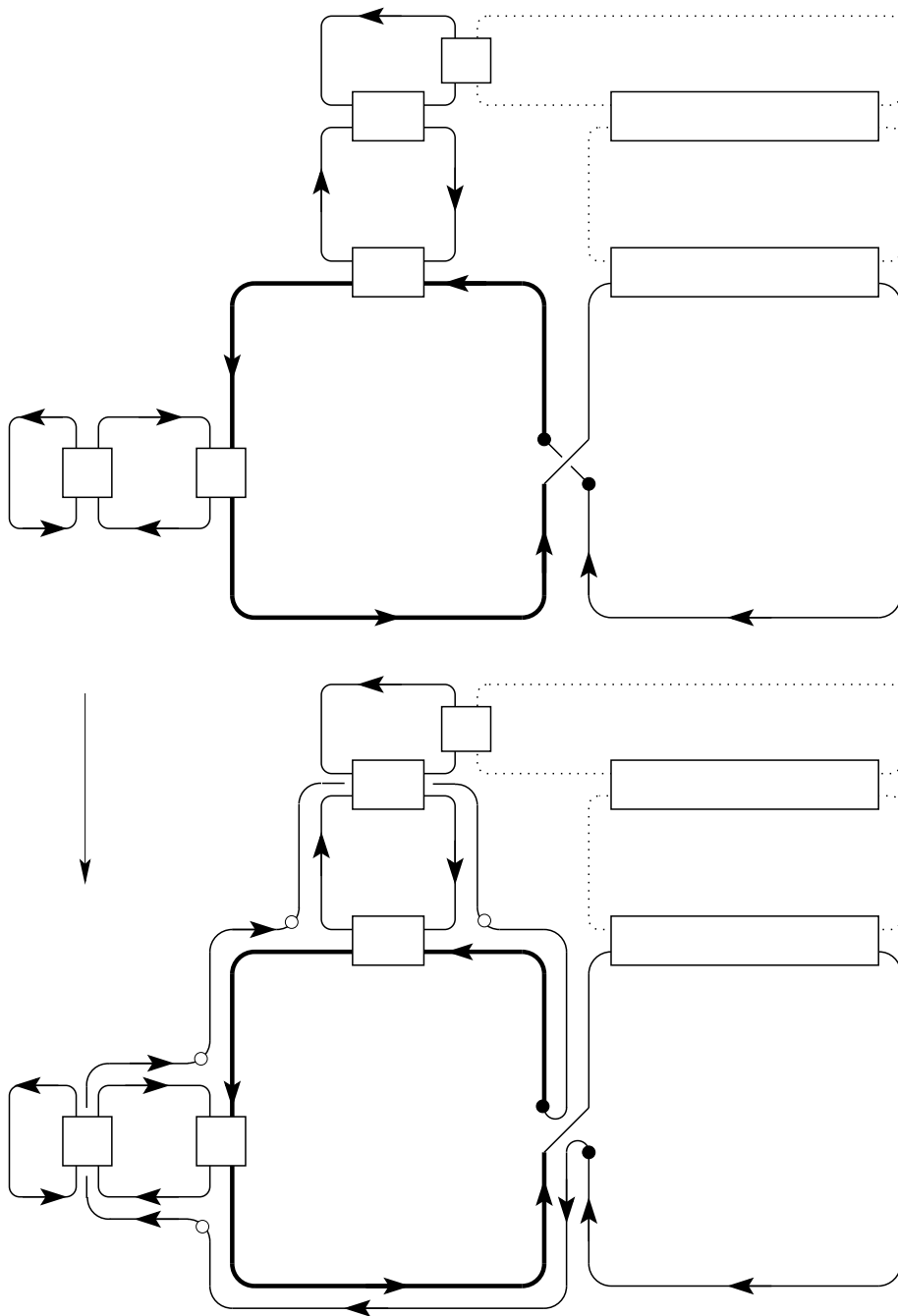


figure 2

Before explaining figure 2 in detail let us remark that the dotted part of the figure plays no active role in the operation. It is added to the picture to avoid illusion that the situation

is simpler than it might really be.

The operation is performed in the following manner: first, we start from the point at which the short arc begins (one of the black dots in figure 2 — there are two of them because each of the two may be considered the starting point of the operation) and we make a U-turn, starting to walk along the basic reduction circle. The idea is to travel all the way around the basic circle keeping close to it, except when this would involve passing through a braid-box with the orientation incompatible with the orientation of our arc. Whenever we come close to a braid-box like this we make another sharp turn (there are two such turns in figure 2, starting at the white dots) just before entering the box and we circle round another Seifert circle, eventually coming back to the basic circle at two other white points. When we circle round any Seifert circle adjacent to the basic one, then it is automatically true that any braid-box encountered on the way has the orientation compatible with that of our arc. This operation (of replacing a short arc with a long one) does not change the isotopy type of the link, provided that a tunnel is replaced with a tunnel or a bridge with a bridge. The number of Seifert circles is decreased by exactly 1 : the two large Seifert circles of figure 2 are connected to form just one circle in the new diagram. This new circle is rather complicated so we show it separately in figure 3. Let us observe that the writhe is decreased by one if the crossing at the reduction site is positive or increased by one if the crossing at the reduction site is negative.

