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THE STRONG ISOMETRIC DIMENSION OF GRAPHS OF DIAMETER TWO

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The strong isometric dimension of graphs of diameter two

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Abstract

The strong isometric dimension $\text{idim}(G)$ of a graph G is the least number k such that G can be isometrically embedded into the strong product of k paths. The problem of determining $\text{idim}(G)$ for graphs of diameter two is reduced to a covering problem of the complement of G with complete bipartite graphs. As an example it is shown that $\text{idim}(P) = 5$, where P is the Petersen graph.

1 Introduction

Graph products offer a variety of possibilities to introduce the concept of a graph dimension. Nešetřil and Rödl [7] presented a general framework that for any class of graphs and for any graph product gives a different dimension concept. Slightly more precisely, the dimension of G is defined as the minimum number of factor graphs (from a selected class of graphs and with respect to a selected graph product) such that G embeds as an *induced subgraph* into their product. Then Nešetřil and Rödl proved a nice general result that either a fixed dimension is equal to 1 or tends to infinity. Earlier, Poljak and Pultr [8] introduced three

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specific related dimensions: the dimension of bipartite graphs with respect to induced embeddings into the direct product of paths of length 3, the dimension with respect to induced embeddings into the strong product of paths of length two, and the dimension with respect to induced embeddings into the direct product of complete graphs. For the latter dimension see also [2] and for the bipartite dimension [9].

Besides induced embeddings it is also interesting to consider isometric ones. A classical result of Graham and Winkler [5] asserts that any graph can be canonically isometrically embedded into the Cartesian product of graphs. Since this embedding is unique among all irredundant isometric embeddings with respect to the largest possible number of factors, the latter number is called the *isometric dimension* of a graph. We also add that four different dimensions (product dimension, isometric dimension, induced dimension, and dimension) with respect to the Cartesian product are treated in [1].

Back in 1938 Schönberg [10] proved that every connected graph admits an isometric embedding into the strong product of paths, cf. [6, Proposition 5.2]. It is hence natural to define the *strong isometric dimension*, $\mathrm{idim}(G)$, of a graph G as the least number k such that G embeds isometrically into the strong product of k paths. Recently Fitzpatrick and Nowakowski [3] extensively studied this concept and obtained several interesting results, see also [4].

In this note we consider the strong isometric dimension of graphs of diameter two. In the rest of this section we define the needed concepts and fix the notation. In the next section we reduce the computation of the isometric dimension of graphs of diameter two to the covering problem with complete bipartite graphs. As an application of this approach we show in the concluding section that the isometric dimension of the Petersen graph is five.

For P_n we will always assume $V(P_n) = \{0, 1, \ldots, n-1\}$, where i is adjacent to $i+1$ for $i = 0, 1, \ldots, n-2$. To distinguish the path on n vertices from a path that appears in a sequence of paths we will write $P^{(n)}$ to denote the nth term of such a sequence. By $d_G(u, v)$ we mean the standard graph distance, that is, the number of edges on a shortest u, v-path. The *diameter*, $\mathrm{diam}(G)$, of a connected graph G is the maximum distance between any two vertices of G. The *complement* $\overline{G}$ of a graph G is the graph on $V(G)$ with the edge set $E(\overline{G}) = \{xy \mid x, y \in V(G), x \neq y, xy \notin E(G)\}$. A mapping $f : V(G) \to V(H)$ is an *isometric embedding* if $d_H(f(u), f(v)) = d_G(u, v)$ for any $u, v \in V(G)$.

The strong product $G = \boxtimes_{i=1}^{k} G_i$ of graphs $G_1, G_2, \ldots, G_k$ is the graph defined on the Cartesian product of the vertex sets of the factors, two distinct vertices $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ being adjacent if and only if u_i is equal or adjacent to v_i in G_i for $i = 1, 2, \ldots, k$. The *strong isometric dimension*, $\mathrm{idim}(G)$, of a graph G is the least number k such that there is a set of k paths $\{P^{(1)}, P^{(2)}, \ldots, P^{(k)}\}$ such that G isometrically embeds into $\boxtimes_{i=1}^{k} P^{(i)}$.

