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# HOMOMORPHISMS OF HEXAGONAL GRAPHS TO ODD CYCLES

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# Homomorphisms of hexagonal graphs to odd cycles


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**Abstract**

The problem of deciding whether an arbitrary graph $G$ has a homomorphism into a given graph $H$ has been widely studied and has turned out to be very difficult. Hell and Nešetril proved that the decision problem is NP-complete unless $H$ is bipartite. We consider a restricted problem where $H$ is an odd cycle and $G$ an arbitrary hexagonal graph. We show that any triangle-free hexagonal graph has a homomorphism into cycle $C_5$.



**Keywords:** homomorphism, H-coloring, triangle-free hexagonal graph.


## 1 Introduction

Graph homomorphisms in the current sense were first studied by Sabidussi in the late fifties and early sixties. This was followed by much activity in different areas, see [3], [2] and the references there.


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One of the approaches is deciding whether an arbitrary graph $G$ has a homomorphism into a fixed graph $H$. Since an $n$-coloring of a graph $G$ is a homomorphism of $G$ to $K_n$, the term *$H$-coloring* of $G$ has been employed to describe the existence of a homomorphism of a graph $G$ into graph $H$. In such a case the graph $G$ is said to be *$H$-colorable.* Many authors have studied the complexity of the $H$-coloring problem. The main result was given by Hell and Nešetril in 1990 [6]. They proved that $H$-coloring problem is NP-complete, if $H$ is non-bipartite graph and polynomial otherwise, assuming P$\neq$NP. Several restrictions of the $H$-coloring problem have been studied [3]. The case when the input graph $G$ is restricted to have degree bounded by a small constant is investigated in [1]. The paper proves, among other things, that the $H$-coloring problem remains NP-complete for $H = C_{2k+1,}$ $k \geq 2$ even if the input is restricted to graphs of maximum degree three. The $H$-coloring problem also remains NP-complete for any non-bipartite graph $H$, when the input is restricted to *odd girth* (length of a shortest odd cycle) at least seven [7].

We investigate a restricted $H$-coloring problem, where $H$ is an odd cycle and $G$ an arbitrary, so called, *hexagonal graph*, which is an induced subgraph of a triangular lattice. We will consider only triangle-free hexagonal graphs, because a hexagonal graph which contains a triangle is obviously $C_3$-colorable but it is not $C_5$-colorable. It is not difficult to see that the odd girth of triangle-free hexagonal graphs is at least nine, therefore it is interesting to ask whether there is a $C_5$, $C_7$, or $C_9$-coloring of such a graph.

A $C_{2k+1}$-coloring problem is also interesting because it is connected with the so called *$n$-[$k$] coloring* of a weighted graph $G$, where a weight of every vertex $v \in G$ equals $k$. An *$n$-[$k$] coloring* of $G$ is a multicoloring of graph $G$, where every vertex $v \in G$ is assigned a subset of $k$ different colors from the set of $n$ colors, such that subsets assigned to any two adjacent vertices are disjoint. In terms of homomorphisms, an $n$-[$k$] coloring is equivalent to a homomorphism to the Kneser graph $K(k, n)$. A motivation for interest in $n$-[$k$] colorings of hexagonal graphs comes from its relation to frequency assignment problems, see [9]. Namely, the existence of an algorithm for $n$-[$k$] coloring of a hexagonal graph implies existence of an algorithm for multi-coloring of weighted hexagonal graph with competitive ratio $\frac{n}{2k}$. For more details see [5]. Furthermore, if we can prove that a graph $G$ is $C_{2k+1}$-colorable, this implies that there exist a $(2k+1)$-[$k$] coloring of $G$. (As there exists a $(2k+1)$-[$k$] coloring, let say $F$, of $C_{2k+1}$ and there exists a homomorphism

$f : G \to C_{2k+1}$, then $F \circ f : G \to \{1, 2, ..., 2k-1\}$ is a $(2k+1)$-$[k]$ coloring of $G$.) Several results on $n$-$[k]$ colorings of hexagonal graphs have been given recently. Havet and Žerovnik [5] proved that every triangle-free hexagonal graph is 5-[2] colorable, Havet proved that there exists a 7-[3] coloring of every triangle-free hexagonal graph [4]. McDiarmid and Reed conjectured that every triangle-free hexagonal graph is 9-[4] colorable [10].

