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Abstract

Let Γ denote a distance-regular graph with diameter $d \geq 3$. Assume Γ has classical parameters (d, b, α, β) with $b < -1$.

We investigate the extent to which Γ is 1-homogeneous in the sense of Nomura. We show that either Γ is 1-homogeneous, or else Γ has a certain equitable partition of its vertex set which involves $4d - 1$ cells.

Key words: distance-regular graph, Q-polynomial, classical parameters, kite-free

1 Introduction

Let Γ denote a distance-regular graph with diameter $d \geq 3$. Then Γ is said to have *classical parameters* (d, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \quad (1 \leq i \leq d), \tag{1}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \quad (0 \leq i \leq d-1), \tag{2}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}. \tag{3}$$

In this case b is an integer and $b \notin \{0, -1\}$. We say that Γ has *negative type* whenever Γ has classical parameters (d, b, α, β) such that $b < -1$.

For the rest of this Introduction assume Γ has negative type. We are interested in the extent to which Γ is 1-homogeneous in the sense of Nomura [4]. If the intersection number a_1 of Γ is 0, then Γ is 1-homogeneous by [3]. For the rest of this Introduction assume $a_1 \neq 0$.

Let $V\Gamma$ denote the vertex set of Γ and fix adjacent vertices $x, y \in V\Gamma$. We define

$$D^i_j = D^i_j(x,y) = \{z \in V\Gamma | \partial(x,z) = i \text{ and } \partial(y,z) = j\} \quad (0 \leq i, j \leq d).$$

For each i $(1 \leq i \leq d)$ and for $z \in D^i_i$ define $\sigma(z) := |\{w \in D^1_1 | \partial(z,w) = i-1\}|$. It turns out that Γ is kite-free in the sense of Terwilliger [6]; using this we find $\sigma(z) \in \{0, 1\}$. For $j \in \{0, 1\}$ we define $D^i_i(j) = \{z \in D^i_i | \sigma(z) = j\}$.

Assume for the moment that Γ is a near polygon [4]. In this case $D^i_i(0) = \emptyset$ $(1 \leq i \leq d)$. Morover Γ is 1-homogeneous by Nomura [4].

Next assume Γ is not a near polygon. In this case we will show

(i) The sets D^{i-1}_i, D^i_{i-1}, $D^i_i(1)$ $(1 \leq i \leq d)$ and the sets $D^i_i(0)$ $(2 \leq i \leq d)$ are all nonempty;

(ii) The partition of $V\Gamma$ into the sets $D^i_{i-1}, D^{i-1}_i, D^i_i(1)$ $(1 \leq i \leq d)$ and $D^i_i(0)$ $(2 \leq i \leq d)$ is equitable;

(iii) The corresponding parameters are independent of the choice of x, y.

We will prove (ii) using Terwilliger's "balanced set" characterization of the Q-polynomial property [5]. We will prove (iii) by displaying explicit formulae for the corresponding parameters in terms of the intersection numbers of Γ. The results (i)–(iii) are the main results of the paper.

Our paper is organized as follows. In Sections 2–4, we set up the necessary tools for the proof of our main results. More precisely, in Section 2, we review basic definitions and concepts about distance-regular graphs. In Section 3 we discuss the 1-homogeneous property and the Q-polynomial property. In Section 4 we discuss kites and parallelograms. We prove our main results in Sections 5 and 6.

2 Preliminaries

In this section, we review some definitions and basic concepts. See the book of Brouwer et al. [1] for more background information.

Throughout this paper, Γ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $V\Gamma$, edge set $E\Gamma$, path-length distance function ∂, and diameter $d := \max\{\partial(x,y) | x, y \in V\Gamma\}$. For $x \in V\Gamma$ and for an integer i define $\Gamma_i(x)$ to be the set of vertices of Γ at distance i from x. We abbreviate $\Gamma(x) := \Gamma_1(x)$. The graph Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le d)$, and all $x, y \in V\Gamma$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)| \tag{4}$$

is independent of x, y. The constants p^h_{ij} $(0 \le h, i, j \le d)$ are known as the *intersection numbers* of Γ. For notational convenience define $c_i := p^i_{1,i-1}$ $(1 \le i \le d)$, $a_i := p^i_{1i}$ $(0 \le i \le d)$, $b_i := p^i_{1,i+1}$ $(0 \le i \le d-1)$, $k_i := p^0_{ii}$ $(0 \le i \le d)$, and $c_0 = b_d = 0$. We observe $a_0 = 0$ and $c_1 = 1$. Moreover,

$$c_i + a_i + b_i = k \quad (0 \le i \le d), \tag{5}$$

where $k := k_1$. From now on we assume Γ is distance-regular with diameter $d \ge 3$.