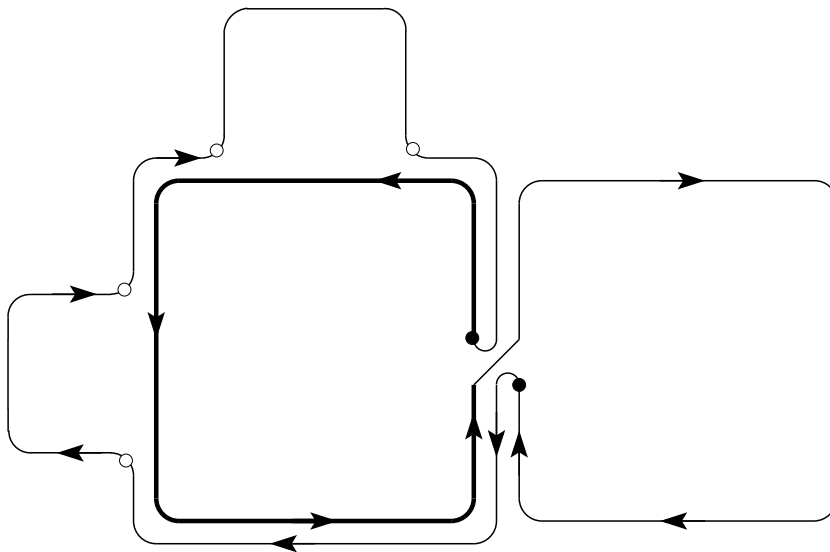


figure 3

It is obvious that if in the given diagram we have more than one reduction site, then there is a chance that several MP-operations may be performed. However, when the operation at one reduction site is performed it can damage the other reduction sites. We

will now recall the method to avoid this damage as much as possible, as introduced in [M–P]. To find an effective order in which to perform the sequence of MP–operations it is convenient to consider the *Seifert graph* $\Gamma(D)$ of the given diagram D . The graph $\Gamma(D)$ consists of a vertex for each Seifert circle and an edge for each crossing involving a pair of Seifert circles. It is a planar bipartite graph (though it is not planar in a canonical way). We recall now the definition of the cyclic index of a graph. A set of single edges is called independent if in every cycle the number of edges from the considered set is less than half. Here *cycle* means a simple closed path in the graph. The cyclic index of a graph is the maximum number of independent edges. In [M–P] two notions of index are considered: the index and the cyclic index described above. However, in [T] it is proved that for bipartite graphs (and a Seifert graph of a link diagram is always bipartite) the two notions coincide.

We can now recall the following result from [M–P].

Proposition 2.1. *The number of the Seifert circles in a diagram D may be reduced by $\text{ind}(\Gamma(D))$ by performing a sequence of MP–operations.*

Every edge in the Seifert diagram can have a sign assigned in a natural way (as the sign of the corresponding crossing). The symbols $\text{ind}_-(\Gamma(D))$ and $\text{ind}_+(\Gamma(D))$, as defined in [M–P] denote the maximum number of edges in an independent set of single negative (positive) edges in the considered graph. It is obvious that the inequality

$$(1) \quad \text{ind}(\Gamma(D)) \leq \text{ind}_-(\Gamma(D)) + \text{ind}_+(\Gamma(D))$$

holds for all diagrams. It is usually sharp, so Proposition 2.1 does not imply a possibility of reducing the number of Seifert circles by $\text{ind}_-(\Gamma(D)) + \text{ind}_+(\Gamma(D))$.

The question, whether a reduction of the number of Seifert circles by $\text{ind}_-(\Gamma(D)) + \text{ind}_+(\Gamma(D))$ is possible is briefly mentioned in [M–P], with a suggestion that this is *not* always possible. In this paper we actually conjecture that such reduction *is* always possible. More precisely:

Conjecture 2.2. *Let D be a diagram of an oriented link. Assume that the number of Seifert circles is s and the algebraic crossing number is c . Then it is possible to find another diagram of the same link with the number of Seifert circles equal to $s - \text{ind}_-(\Gamma(D)) - \text{ind}_+(\Gamma(D))$ and the algebraic crossing number equal to $c + \text{ind}_-(\Gamma(D)) - \text{ind}_+(\Gamma(D))$.*

In the next section we will describe certain operations that we expect to be sufficient for the reduction. Here, let us briefly mention the connection between conjectures **1.1** and **2.2**. Let us consider, just for the sake of illustration the simplest possible case. Suppose there is a diagram D with the number of Seifert circles equal to s , the algebraic crossing number equal to c and $\text{ind}_-(\Gamma(D)) = \text{ind}_+(\Gamma(D)) = 1$. Assume that it is *not* possible to reduce the number of Seifert circles by 2 (contrary to the conjecture above). Then the braid index of the link L represented by D is equal to $s - 1$. Moreover Proposition 2.1 guarantees that there is a diagram representing L with $s - 1$ Seifert circles and the algebraic crossing number equal to $c - 1$ and another with $s - 1$ Seifert circles and the algebraic

crossing number equal to $c + 1$. Such a situation would contradict Conjecture 1.1. It is easy to see that any counterexample to Conjecture 2.2 would yield a counterexample to Conjecture 1.1 (at least to its second part): Let D be a diagram of a link L . Then there exists a diagram D_+ of L with the same number of Seifert circles ($s(D)$) and $w(D_+)$ equal to $w(D) + 2ind_-(D)$, and another diagram D_- of L with the same number of Seifert circles ($s(D)$) and $w(D_-)$ equal to $w(D) - 2ind_+(D)$. These are easily obtain: to construct D_+ we first decrease the number of Seifert circles by applying $ind_-(D)$ negative MP-operations (we know by [MP] that this is possible). Then we add $ind_-(D)$ trivial positive loops to the diagram. In the resulting diagram the number of Seifert circles is equal to the original number $s(D)$ and the algebraic crossing number is increased first by $ind_-(D)$, when performing MP-operations, then again by $ind_-(D)$, when adding positive loops.