In this note we prove

**Theorem 1** *Any triangle-free hexagonal graph is $C_5$-colorable.*

The rest of the paper is organized as follows. In Section 2 some basic definitions and results are given, in Section 3 an algorithm for finding a homomorphism of any triangle-free hexagonal graph to cycle $C_5$ is given. In the last section a triangle-free hexagonal graph with no homomorphism to $C_9$ is given while the $C_7$-colorability of triangle-free graphs is left as an open problem.

# 2 Preliminaries

Let $G$ and $H$ be graphs. A function $\varphi : G \to H$ is a *homomorphism* from $G$ to $H$ if it is an adjacency preserving mapping from $V(G)$ to $V(H)$, namely a mapping for which $[\varphi u, \varphi v] \in E(H)$ whenever $[u, v] \in E(G)$. We write simply $\varphi : G \to H$ and $u \mapsto \varphi(u)$.

Let $P_n = (v_0, v_1, v_2, ..., v_n)$ be a path of length $n$ and let $C_n = (v_0, v_1, ..., v_{n-1})$ be a cycle on $n$ vertices. In the continuation we will denote vertex $v_i$ of a cycle $C_n$ simply by $i$, where vertex $i$ is adjacent to vertices $i-1 \pmod n$ and $i+1 \pmod n$, for $i = 0, 1, ..., n-1$.

Let $u$ and $v$ be two vertices of a graph $G$. The *distance* between $u$ and $v$ in $G$ is the length of the shortest path from $u$ to $v$ in $G$ and will be denoted as $dist_G(u, v)$.

The following facts are well known:

**Lemma 2** *Let $P_{2k+1}$ be a path with endvertices $u$ and $v$ and let $f$ be a mapping $f : V(P_{2k+1}) \to V(C_{2n+1})$, where $1 \leq k < n$. If $f(u) = f(v)$, then $f$ is not a homomorphism.*

**Proof.** For vertices $u$ and $v$ holds $dist_{P_{2k+1}}(u, v) = 2k + 1$. Suppose that $f$ is a homomorphism. As homomorphism $\varphi$ maps an $v_0, v_1, ..., v_k$ walk into an

$\varphi(v_0), \varphi(v_1), ..., \varphi(v_k)$ walk, it is obvious that in the cycle $C_{2n+1}$ we can came from a vertex $x$ back to the vertex $x$ in odd number of steps only if we go through the whole cycle. Therefore, number of steps must be greater than or equal to $2n+1$, thus $k \geq n$, which is a contradiction. ■

**Corollary 3** *There is no homomorphism $f : C_{2k+1} \to C_{2n+1}$ if $k < n$.*

**Corollary 4** *Let $C_{2k+1}$ be a cycle on vertices $u_1, u_2, ..., u_{2k+1}$. If for a mapping $f : V(C_{2k+1}) \to V(C_{2k+1})$ we have $f(u_1) = u_1$ and $f(u_{2k+1}) = u_{2k+1}$, then $f$ is a homomorphism if and only if $f(u_i) = u_i$ for $2 \leq i \leq 2k$.*

As we already mentioned we will consider the existence of a homomorphism from a given graph $G$ to an odd cycle $C_{2k+1}$. In general, there does not exist a homomorphism from an arbitrary graph to an odd cycle. Fig. 1 shows a planar graph $G$ of girth 5 which is not $C_5$-colorable.

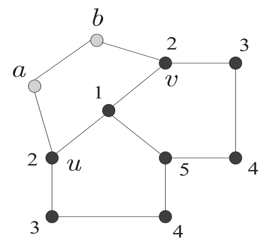


Figure 1: An example of a planar graph, which is not $C_5$-colorable.

Namely, if there existed such a homomorphism $f : G \to C_5$, the only possibility for vertices $u$ and $v$, by Corollary 4, would be $f(u) = f(v)$, but in this case the gray vertices $a$ and $b$ can not be properly mapped.