2 Reduction to coverings with complete bipartite graphs

Before stating our main result we give two lemmas, the first one being well-known, cf. [6, Lemma 5.1].

Lemma 2.1 *Let $G = \boxtimes_{i=1}^{k} G_i$ be the strong product of connected graphs. Then*

$$d_G(u, v) = \max_{1 \le i \le k} d_{G_i}(u_i, v_i).$$

Lemma 2.2 *Let G be a graph isometrically embedded into the strong product of k paths. If $\mathrm{diam}(G) = 2$ then G isometrically embeds into the strong product of k paths each of length at most two.*

Proof. Let $f : G \to \boxtimes_{i=1}^{k} Q^{(i)}$ be an isometric embedding into the strong product of paths $Q^{(i)}$. Hence $f(u) = (u^{(1)}, \ldots, u^{(k)})$, where $u^{(i)} \in \{0, \ldots, |V(Q^{(i)})| - 1\}$. Set $M_i = \max\{u^{(i)} \mid u \in V(G)\}$ and $m_i = \min\{u^{(i)} \mid u \in V(G)\}$. Since $\mathrm{diam}(G) = 2$, Lemma 2.1 implies that $M_i - m_i \le 2$ for $1 \le i \le k$. Let $P^{(i)}$ be a path with the vertex set $\{0, \ldots, M_i - m_i\}$. Clearly, $P^{(i)}$ is of length at most two. Then the mapping $G \to \boxtimes_{i=1}^{k} P^{(i)}$ induced by f is an isometry. □

In the next theorem, our main result, we treat also $K_1 = K_{1,0}$ as a complete bipartite graph.

Theorem 2.3 *Let G be a graph with $\mathrm{diam}(G) = 2$. Then $\mathrm{idim}(G)$ is equal to the smallest r for which the edges of $\overline{G}$ can be covered with complete bipartite subgraphs $B_1, \ldots, B_r$ of $\overline{G}$, such that for any edge e of G there exists a B_i with one end of e belonging to B_i but not the other.*

Proof. Suppose that $\mathrm{idim}(G) = r$. By Lemma 2.2 there is an isometric embedding $f : G \to H = \boxtimes_{i=1}^{r} P_3^{(i)}$. For $i = 1, 2, \ldots, r$ let B_i be a complete bipartite graph with the bipartition $X_i + Y_i$, where $X_i = \{u \in V(G) \mid (f(u))_i = 0\}$ and $Y_i = \{u \in V(G) \mid (f(u))_i = 2\}$. Note that if $u \in X_i$ and $v \in Y_i$ then $uv \in E(\overline{G})$. Consider now an arbitrary edge uv of $\overline{G}$. Since $\mathrm{diam}(G) = 2$ we have $d_G(u, v) = 2$ and as f is isometry, we also have $d_H(f(u), f(v)) = 2$. Hence, applying Lemma 2.1, there exists an index i such that $(f(u))_i = 0$ and $(f(v))_i = 2$ or vice versa. But then uv is an edge of B_i which implies that all the edges of $\overline{G}$ are covered with the B_i's, $1 \le i \le r$. Let $uv \in E(G)$. Then since f is an isometry, there exists an index i such that $\{(f(u))_i, (f(v))_i\}$ is equal either to $\{0, 1\}$ or to $\{1, 2\}$. Assuming without loss of generality $(f(u))_i = 0$ and $(f(v))_i = 1$ we infer that $u \in B_i$ and $v \notin B_i$.