It follows that for diagrams with $s(D)$ Seifert circles representing L the span of possible values of algebraic crossing number is at least $2(ind_-(D) + ind_+(D))$. However, assuming Conjecture 1.1, such span in algebraic crossing number is only possible if $b(L)$ is equal to $s(D) - (ind_-(D) + ind_+(D))$ or in case it is bigger, the span of the algebraic crossing number at the $b(L)$ level is non-zero. It follows that a counterexample to Conjecture 2.2 would imply a counterexample to Conjecture 1.1.

Of course the most convincing way to find a counterexample to Conjecture 2.2 would be to find such a diagram D that the number of components would exceed $s(D) - ind_-(D) - ind_+(D)$ — it is obvious that the number of components of a link cannot exceed the number of Seifert circles in its diagram. The following theorem is an aside given here as supporting evidence for Conjecture 2.2.

Theorem 2.3. *The number of components m of a link L represented by a diagram D is smaller than or equal to $s(D) - ind_-(D) - ind_+(D)$.*

Proof. By Theorem 8.3 of [M-P] the span in variable l of the Homfly polynomial $P(L)$ of L satisfies the condition

$$2.4 \quad span_l \leq 2(s(D) - 1 - ind_+(D) - ind_-(D)).$$

But the l -span of the Homfly polynomial of a link of m components is always greater than or equal to $2(m - 1)$. This is true because of the special form of the P_0 polynomial (the part of the polynomial P that gathers all terms whose degree in variable m equals $-m + 1$) proved in [L-M], namely: P_0

$$\left(-\frac{l^{-1} + l}{m} \right)^{-m+1} \cdot P_0(L_1) \cdots P_0(L_m),$$

where $P_0(L_1), \dots, P_0(L_m)$ are the polynomials of components of L . Obviously, any polynomial that splits like that has the l -span at least $2(m - 1)$. Putting together this and 2.4 we obtain

$$2(m - 1) \leq span_l \leq 2(s(D) - 1 - ind_+(D) - ind_-(D)),$$

which completes the proof. \square

Let us recall that the effect of the MP-operation on the Seifert graph is that the star of the vertex corresponding to the basic Seifert circle is contracted. In fact, it is more accurate to say that in the resulting Seifert graph there is also a part containing vertices corresponding to those Seifert circles that were originally contained in the basic reduction circle. However, we omit this part of the graph because we adopt a strategy of using operations involving those Seifert circles that are less nested first.

We now describe five operations which we expect to be sufficient for the desired reduction.

Operation 1. Cancellation of a trivial loop, as shown in figure 4.

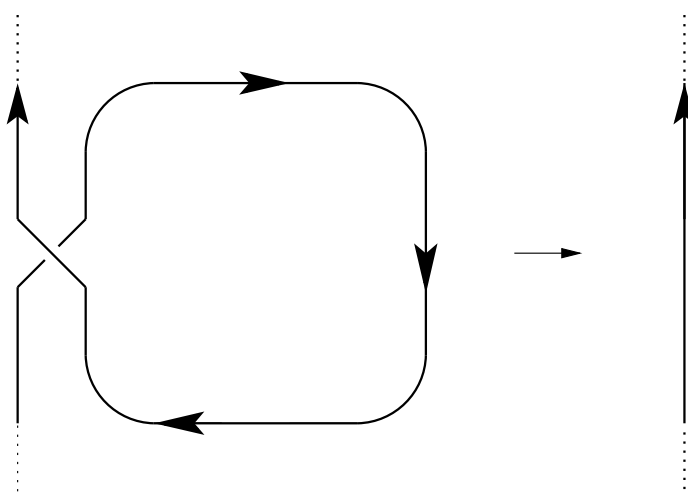


figure 4

This operation obviously decreases the number of Seifert circles by one, and changes the algebraic crossing number of the diagram also by one in the direction determined by the sign of the crossing involved. In terms of the Seifert graph the corresponding operation is a cancellation of a signed stump in the Seifert graph.

Operation 2. This operation is the most similar to the MP-operation. To perform operation 2 we need a reduction site like for an MP-operation. Moreover, we assume (for the time being) that there are no Seifert circles nested in the basic reduction circle. We start in the same manner as in the MP-operation but when the first circle adjacent to the basic circle is met and circled round and we come back close to the basic circle, we do not follow the basic circle any further (like in the MP-operation). Instead we create a crossing of our new arc and the basic circle and we proceed to the other end of the short arc that is being replaced by the long arc. In doing this we keep close to the reduction circle following its orientation as shown in figure 5. In figure 5 we did not show any part of the diagram

that might be nested in the reduction circle but this is not excluded.