In the following two lemmas, we cite some well known facts about the intersection numbers; see for example Brouwer et al. [1, p. 127, 134].

Lemma 2.1 *Let Γ denote a distance-regular graph with diameter $d \ge 3$. Then for all integers h, i, j $(0 \le h, i, j \le d)$ the following (i), (ii) hold.*

(i) If one of h, i, j is greater then the sum of the other two, then $p^h_{ij} = 0$.

(ii) If one of h, i, j is equal to the sum of the other two, then $p^h_{ij} \neq 0$. ∎

Lemma 2.2 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Then the following (i)–(iii) hold.*

(i) $k_i = \dfrac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$ $(0 \leq i \leq d)$,

(ii) $p^1_{i,i-1} = \dfrac{c_i k_i}{k}$ $(1 \leq i \leq d)$,

(iii) $p^1_{ii} = \dfrac{a_i k_i}{k}$ $(0 \leq i \leq d)$. ∎

Lemma 2.3 *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), $d \geq 3$. Then the following (i), (ii) hold.*

(i) If $b < -1$ then $\alpha \neq 0$.

(ii) For each integer i $(1 \leq i \leq d)$,

$$a_i - a_1 c_i = -\alpha(1 + b + a_1) \begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix}.$$

Proof. (i) If $b < -1$ and $\alpha = 0$ then $c_2 = 1 + b < 0$, a contradiction.

(ii) Straighforward from (1), (2) and (3). ∎

Let Γ denote a distance-regular graph with diameter $d \geq 3$. We recall the Bose-Mesner algebra of Γ. For each i $(0 \leq i \leq d)$ let A_i denote the matrix with rows and columns indexed by $V\Gamma$, and x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in V\Gamma). \tag{6}$$

We call A_i the *ith distance matrix* of Γ. We observe

$$A_0 = I, \tag{7}$$

$$A_0 + A_1 + \cdots + A_d = J \ (J = \text{ all 1's matrix}), \tag{8}$$

$$A_i^t = A_i \quad (0 \le i \le d), \tag{9}$$

and

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \quad (0 \le i, j \le d). \tag{10}$$

By (7), (9) and (10), the matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative semi-simple $\mathbb{R}$-algebra M, known as the *Bose-Mesner algebra.* By Godsil [2, Theorem 12.2.1], the algebra M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |V\Gamma|^{-1} J, \tag{11}$$

$$E_i E_j = \delta_{ij} E_i \ (0 \le i, j \le d), \tag{12}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{13}$$

$$E_i^t = E_i \ (0 \le i \le d). \tag{14}$$

The $E_0, E_1, \ldots, E_d$ are known as the *primitive idempotents* of Γ, and E_0 is the *trivial* idempotent.

Set $A := A_1$, and define the real numbers $\theta_i \ (0 \le i \le d)$ by

$$A = \sum_{i=0}^{d} \theta_i E_i. \tag{15}$$

Then $AE_i = E_i A = \theta_i E_i \ (0 \le i \le d)$, and $\theta_0 = k$. The scalars $\theta_0, \theta_1, \ldots, \theta_d$ are distinct, since A generates M [1, p. 128]. The $\theta_0, \theta_1, \ldots, \theta_d$ are known as the *eigenvalues* of Γ.

For notational convenience, we identify $V\Gamma$ with the standard orthonormal basis in the Euclidean space $V, \langle \, , \, \rangle$, where $V = \mathbb{R}^{|V\Gamma|}$ (column vectors), and where $\langle \, , \, \rangle$ is the dot product

$$\langle u, v \rangle = u^t v \ (u, v \in V).$$

Observe M acts on V by left multiplication. The Euclidean space $V, \langle \, , \, \rangle$ is known as the *standard module* of Γ.

In the following lemma, we cite some well known results about primitive idempotents.