It can be proved that for special planar graphs, so called *hexagonal graphs* without triangles, there exists a homomorphism to $C_5$. In Section 3, an algorithm for finding such a homomorphism is given. Recall that *hexagonal graphs* are induced subgraphs of a triangular lattice. Vertices of a triangular lattice may be described as follows. The position of each vertex is an integer linear combination $x\vec{p} + y\vec{q}$ of two vectors $\vec{p} = (1, 0)$ and $\vec{q} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus, we may identify the vertices of a triangular grid with the pairs $(x, y)$ of integers. Two vertices are adjacent when the Euclidean distance between them is one. Therefore each vertex $(x, y)$ has six neighbors $(x \pm 1, y)$, $(x, y \pm 1)$,

$(x+1, y-1)$ and $(x-1, y+1)$. For the simplicity, we will refer to the neighbors as R (right), L (left), UR (up-right), DL (down-left), DR (down-right) and UL (up-left), respectively. There is an obvious 3-coloring of the infinite triangular lattice which gives rise to a partition of vertex set of any triangular lattice graph into three independent sets $I_0$, $I_1$ and $I_2$. The partition can be decided from the coordinates by the rule: a vertex with coordinates $(x, y)$ is in the independent set $I_i$ where $i = x + 2y \pmod 3$. According to the partition $I_0$, $I_1$ and $I_2$, vertices are assigned their *base colors*, which are denoted by $r$ (red), $b$ (blue) and $g$ (green) respectively.

# 3 The main result

In this section an algorithm which finds a homomorphism from an arbitrary hexagonal graph to cycle $C_5$ is given.

**Lemma 5** *Let $P = (x_0, x_1, ..., x_m)$ be a path of length $m$ and $c_0$, $c_m$ two different vertices from cycle $C_5$. There exists a homomorphism $\varphi : P \to C_5$ such that $\varphi(x_0) = c_0$ and $\varphi(x_m) = c_m$ if and only if ($m = 1$ and $d_{C_5}(c_0, c_1) = 1$) or ($m = 2$ and $d_{C_5}(c_0, c_2) \neq 1$) or ($m = 3$ and $d_{C_5}(c_0, c_3) \geq 1$) or $m \geq 4$.*

**Proof.** We will prove the result by induction on $m$.
- for $m = 1$ the result is obvious.
- $m = 2$ : we have two possibilities. If $c_2 = c_0$ or $c_2 = c_0 + 2 (\text{mod } 5)$, then for $x_1 \mapsto c_0 + 1 (\text{mod } 5)$, $\varphi$ is a homomorphism. If $c_2 = c_0 - 2 (\text{mod } 5)$, then for $x_1 \mapsto c_0 - 1 (\text{mod } 5)$, $\varphi$ is a homomorphism. If $d_{C_5}(c_0, c_2) = 1$, then neighbors $c_0$ and $c_2$ do not have a common neighbor in $C_5$ which could be an image of $x_1$.
- $m = 3$ : if $d_{C_5}(c_0, c_3) = 1$, then for $x_1 \mapsto c_3$ and $x_2 \mapsto c_0$, $\varphi$ is a homomorphism. If $c_3 = c_0 + (-1)^k 2 (\text{mod } 5)$, where $k = 1, 2$, then for $x_1 \mapsto c_0 + (-1)^{k+1} (\text{mod } 5)$ and $x_2 \mapsto c_0 + (-1)^{k+1} 2 (\text{mod } 5)$, $\varphi$ is a homomorphism. If $c_0 = c_3$ then, by Lemma 2, $\varphi$ is not a homomorphism.
- $m = 4$ : we have three different possibilities:
(a) if $c_0 = c_4$, then $\varphi(x_1) = \varphi(x_3) = c_0 (\text{mod } 5)$ and $\varphi(x_2) = c_0$ yield a homomorphism,
(b) if $c_4 = c_0 + (-1)^k (\text{mod } 5)$, where $k = 1, 2$, then $\varphi(x_i) = c_0 + (-1)^{k+1} i (\text{mod } 5)$, for $i = 1, 2, 3$ yield a homomorphism,
(c) if $c_4 = c_0 + (-1)^k 2 (\text{mod } 5)$, where $k = 1, 2$, then $\varphi(x_1) = \varphi(x_3) =$

$c_0 + (-1)^k (\text{mod } 5)$ and $\varphi(x_2) = c_0$ yield a homomorphism.

- Let now be $m > 4$ and $c_{m-1}$ such a vertex of $C_5$ that $d_{C_5}(c_{m-1}, c_m) = 1$. By induction hypothesis there exists a homomorphism $\overline{\varphi} : P_{m-1} \to C_5$, such that $\overline{\varphi}(x_0) = c_0$ and $\overline{\varphi}(x_{m-1}) = c_{m-1}$. Because $c_{m-1}$ and $c_m$ are neighbors in $C_5$, we can extend $\overline{\varphi}$ to homomorphism $\varphi : P_m \to C_5$, such that $\varphi / P_{m-1} = \overline{\varphi}$ and $\varphi(x_m) = c_m$. ■

According to one of the referees, Lemma 5 has been used by several other authors. Its first use occures in [8].