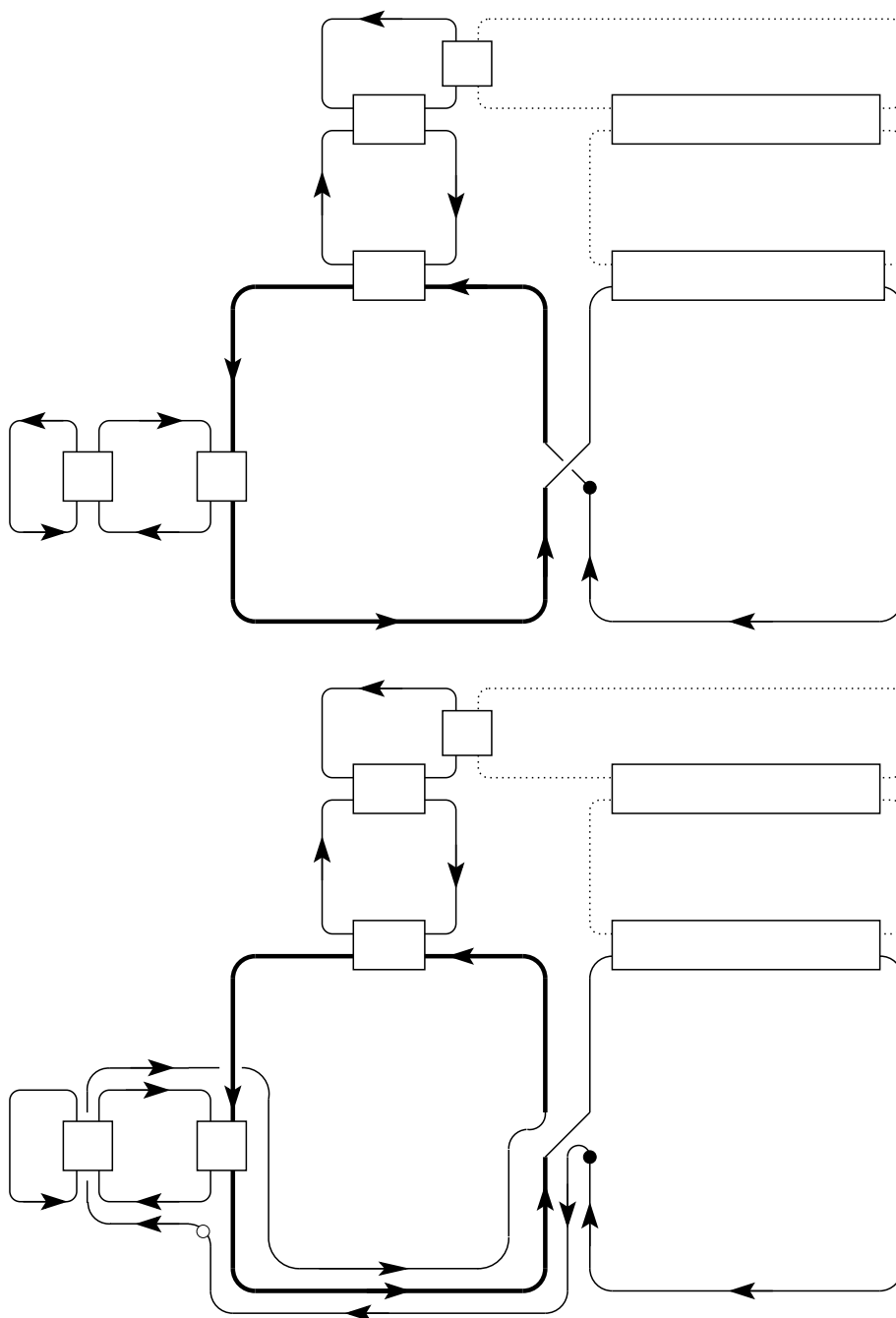


figure 5

In the following figure 6 we show how the two Seifert circles directly involved in the operation are changed into two new Seifert circles. In particular, one of the new Seifert

circles engulfs the Seifert circle that was circled round.

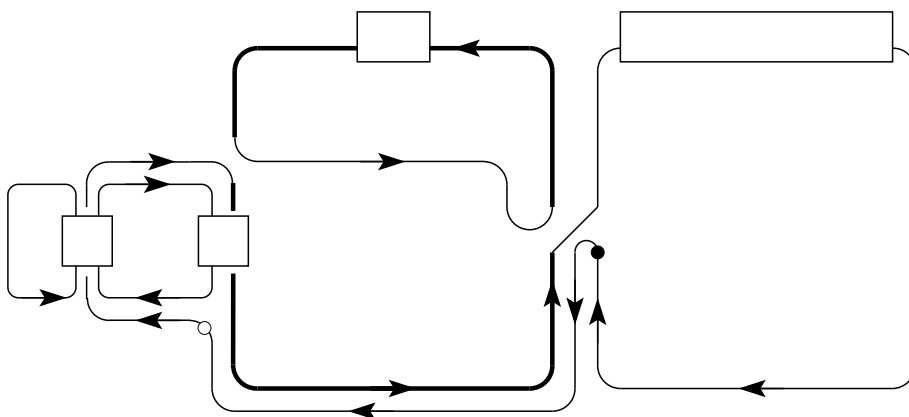


figure 6

This operation does not change the number of Seifert circles or the algebraic crossing number. It is meant as an operation used to prepare reduction sites for the other operations. Clearly, to be of any use it must be used in a way that wouldn't change the positive index or the negative index of the corresponding Seifert graph.

Operation 3. Cancellation of a pair of one positive and one negative crossing in a situation shown in figure 7.