Lemma 2.4 (Terwilliger [5, Lemma 1.1]) *Let Γ denote a distance-regular graph with diameter $d \ge 3$. Pick any $\theta, \theta_0^*, \theta_1^*, \ldots, \theta_d^* \in \mathbb{R}$, and set*

$$E := |V\Gamma|^{-1} \sum_{i=0}^{d} \theta_i^* A_i. \tag{16}$$

Then the following (i)–(iii) are equivalent:

(i) θ is an eigenvalue of Γ, and E is the associated primitive idempotent.

(ii) For all $x, y \in V\Gamma$,

$$\langle Ex, Ey \rangle = |V\Gamma|^{-1}\theta_i^* \quad \textit{whenever } \partial(x,y) = i,$$

and

$$\sum_{\substack{z \in V\Gamma \\ \partial(x,z)=1}} Ez = \theta Ex.$$

(iii) The intersection numbers of Γ satisfy

$$c_i\theta_{i-1}^* + a_i\theta_i^* + b_i\theta_{i+1}^* = \theta\theta_i^* \quad (0 \le i \le d),$$

and $\theta_0^* =$ rank E. ∎

If (i)–(iii) hold, we call the sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ the *dual eigenvalue sequence* associated with θ, E. The sequence is *trivial* whenever $E = E_0$ (in which case $\theta_0^* = \theta_1^* = \cdots = \theta_d^* = 1$).

3 The 1-homogeneous property and the Q-polynomial property

We begin this section with a definition.

Definition 3.1 Let Γ denote a distance-regular graph with diameter $d \ge 3$ and let x, y denote adjacent vertices in $V\Gamma$. For all integers i and j we define $D_j^i = D_j^i(x,y)$ by

$$D_j^i = \Gamma_i(x) \cap \Gamma_j(y).$$

We observe $D_j^i = \emptyset$ unless $0 \le i, j \le d$. Moreover $|D_j^i| = p_{ij}^1$ $(0 \le i, j \le d)$.

Lemma 3.2 *Let Γ denote a distance-regular graph with diameter $d \ge 3$ and let x, y denote adjacent vertices in $V\Gamma$. Then with the reference to Definition 3.1 the following (i), (ii) hold.*

(i) For all i, j $(0 \le i, j \le d)$, if $|i-j| > 1$ then $D^i_j = \emptyset$. If $|i-j| = 1$ then $D^i_j \ne \emptyset$.

(ii) For each i $(0 \le i \le d)$ we have $D^i_i = \emptyset$ if and only if $a_i = 0$.

Proof. Immediate from Lemma 2.1 and Lemma 2.2. ∎

An *equitable partition* of a graph is a partition $\pi = \{C_1, C_2, \ldots, C_s\}$ of its vertex set into nonempty cells, so that for all i, j $(1 \le i, j \le s)$ the number c_{ij} of neighbours, which a vertex in the cell C_i has in the cell C_j, is independent of the choice of the vertex in C_i. We call the c_{ij} the *corresponding parameters.*

Let Γ denote a distance-regular graph with diameter $d \ge 3$. Then Γ is said to be *1-homogeneous*, whenever for all pairs x, y of adjacent vertices, the partition of $V\Gamma$ given by $\{D^i_j(x,y) \,|\, 0 \le i, j \le d,\ D^i_j(x,y) \ne \emptyset\}$ is equitable, and moreover the corresponding parameters are independent of the choice of x, y.

Let Γ denote a distance-regular graph with diameter $d \ge 3$. The *Krein parameters* q^h_{ij} $(0 \le h, i, j \le d)$ of Γ are defined by

$$E_i \circ E_j = |V\Gamma|^{-1} \sum_{h=0}^{d} q^h_{ij} E_h \quad (0 \le i, j \le d), \tag{17}$$

where $\circ$ denotes entrywise multiplication. We say Γ is *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents), whenever for all distinct integers i, j $(0 \le i, j \le d)$,

$$q^1_{ij} \ne 0 \text{ if and only if } |i-j| = 1.$$

Let E denote a nontrivial primitive idempotent of Γ. We say Γ is *Q-polynomial with respect to E* whenever an ordering $E_0, E_1 = E, \ldots, E_d$ of the primitive idempotents of Γ exists, with respect to which Γ is Q-polynomial.

We have the following lemma about Q-polynomial distance-regular graphs.