Assume now that the edges of $\overline{G}$ can be covered with r complete bipartite graphs B_i with bipartitions $V(B_i) = X_i + Y_i$, $i = 1, 2, \ldots, r$, such that for any edge uv of G there is an i with $u \in B_i$ and $v \notin B_i$. Define a mapping $f : G \to \boxtimes_{i=1}^{r} P_3^{(i)}$ with

$$(f(u))_i = \begin{cases} 0; & u \in X_i, \\ 2; & u \in Y_i, \\ 1; & \text{otherwise}. \end{cases}$$

We claim that f is an isometry. Let $uv \in E(G)$. We need to show that then $\max_i\{|(f(u))_i - (f(v))_i|\} = 1$. If for some i we have $u, v \in B_i$, then since u and v are not adjacent in $\overline{G}$, we have either $(f(u))_i = (f(v))_i = 0$ or $(f(u))_i = (f(v))_i = 2$. On the other hand we have assumed that there exists an i such that $u \in B_i$ and $v \notin B_i$. But then $(f(u))_i$ and $(f(v))_i$ differ by one. It follows that f maps edges to edges. Suppose next that $d_G(u, v) = 2$. Then uv is an edge of $\overline{G}$ and hence uv is covered with at least one graph B_i. Hence $|(f(u))_i - (f(v))_i| = 2$. It follows that f is an isometry, hence $\mathrm{idim}(G) \le r$ and the argument is complete. □

Theorem 2.3 and its proof are illustrated in Fig. 1. The graph G is of diameter two and its complement can be covered with two complete bipartite graphs: a $K_{1,3}$ induced by the partition $\{z\} + \{u, v, w\}$ and a $K_{2,2}$ induced by the partition $\{x, u\} + \{y, w\}$. Then the proof gives the embedding into $P_3 \boxtimes P_3$ with $f(u) = (2, 0)$, $f(v) = (2, 1)$, $f(w) = (2, 2)$, $f(x) = (1, 0)$, $f(y) = (1, 2)$, and $f(z) = (0, 1)$, see the right hand side of Fig. 1.

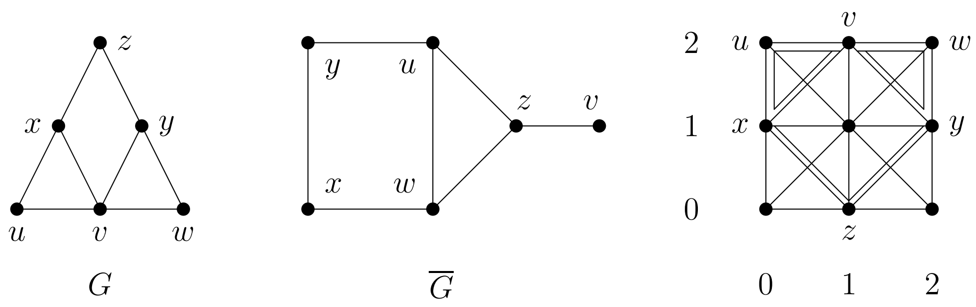

Figure 1: Graph G of diameter two with $\mathrm{idim}(G) = 2$

Consider next the complete graph on four vertices minus an edge $K_4 - e$. It is of diameter two and its complement consists of an edge and two isolated vertices so that its edge(s) can be covered with one complete bipartite graph $K_{1,1}$. Since $\mathrm{idim}(K_4 - e) = 2$, we see that the condition of Theorem 2.3 requiring that for any edge uv of G there is an i with $u \in B_i$ and $v \notin B_i$ cannot be dropped. Moreover, this example also shows that for an optimal embedding we may (and must) use a K_1 in a covering with complete bipartite graphs. On the other hand, we have:

Corollary 2.4 *Let G be a graph with $\mathrm{diam}(G) = 2$ and let any edge of G be contained in an induced path on three vertices. Then $\mathrm{idim}(G)$ is equal to the smallest r such that the edges of $\overline{G}$ can be covered with r complete bipartite subgraphs.*