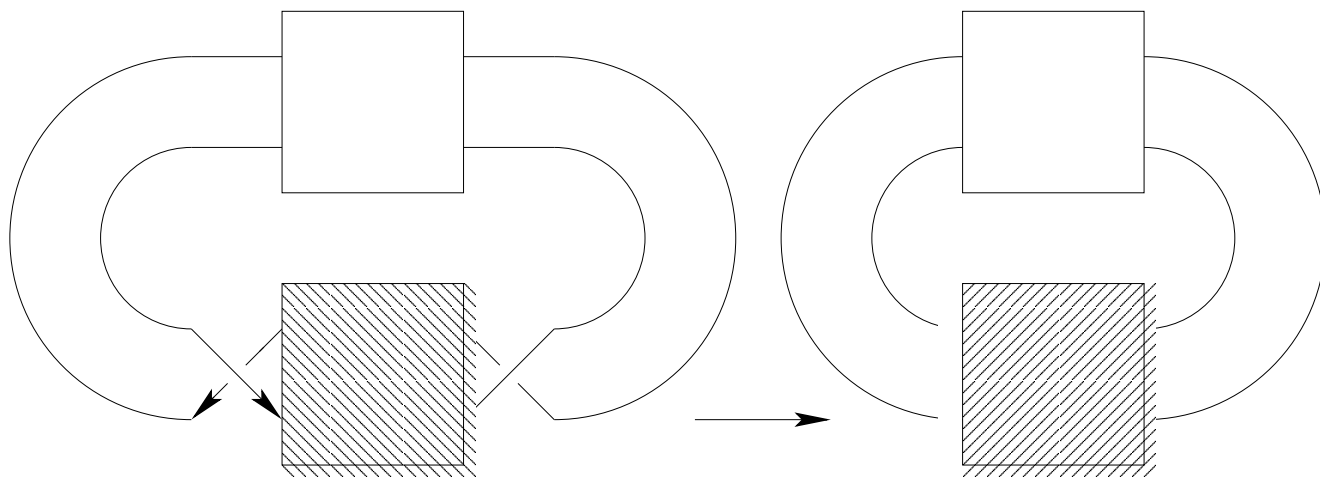


figure 7

This operation decreases the number of Seifert circles by two while preserving the algebraic

crossing number. In terms of the Seifert graph the corresponding operation is a simultaneous contraction of two signed edges, one positive and one negative in a situation shown in figure 8.

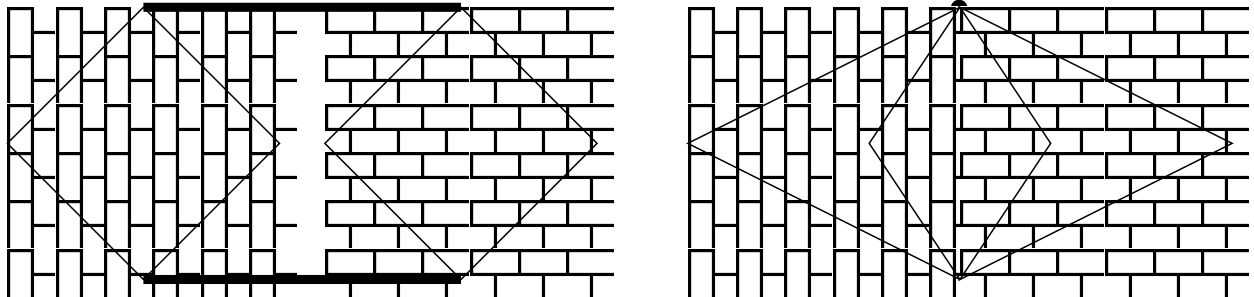


figure 8

The reason that prevents us from performing MP-operations in an unrestricted manner is that they are getting into each others way — a single crossing between two Seifert circles may lose its quality after an operation involving another crossing is performed. The same applies to operation 2. However, in some special cases we can work with two single crossings at a time. The operation 3 described above is the first example of such a situation. We will add two more:

Operation 4. To perform this operation we need four Seifert circles in a cycle like in figure 9. It is assumed that there is nothing more inside of the cycle but the diagram may be complicated outside. We rearrange one short tunnel and one short bridge to obtain one long tunnel and one long bridge. The two single crossings involved are of opposite signs. This is why the long tunnel and the long bridge do not go into each other's way. As a result one Seifert circle of the new diagram is now nested. The two single crossings do survive but the whole configuration of the Seifert graph is simplified.

Operation 5. This is really a variant of operation 4. A similar configuration of four Seifert circles is needed with a different placement of the two single crossings.