Lemma 3.3 (Brouwer et al. [1, Thm. 8.1.1]) *Let Γ denote a distance-regular graph with diameter $d \ge 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ denote the corresponding dual eigenvalue sequence. Suppose Γ is Q-polynomial with respect to E. Then $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are mutually distinct.* ∎

The following result of Terwilliger will play a crucial role in our investigation.

Lemma 3.4 (Terwilliger [5, Thm. 3.3]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Then the following (i), (ii) are equivalent:*

(i) Γ *is Q-polynomial with respect to* E.

(ii) $\theta_0^* \neq \theta_i^*$ $(1 \leq i \leq d)$, *and for all integers* h, i, j $(1 \leq h \leq d)$, $(0 \leq i, j \leq d)$ *and for all vertices* $x, y \in V\Gamma$ *with* $\partial(x, y) = h$ *the following hold:*

$$\sum_{\substack{z \in V\Gamma \\ \partial(x,z)=i \\ \partial(y,z)=j}} Ez - \sum_{\substack{z \in V\Gamma \\ \partial(x,z)=j \\ \partial(y,z)=i}} Ez \in span\{Ex - Ey\}.$$

Suppose (i), (ii) hold. Then for all integers h, i, j $(1 \leq h \leq d)$, $(0 \leq i, j \leq d)$ *and for all vertices* $x, y \in V\Gamma$ *with* $\partial(x, y) = h$,

$$\sum_{\substack{z \in V\Gamma \\ \partial(x,z)=i \\ \partial(y,z)=j}} Ez - \sum_{\substack{z \in V\Gamma \\ \partial(x,z)=j \\ \partial(y,z)=i}} Ez = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*}(Ex - Ey). \tag{18}$$

∎

Lemma 3.5 (Brouwer et al. [1, Cor. 8.4.2]) *Let Γ denote distance-regular graph with classical parameters (d, b, α, β), $d \geq 3$. Then the following (i)–(iii) hold.*

(i) The scalar $\theta = \left[{d-1 \atop 1}\right](\beta - \alpha) - 1$ *is an eigenvalue of* Γ.

(ii) Γ *is Q-polynomial with respect to* θ.

(iii) Let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ *denote the dual eigenvalue sequence corresponding to* θ. *Then there exist real numbers* γ, δ *such that*

$$\theta_i^* = \gamma \begin{bmatrix} d-i \\ 1 \end{bmatrix} + \delta \quad (0 \leq i \leq d).$$

∎

4 Kites and parallelograms

Let Γ denote a distance-regular graph with diameter $d \geq 3$. We recall the notion of a kite in Γ.

Pick an integer i $(2 \leq i \leq d)$. By a *kite of length* i (or i*-kite*) in Γ we mean a 4-tuple $xyzu$ of vertices of Γ, such that x, y and z are mutually adjacent, and $\partial(u,x) = i$, $\partial(u,y) = \partial(u,z) = i-1$. We say Γ is *kite-free* whenever Γ has no kites of any length. We have the following result about kite-free distance-regular graphs.

Theorem 4.1 (Terwilliger, [6, Thm. 2.12], Weng, [9, Thm. 2.6]) *Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 \neq 0$. Then the following (i) and (ii) are equivalent.*

(i) Γ has classical parameters (d, b, α, β), and either $b < -1$, or Γ is a dual polar graph or a Hamming graph.

(ii) Γ has no kites of any length. ∎

Let Γ denote a distance-regular graph with diameter $d \geq 3$. Pick an integer i $(2 \leq i \leq d)$. By a *parallelogram of length* i (or i*-parallelogram*) in Γ, we mean a 4-tuple $xyzu$ of vertices in $V\Gamma$, such that $\partial(x,y) = \partial(z,u) = 1$, $\partial(x,u) = i$, and $\partial(x,z) = \partial(y,z) = \partial(y,u) = i-1$.

Theorem 4.2 *Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 \neq 0$. Then the following (i) and (ii) are equivalent.*

(i) Γ has no kites of any length.

(ii) Γ has no parallelogram of any length.