Proof. By Theorem 2.3 we only need to prove that if $\overline{G}$ is covered with r complete bipartite graphs B_i with bipartitions $V(B_i) = X_i + Y_i$, for $i = 1, 2, \ldots, r$, then G embeds isometrically into $H = \boxtimes_{i=1}^{r} P_3^{(i)}$. We define f as in the proof of Theorem 2.3. In addition, if $d_G(u, v) = 2$, the same argument implies $d_H(f(u), f(v)) = 2$. Let now u and v be vertices with $d_G(u, v) = 1$. Again, as in the previous proof we see that $\max_i\{|(f(u))_i - (f(v))_i|\} \leq 1$. To see that this maximum equals 1, let $u \to v \to w$ be an induced path that exists by the theorems assumption. Then $uw \in E(\overline{G})$. Let B_i be a complete bipartite graph that covers the edge uw. Then $v \notin B_i$, hence by Theorem 2.3 G isometrically embeds into H. □

3 The dimension of Petersen graph

Fitzpatrick and Nowakowski asked in [3] whether there exists a graph G with $\mathrm{idim}(G) > \lceil |V(G)|/2 \rceil$. On the other hand they proved that for a graph G,

$\mathrm{idim}(G) \leq |V(G)| - \mathrm{diam}(G)$. These findings were one of our motivations for the present study. In this section we show that the Petersen graph is a graph with the strong isometric dimension equal to one half of its size.

Proposition 3.1 *Let P be the Petersen graph, then* $\mathrm{idim}(P) = 5$.

Proof. Let a_i and b_i, $1 \leq i \leq 5$, be the vertices of P as shown on Fig. 2. Set $A = \{a_i \mid i = 1, \ldots, 5\}$ and $B = \{b_i \mid i = 1, \ldots, 5\}$.

Clearly, any edge of P is contained in an induced path on three vertices, hence we may apply Corollary 2.4. Let $\mathcal{C}$ be a collection of complete bipartite subgraphs of $\overline{P}$ that cover the edges of $\overline{P}$. Suppose first that there is a copy of $K_{2,5}$ in $\mathcal{C}$. Let $V(K_{2,5}) = X + Y$, where $X = \{x, y\}$. If $xy \in E(P)$ then $|Y| \leq 4$. On the other hand, if x is not adjacent to y, then $y \cup \{Y\} = \{z \in V(P) \mid d_P(x, z) = 2\}$. But then y is in $\overline{P}$ not adjacent to all vertices of Y. It follows that $\mathcal{C}$ contains no copy of $K_{2,5}$.

Suppose next that there is a copy of $K_{3,3}$ in $\mathcal{C}$. Let $V(K_{3,3}) = X + Y$. Note that $|X \cap A| \leq 2$ and $|X \cap B| \leq 2$, for otherwise Y would contain less than three vertices. Similarly $|Y \cap A| \leq 2$ and $|Y \cap B| \leq 2$. So we may without loss of generality assume $|X \cap A| = 2$. If the two vertices of $X \cap A$ would not be adjacent, then we would have $|Y \cap B| = 3$, which is not possible. Hence, we may in addition without loss of generality assume $X \cap A = \{a_1, a_2\}$. Because of adjacencies in P we see that $b_1, b_2, a_3, a_5 \notin Y$ and therefore $a_4 \in Y$, for otherwise we would again have $|Y \cap B| = 3$. This implies that $a_3, a_5, b_4 \notin X$. Hence the third vertex of X must be one of the vertices b_1, b_2, b_3, b_5. However, if X contains any of these vertices, then, using the adjacencies in P again, Y cannot contain three elements. For instance, if $b_1 \in X$ then besides a_4 only b_5 can be in Y.

We have thus shown that $\mathcal{C}$ contains only subgraphs isomorphic to $K_{2,2}$, $K_{2,3}$, $K_{2,4}$, $K_{1,6}$, and smaller ones. Since $|E(\overline{P})| = 30$, it follows that $|\mathcal{C}| \geq 4$. Moreover, if $|\mathcal{C}| = 4$, it necessarily contains at least three copies of $K_{2,4}$. So assume that $\mathcal{C}$ is indeed such and consider an arbitrary copy of $K_{2,4} =: B$ with the bipartition $X + Y$, where $|X| = 2$. Then the two vertices of X must be adjacent. Moreover, the vertices of Y are uniquely determined and in P they induce two independent edges. It follows that the vertices of B induce three independent edges of P. Hence there are precisely five possibilities to select $X \cup Y$, and thus there are 15 different subgraphs B isomorphic to $K_{2,4}$.