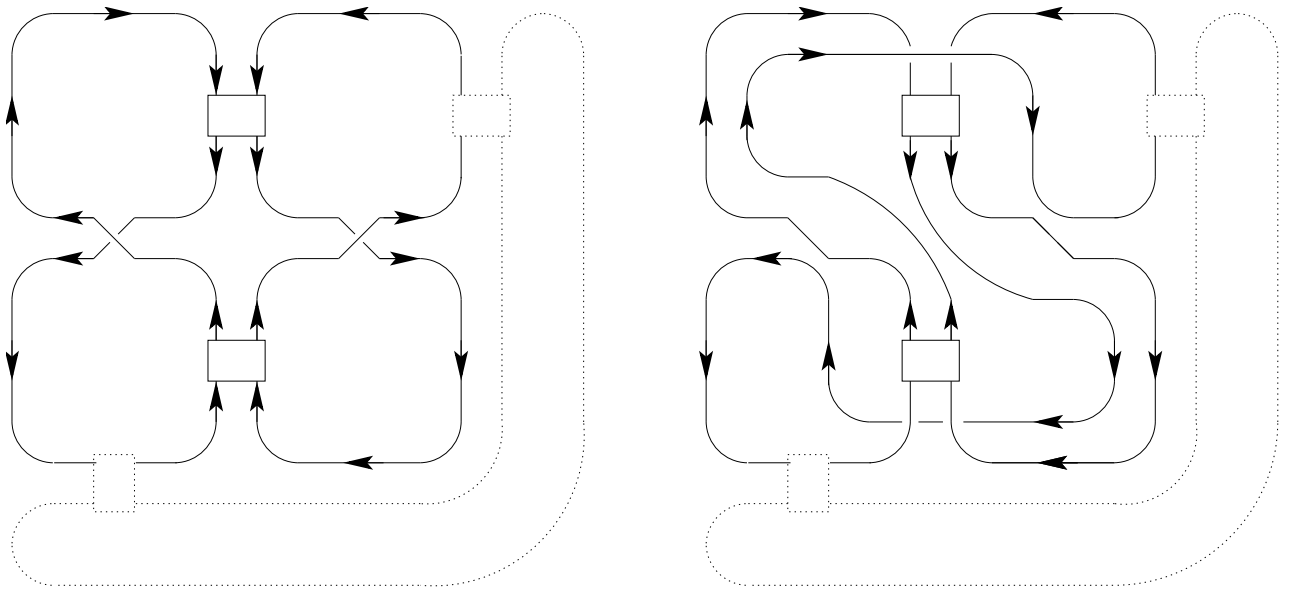


figure 9

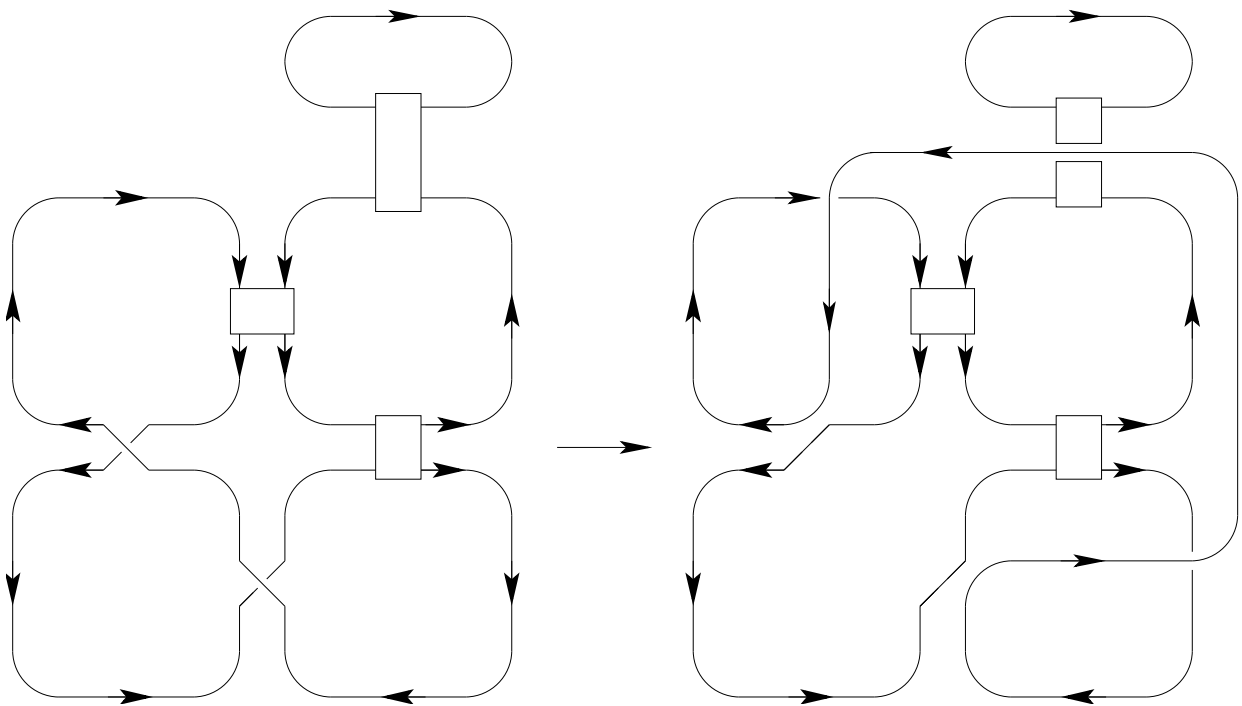


figure 10

Now we show two examples demonstrating how in some particular cases the operations 1–3 allow the required reduction, while the index inequality is sharp (which means that Proposition 2.1 is not sufficient).

The first example is quite an obvious one. The link defined by the diagram given in figure 11 can be represented by a diagram with only two Seifert circles, while the original number of Seifert circles is four and $ind = 1$. Operation 3 can be applied.