Proof. $(i) \rightarrow (ii)$ By Theorem 4.1, Γ has classical parameters (d, b, α, β). Pick an integer i $(2 \leq i \leq d)$. Let x, y, u denote vertices of Γ, such that $\partial(x,y) = 1$, $\partial(x,u) = i$ and $\partial(y,u) = i-1$. Define

$$f_i := |\{z \mid z \in V\Gamma, xyzu \text{ is an } i\text{-parallelogram }\}|$$

and

$$e_i := |\{z \mid z \in V\Gamma, xyzu \text{ is an } i\text{-kite }\}|.$$

By Weng [8, Lemma 7.3], we have

$$f_i = b^{i-2} e_i.$$

Since $e_i = 0$ by the assumption, we obtain $f_i = 0$ as well.

(ii) $\to$ *(i)* This follows from Weng [8, Lemma 6.12]. ∎

Let Γ denote a kite-free distance-regular graph. Pick an integer i $(1 \leq i \leq d)$. With reference to Definition 3.1, for $z \in D_i^i$ define $\sigma(z) := |\Gamma_{i-1}(z) \cap D_1^1|$. Observe in this case that $\sigma(z) \in \{0, 1\}$; otherwise Γ has an i-kite or a 2-kite. This allows us to make the following definition.

Definition 4.3 Let Γ denote a kite-free distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 \neq 0$. Pick an integer i $(1 \leq i \leq d)$. Then with reference to Definition 3.1, for $j \in \{0, 1\}$ we define $D_i^i(j) = \{z \in D_i^i \mid \sigma(z) = j\}$. We observe $D_i^i = D_i^i(1) \cup D_i^i(0)$. We further observe $D_1^1(1) = D_1^1$ and $D_1^1(0) = \emptyset$.

Lemma 4.4 *Let Γ denote a kite-free distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 \neq 0$. Then with reference to Definition 3.1 and Definition 4.3 the following (i) and (ii) hold.*

(i) $\partial(u, z) = 1$ *for all distinct* $u, z \in D_1^1$.

(ii) *For an integer i $(2 \leq i \leq d)$ we have $\partial(u, z) = i$ for all $u \in D_1^1$ and all $z \in D_i^i(0)$.*

Proof. (i) If u and z are nonadjacent, then $zxyu$ is a 2-kite, a contradiction.

(ii) Observe that $\partial(u, z) \in \{i, i+1\}$. If $\partial(u, z) = i + 1$, then the 4-tuple $uxyz$ is an $(i+1)$-kite, a contradiction. ∎

5 The main result I

We will be discussing the following situation.

Definition 5.1 Let Γ denote a Q-polynomial kite-free distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 \neq 0$, which is not a near polygon. By Theorem 4.1, Γ has classical parameters (d, b, α, β) with $b < -1$. We fix adjacent vertices x, y of Γ and let $D_j^i, D_i^i(0), D_i^i(1)$ denote the corresponding sets as in Definition 3.1 and Definition 4.3.

Our goal for the next two sections is to establish points (i)–(iii) from the Introduction. We begin with a lemma.

Lemma 5.2 *With reference to Definition 5.1 the following (i)–(iii) hold.*

(i) There are no edges between $D_i^{i-1} \cup D_{i-1}^i$ *and* $D_{i-1}^{i-1}(0) \cup D_{i-1}^{i-1}(1)$ $(2 \le i \le d)$.

(ii) There are no edges between $D_i^i(1)$ *and* $D_{i+1}^{i+1}(0)$ $(1 \le i \le d-1)$.

(iii) There are no edges between $D_i^i(1)$ *and* $D_{i-1}^{i-1}(0)$ $(3 \le i \le d)$.

PROOF. (i) Let $v \in D_i^{i-1}$ (resp. $v \in D_{i-1}^i$) and $w \in D_{i-1}^{i-1}(0) \cup D_{i-1}^{i-1}(1)$. Suppose that v and w are adjacent. Then the 4-tuple $yxwv$ (resp. $xywv$) is an i-parallelogram, contradicting Theorem 4.2.

(ii) There are no edges between $D_i^i(1)$ and $D_{i+1}^{i+1}(0)$ by the definition of the set $D_{i+1}^{i+1}(0)$.