Let B and B' be two different copies of $K_{2,4}$ from $\mathcal{C}$. If $V(B)$ contains two vertices of $V(B')$ that are adjacent in P, then $V(B) = V(B')$. Then, as $B \neq B'$, B and B' have 4 common edges. But then the subgraphs from $\mathcal{C}$ cannot cover all the 30 edges of $\overline{P}$. So we may assume in the rest that $V(B) \neq V(B')$. Then it is straightforward to verify that $|V(B) \cap V(B')| = 3$. Let e, f, and g be the edges of P induced by $V(B)$. Since $|V(B) \cap V(B')| = 3$, $V(B')$ contains exactly one of the ends of each e, f, and g. Then B and B' have at least one edge in common. Consider now three copies of $K_{2,4}$ from $\mathcal{C}$: B_1, B_2, and B_3. Then B_1 covers 8 edges of $\overline{P}$, B_2 covers at most 7 additional edges, and B_3 covers at most 6 new edges. Hence any such three subgraphs cover at most 21 edges of $\overline{P}$. It follows that we cannot cover all the 30 edges of $\overline{P}$ with 4 complete bipartite graphs and therefore $\mathrm{idim}(P) \geq 5$.

To complete the proof we show that the edges of $\overline{P}$ are covered with the following complete bipartite subgraphs B_i, $1 \leq i \leq 5$. Let $V(B_i) = X_i + Y_i$, where $X_i = \{a_i, b_i\}$ and Y_i is the set of vertices that are on distance two from both a_i and b_i in P. For instance, $Y_1 = \{a_3, a_4, b_2, b_5\}$. Then it is straightforward to check that the subgraphs B_i cover the edges of $\overline{P}$, hence Corollary 2.4 implies $\mathrm{idim}(P) \leq 5$. The corresponding embedding is shown on Fig. 2. □

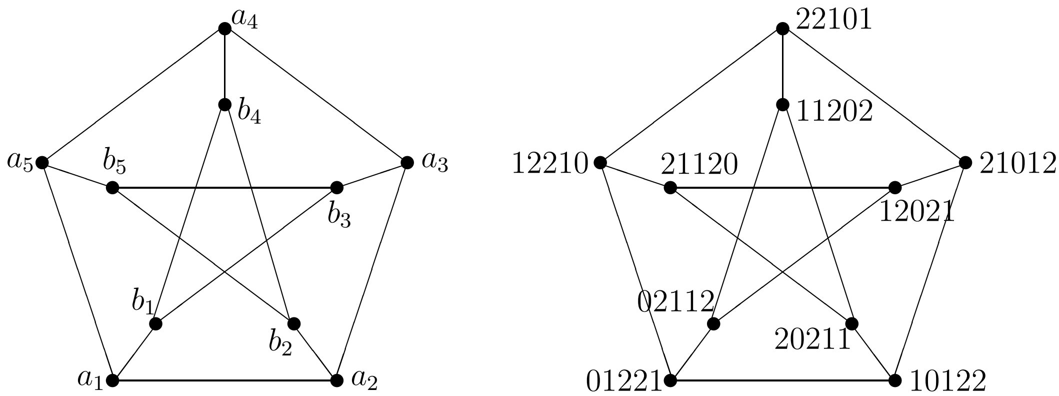

Figure 2: The Petersen graph and its isometric embedding into $\boxtimes_{i=1}^{5} P_3^{(i)}$

We conclude by noting that any independent set of edges of P of size five yields an isometric embedding into $\boxtimes_{i=1}^{5} P_3^{(i)}$ similar to the one from Fig. 2.

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