## 3.1 The algorithm

Let $G$ be a hexagonal graph without triangles. A vertex $v \in G$ is said to be *suitable* if it has neither L, RD nor RU neighbor in a triangular lattice. It is clear that the set of suitable vertices is independent.

A path $(x_0, x_1, ..., x_n)$ is *left (*resp. *rightup, rightdown)* if $x_{i+1}$ is the L (resp. RU, RD) neighbor of $x_i$, for $i = 0, 1, ..., n-1$, in a triangular lattice. A *tristar* of centre $x$ is the union of one left, one rightup and one rightdown path emerging from a common origin $x$.

**Definition 6** *Let us define a mapping* $\overline{F} : S \to V(C_5)$*, where* $S$ *is a set of vertices, as follows:*
*- if a base color of* $v \in S$ *is red, then* $\overline{F}(v) := 1$*,*
*- if a base color of* $v \in S$ *is blue, then* $\overline{F}(v) := 2$*,*
*- if a base color of* $v \in S$ *is green, then* $\overline{F}(v) := 3$*.*

We will denote the restriction of a mapping $F$ to a set $S$ with $F|_S$.

The input of the algorithm is a triangle-free hexagonal graph where each vertex knows its base color. The algorithm consists of two steps.

**Step 1:** Map every vertex of the set of all suitable vertices $S_G$ of $G$ by the mapping $\overline{F} : S_G \to V(C_5)$.

**Step 2:** Extend the mapping $\overline{F} : S_G \to V(C_5)$ to a mapping $F : V(G) \to V(C_5)$, such that $F|_{S_G} = \overline{F}$.

Details of the algorithm and its correctness proof are given below.

**Remark.** A straightforward implementation of the algorithm runs in linear time. It is also possible, analogous to [5] or [11], to implement a distributed version, using $O(n)$ procesors, with constant time complexity.

## 3.2 The correctness of the algorithm

Because suitable vertices are independent, $\overline{F}$ is trivially a homomorphism.

Let $G'$ be the graph induced on vertices of $V(G)\setminus S_G$. It is easy to see that the only connected components of $G'$ are paths and tristars. According to Lemma 5 the homomorphism $\overline{F} : S_G \to C_5$ can be extended to homomorphism $F : S_G \cup P \to C_5$, such that $F|_{S_G} = \overline{F}$ for any unmapped path $P$ after Step 1.

Let us show now how the homomorphism $\overline{F} : S_G \to C_5$ can be extended to an unmapped tristar after Step 1. Let $T$ be a tristar of $G'$ of centre $x = u_0 = v_0 = w_0$. Let $(u_0, u_1, ..., u_k)$ be the left, $(v_0, v_1, ..., v_l)$ the rightup and $(w_0, w_1, ..., w_m)$ the rightdown path of $T$. Let $u$ be the L neighbor of $u_k$, $v$ the RU neighbor of $v_l$ and $w$ the RD neighbor of $w_m$ in $G$. (Note that vertices $u, v$ and $w$ belong to $G$. Otherwise at least one of the vertices $u_k$, $v_l$ or $w_m$ would be suitable.) Let $G_T$ be the subgraph of $G$ induced on vertices $V(T) \cup \{u, v, w\}$.

In the following Lemma tristar $T$, vertices $u, v, w$, subgraph $G_T$ and homomorphism $\overline{F}$ have the same meaning as above.

**Lemma 7** *For any subgraph $G_T = T \cup \{u, v, w\}$ of a hexagonal graph $G$ there exists a homomorphism $F : G_T \to C_5$, such that $F|_{\{u,v,w\}} = \overline{F}$.*

**Proof.** Suppose first that two paths of $T$ have the same length. By symmetry, we may suppose that $l = m$. Then $v$ and $w$ have the same base color, thus $\overline{F}(v) = \overline{F}(w)$. We have three possible situations:

- if $k + l \geq 2$, then the length of a path $P = (u, u_k, ..., u_1, x, v_1, ...v_l, v)$ is at least four in $G_T$. So, setting $F(w_i) = F(v_i)$ and using Lemma 5, there exists a homomorphism $F : G_T \to C_5$, such that $F|_{\{u,v,w\}} = \overline{F}$.
- if $k + l = 1$, then the length of a path $P = (u, ..., x, ..., v)$ equals three in $G_T$. In this case $\overline{F}(u) \neq \overline{F}(v)$ and $d_{C_5}(\overline{F}(u), \overline{F}(v)) \geq 1$, Thus, due to the same argument as above, there exists a homomorphism $F : G_T \to C_5$, such that $F|_{\{u,v,w\}} = \overline{F}$.
- if $k + l = 0$, then $T = \{x\}$ and $\overline{F}(u) = \overline{F}(v) = \overline{F}(w)$. By $F(x) = \overline{F}(u) + 1 (\mathrm{mod}\ 5)$ we may extend $\overline{F}$ to homomorphism $F : G_T \to C_5$, such that $F|_{\{u,v,w\}} = \overline{F}$.