(iii) Assume $v \in D_i^i(1)$ is adjacent to $w \in D_{i-1}^{i-1}(0)$. Let z be the unique vertex in D_1^1, such that $\partial(z,v) = i-1$. By Lemma 4.4(ii), $\partial(z,w) = i-1$. But now the 4-tuple $xzwv$ is an i-parallelogram, contradicting Theorem 4.2. ∎

Lemma 5.3 *Pick an integer* i $(2 \le i \le d)$. *Then with reference to Definition 5.1 the following (i)–(iii) hold.*

(i) Each $v \in D_{i-1}^i$ *(resp.* D_i^{i-1}*) is adjacent to*

(a) precisely	0	*vertices in* $D_{i-1}^{i-1}(0), D_{i-1}^{i-1}(1)$,
(b) precisely	c_{i-1}	*vertices in* D_{i-2}^{i-1} *(resp.* D_{i-1}^{i-2}*),*
(c) precisely	$c_i - c_{i-1}$	*vertices in* D_i^{i-1} *(resp.* D_{i-1}^i*),*
(d) precisely	a_{i-1}	*vertices in* D_{i-1}^i *(resp.* D_i^{i-1}*),*
(e) precisely	b_i	*vertices in* D_i^{i+1} *(resp.* D_{i+1}^i*),*
(f) precisely	$a_i - a_{i-1} - \lvert\Gamma(v) \cap D_i^i(1)\rvert$	*vertices in* $D_i^i(0)$.

(ii) Each $v \in D_i^i(0)$ *is adjacent to*

(a) precisely	0	*vertices in* $D_{i-1}^{i-1}(1), D_{i+1}^{i+1}(1), D_{i+1}^i, D_i^{i+1}$,
(b) precisely	$c_i - \lvert\Gamma(v) \cap D_{i-1}^{i-1}(0)\rvert$	*vertices in* D_{i-1}^i,
(c) precisely	$c_i - \lvert\Gamma(v) \cap D_{i-1}^{i-1}(0)\rvert$	*vertices in* D_i^{i-1},
(d) precisely	b_i	*vertices in* $D_{i+1}^{i+1}(0)$,
(e) precisely	$a_i + \lvert\Gamma(v) \cap D_{i-1}^{i-1}(0)\rvert - c_i - \lvert\Gamma(v) \cap D_i^i(1)\rvert$	*vertices in* $D_i^i(0)$.

(iii) Each $v \in D^i_i(1)$ *is adjacent to*

(a) precisely	0	*vertices in* $D^{i-1}_{i-1}(0), D^{i+1}_{i+1}(0), D^i_{i+1}, D^{i+1}_i$,
(b) precisely	$c_i - \lvert\Gamma(v) \cap D^{i-1}_{i-1}(1)\rvert$	*vertices in* D^i_{i-1},
(c) precisely	$c_i - \lvert\Gamma(v) \cap D^{i-1}_{i-1}(1)\rvert$	*vertices in* D^{i-1}_i,
(d) precisely	b_i	*vertices in* $D^{i+1}_{i+1}(1)$,
(e) precisely	$a_i + \lvert\Gamma(v) \cap D^{i-1}_{i-1}(1)\rvert -$ $c_i - \lvert\Gamma(v) \cap D^i_i(1)\rvert$	*vertices in* $D^i_i(0)$.

Proof. Routine using Lemma 5.2 and the fact that $D^i_{i-1} \cup D^i_i(0) \cup D^i_i(1) \cup D^i_{i+1} = \Gamma_i(x)$ and $D^{i-1}_i \cup D^i_i(0) \cup D^i_i(1) \cup D^{i+1}_i = \Gamma_i(y)$. ∎

With reference to Definition 5.1 and Lemma 5.3 we visualize D^i_{i-1}, D^{i-1}_i, $D^i_i(0)$, $D^i_i(1)$ in Figure 1.