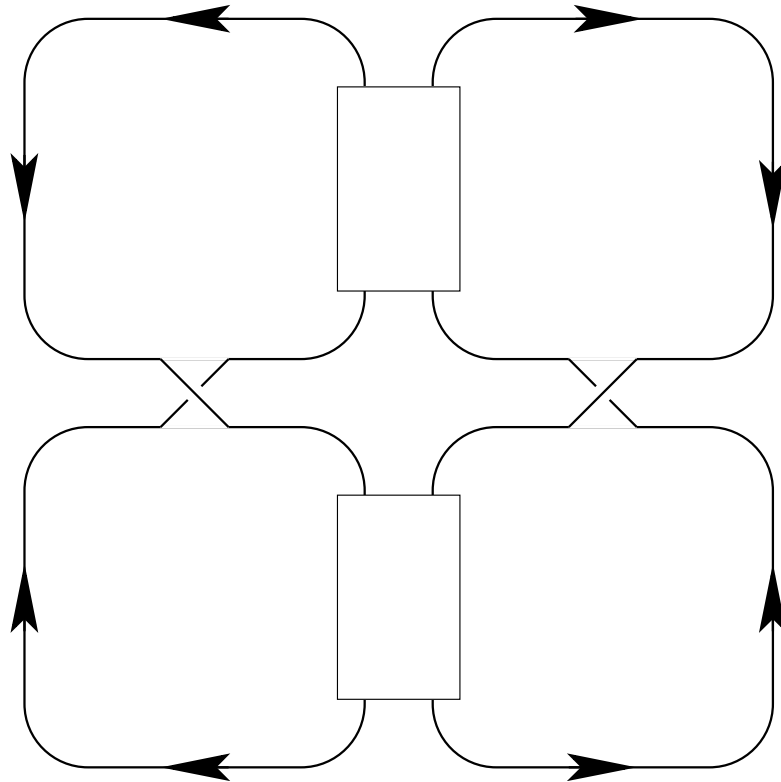


figure 11

In the second example we show a diagram containing initially five Seifert circles. The index is equal to one. It is not possible to cancel the two special crossings by operation 3. Also, it is not possible to perform an MP-operation first at one reduction site and then at the other (this is one of the cases when the two operations get into each others way). However, it is possible to perform operation 2 so that in the resulting diagram operation 3 can be performed, reducing the number of Seifert circles by two.

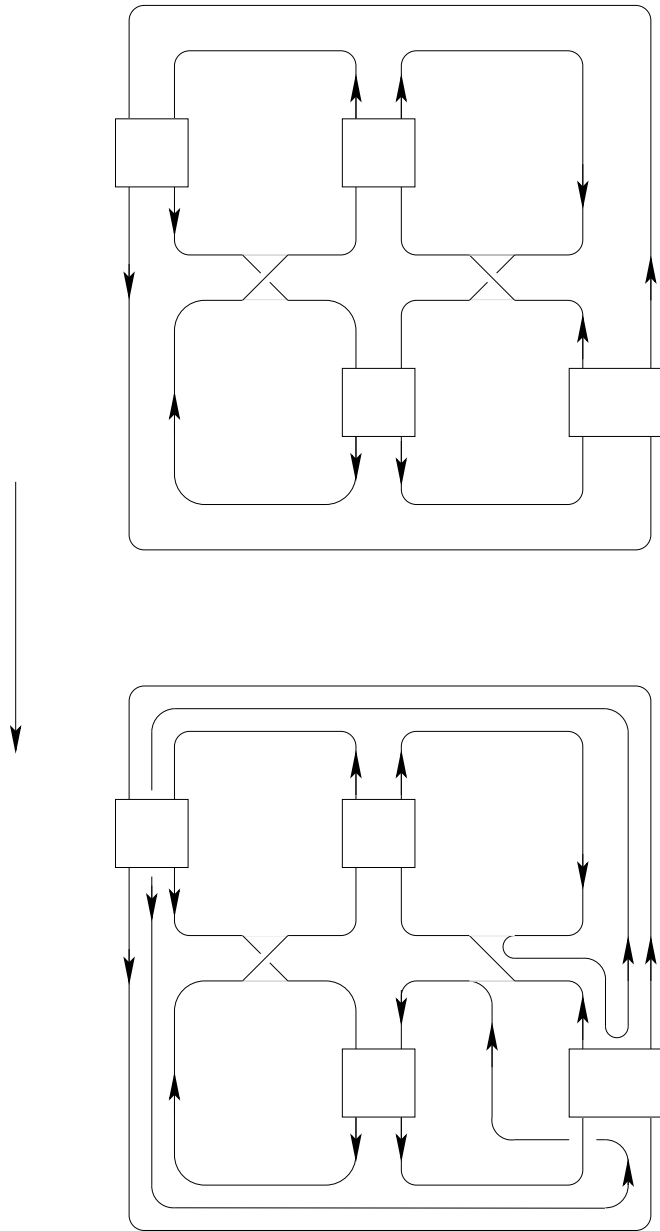


figure 12

Figure 12 shows how operation 2 is performed on the diagram. In the next figure we show the part of the resulting diagram to which operation 3 may now be applied.

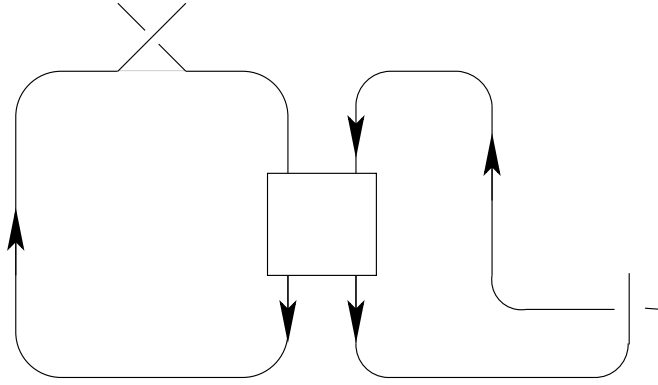


figure 13

3. Graph theory.

In this section we formulate a problem in graph theory. The positive solution would prove Conjecture 2.1. Let Γ be a planar bipartite graph (more precisely: a specific embedding of the graph into the plane). Let E_+ and E_- be two disjoint subsets of the set of edges (positive edges and negative edges). Assume that E_+ and E_- are both cyclically independent sets (which implies, in particular, that all signed edges are single). We will refer to this as independence conditions. A graph endowed with the structure described above (including the independence conditions) will be called an S-graph. As can be seen from this definition, an S-graph is just a Seifert graph corresponding to a diagram with one modification: only single edges retain the signs, and only those that constitute cyclically independent sets. Generally, all edges in a Seifert graph have naturally defined signs but here, in our translation of the problem to graph theory the complete information about the signs does not seem useful. Also, in a Seifert graph corresponding to a diagram the set of the single edges of one sign is not necessarily independent. What we do is *choose* maximal independent subsets for positive edges and for negative edges and *ignore* the signs of the remaining single edges.