Suppose now that all three paths of $T$ have distinct lengths. We will assume that $k > l > m$. The other cases follow by symmetry.

If $k = 2$, then $l = 1$ and $m = 0$. In this case vertices $u, v$ and $w$ have different base colors. We have three different possibilities:
- if $u$ is red, then $v$ is blue and $w$ is green, thus $\overline{F}(u) = 1$, $\overline{F}(v) = 2$ and $\overline{F}(w) = 3$. We may extend $\overline{F}$ to homomorphism $F : G_T \to C_5$, such that $F|_{\{u,v,w\}} = \overline{F}$, as follows: $F(u_2) = F(x) = 2$, $F(u_1) = 1$ and $F(v_1) = 3$.
- if $u$ is blue, then $v$ is green and $w$ is red, thus $\overline{F}(u) = 2$, $\overline{F}(v) = 3$ and $\overline{F}(w) = 1$. We may extend $\overline{F}$ to homomorphism $F : G_T \to C_5$, such that $F|_{\{u,v,w\}} = \overline{F}$, as follows: $F(u_2) = 3$, $F(u_1) = F(v_1) = 4$ and $F(x) = 0$.
- if $u$ is green, then $v$ is red and $w$ is blue, thus $\overline{F}(u) = 3$, $\overline{F}(v) = 1$ and $\overline{F}(w) = 2$. We may extend $\overline{F}$ to homomorphism $F : G_T \to C_5$, such that $F|_{\{u,v,w\}} = \overline{F}$, as follows: $F(u_2) = 4$, $F(u_1) = 0$, $F(x) = 1$ and $F(v_1) = 2$.

Let now be $k \geq 3$. It holds $d_{G_T}(v, w) \geq 3$ in this case. (Note that $\overline{F}(v) \neq \overline{F}(v)$ in the case $d_{G_T}(v, w) = 3$). Thus, by Lemma 5, there exists a homomorphism $F : \{v, ..., x, ..., w\} \to C_5$, such that $F|_{\{v,w\}} = \overline{F}$. Because $d_{G_T}(u, x) \geq 4$, we can extend $F$ to the whole graph $G_T$ by Lemma 5, irrespective of that what has the vertex $x$ been mapped to. ■

Hence Step 2 of the algorithm can be implemented correctly and we have:

**Proposition 8** *Let $G$ be a triangle-free hexagonal graph. Then there exits a homomorphism $\varphi : G \to C_5$.*

This is equivalent to the statement of Theorem 1. Let us note that by arguments sketched in introduction, Theorem 1 implies:

**Corollary 9 ([4, 5])** *Every triangle-free hexagonal graph $G$ is* 5-[2] *colorable.*

# 4 Final remarks

After finding the algorithm that finds a homomorphism of an arbitrary hexagonal graph to cycle $C_5$, a natural question "Is there a homomorphism from a triangle-free hexagonal graph to any odd cycle?" arises. The answer is negative as the example on Fig. 2 shows. If there was a homomorphism $\varphi : G \to C_9$, the only possibility for vertices $u$ and $v$, by Corollary 4, would be $\varphi(u) = \varphi(v)$, but in this case the gray vertices can not be properly mapped (Lemma 2).

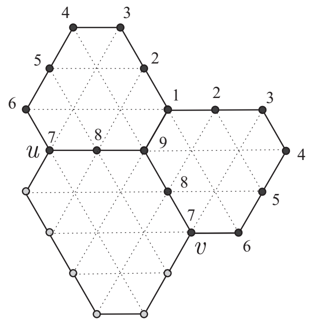


Figure 2: An example of a hexagonal graph, which is not $C_9$-colorable.

Knowing that triangle-free hexagonal graphs are $C_5$-colorable and are not $C_9$-colorable, it is interesting to ask if triangle-free hexagonal graphs are $C_7$-colorable.

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