We will say that two edges in a planar graph are neighbor edges if they have a common vertex and they belong to the boundary of one cell.

We consider five types of operations on graphs. The operations correspond to operations 1–5 on diagrams.

1. A signed stump may be removed from the graph. The resulting graph is always an S-graph.
2. Let u, e_1, v, e_2, w be a sequence of two neighbor edges and their vertices. Assume that e_1 is signed and e_2 neutral. Then u and w may be contracted into one vertex while at the same time e_2 is removed from the graph. The operation is allowed only if the resulting graph is still an S-graph.
3. Assume that e_1 and e_2 are a pair of signed edges, one positive and one negative such that when both of them are removed, then the graph gets disconnected. Then e_1 and

- e_2 may be contracted, each to a point. The resulting graph is always an S-graph.
4. Assume that there is a cell in the plane bounded by $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$. Assume that e_1 negative and e_2 is positive. Then e_3 and e_4 may be erased from the graph while v_2 and v_4 are contracted into one vertex. The resulting graph is always an S-graph.
 5. Assume that there is a cell in the plane bounded by $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$. Assume that e_1 is negative and e_3 is positive. Then e_2 and e_4 may be erased from the graph while v_1 and v_3 are contracted into one vertex. The resulting graph is always an S-graph.

Conjecture 3.1. *For any S-graph, it is possible to perform a sequence of operations of type 1–5 in such a way that in the final graph only neutral edges are left.*

We will now prove that Conjecture 3.1 is true in a special case.

Proposition 3.2. *Let Γ be an S-graph with only one positive and only one negative edge. Then it may be reduced as required in Conjecture 3.1.*

Proof. Let e_1 be the positive and e_3 the negative edge. The two edges may be disjoint or not. These two cases are quite similar. We will consider the case they are disjoint. Let v_1 be one of the vertices of e_1 . If there is no other edge of Γ meeting e_1 at v_1 , then e_1 is a stump and operation 1 may be applied. Assume the opposite. Let e_1, v_1, e_2 be a sequence of two neighbor edges with v_1 being their common vertex. The only possible obstacle to operation 2 in such case is that there is a rectangle $e_1, v_1, e_2, v_2, e_3, v_3, e_4, v_4$ in the considered graph. In particular, if there is no possibility to perform operation 2, then every neighbor edge of any signed edge of Γ is connected directly to the other signed edge of Γ . This proves that the whole graph looks like in figure 14. The shaded areas may contain something complicated but otherwise the graph must look like in the figure. For such a graph operation 2 may be performed.

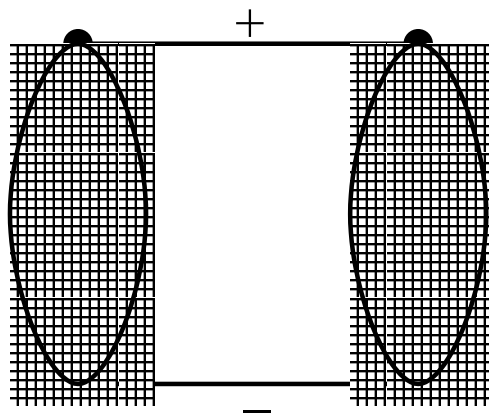


figure 14

It is interesting that in the above restricted case of Conjecture 3.1 only operations 1 – 3 are used. The following example, due to L. Plachta, shows that these operations are not sufficient in general.

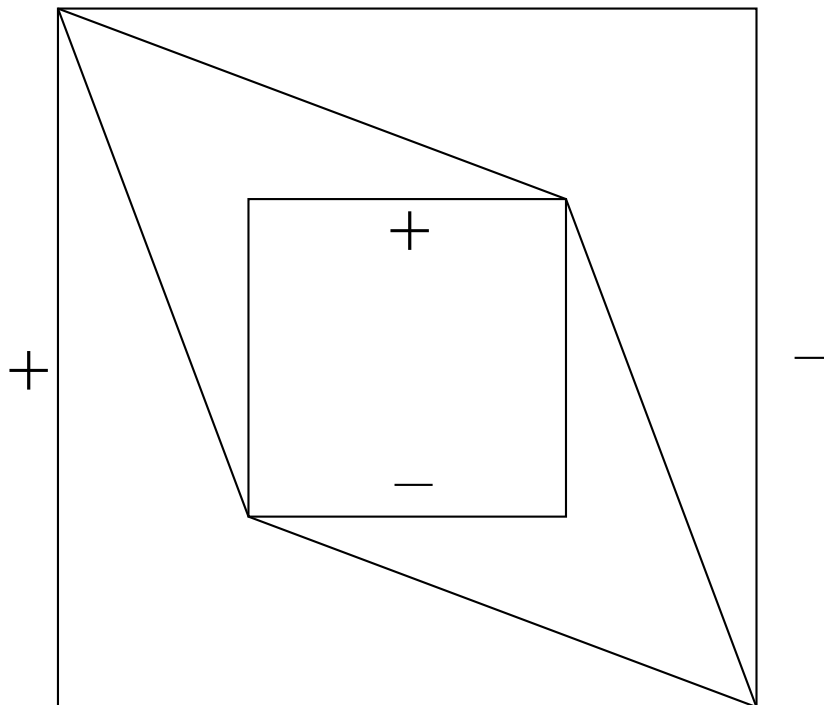


figure 15

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