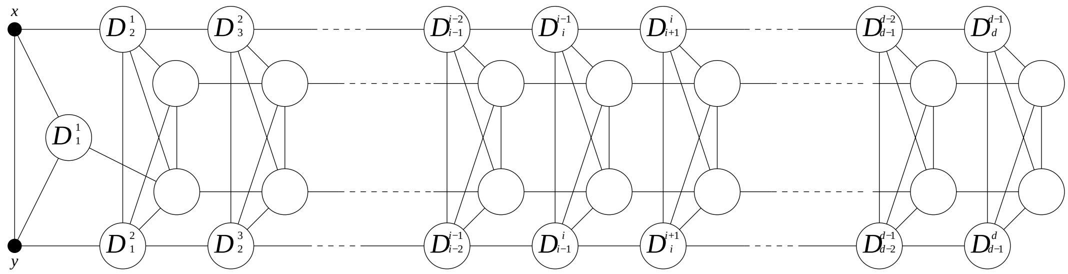

Figure 1: The partition corresponding to a pair of adjacent vertices x and y. The circles in the middle of the figure represent the sets $D^i_i(0)$ $(2 \le i \le d)$ (upper line) and the sets $D^i_i(1)$ $(2 \le i \le d)$ (lower line). Observe that $D^i_{i-1} \cup D^i_i(0) \cup D^i_i(1) \cup D^i_{i+1} = \Gamma_i(x)$ and $D^{i-1}_i \cup D^i_i(0) \cup D^i_i(1) \cup D^{i+1}_i = \Gamma_i(y)$.

6 The main result II

In this section we continue to establish points (i)–(iii) from the Introduction.

Lemma 6.1 *With reference to Definition 5.1 the following holds. For each integer* i $(2 \le i \le d)$ *and for all* $v \in D^{i-1}_i \cup D^i_{i-1}$,

$$|\Gamma(v) \cap D^i_i(1)| = a_1(c_i - c_{i-1}).$$

Proof. Without loss we may assume $v \in D^{i-1}_i$. Then v is at distance i from every vertex in D^1_1. Hence there are exactly $a_1 c_i c_{i-1} \cdots c_1$ paths of length i from

v to D_1^1. Since v has c_{i-1} neighbours in D_{i-1}^{i-2}, exactly $c_{i-1}a_1c_{i-1}c_{i-2}\cdots c_1$ of these paths pass through D_{i-1}^{i-2}. The remaining $a_1(c_i - c_{i-1})c_{i-1}c_{i-2}\cdots c_1$ paths must pass through $D_i^i(1)$. Let $w \in D_i^i(1)$. Since there is a unique vertex $z \in D_1^1$, such that $\partial(z, w) = i-1$, there are exactly $c_{i-1}c_{i-2}\cdots c_1$ paths of length $i-1$ between w and D_1^1. Hence v has exactly $a_1(c_i - c_{i-1})$ neighbours in $D_i^i(1)$. ∎

Lemma 6.2 *With reference to Definition 5.1 the following (i),(ii) hold.*

(i) For each integer i $(2 \leq i \leq d)$ and for all $v \in D_i^i(0)$,

$$|\Gamma(v) \cap D_{i-1}^{i-1}(0)| = c_i \frac{b^{i-2}-1}{b^i - 1}.$$

(ii) For each integer i $(2 \leq i \leq d)$ and for all $v \in D_i^i(0)$,

$$|\Gamma(v) \cap D_i^i(1)| = a_1 c_i \frac{b^i - b^{i-2}}{b^i - 1}.$$

Proof. (i) We abbreviate $\tau = |\Gamma(v) \cap D_{i-1}^{i-1}(0)|$ and $\eta = |\Gamma(v) \cap D_i^{i-1}|$. We observe $\tau + \eta = c_i$. Recall by Definition 5.1 that Γ has classical parameters (d, b, α, β). Let the eigenvalue θ and the dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ be as in Lemma 3.5. Let E denote the corresponding primitive idempotent of Γ. By Lemma 3.4 and since Γ is Q-polynomial with respect to E we find

$$\sum_{\substack{\partial(x,z)=i-1\\ \partial(v,z)=1}} Ez - \sum_{\substack{\partial(x,z)=1\\ \partial(v,z)=i-1}} Ez = c_i \frac{\theta_{i-1}^* - \theta_1^*}{\theta_0^* - \theta_i^*}(Ex - Ev). \tag{19}$$

Observe that $\{z \in V\Gamma | \partial(x, z) = 1,\ \partial(v, z) = i-1\} \subseteq D_2^1$. Taking the inner product of (19) with Ey using Lemma 2.4(ii), we get (after multiplying by $|V\Gamma|$)

$$\tau\theta_{i-1}^* + \eta\theta_i^* - c_i\theta_2^* = c_i \frac{\theta_{i-1}^* - \theta_1^*}{\theta_0^* - \theta_i^*}(\theta_1^* - \theta_i^*).$$

Evaluating the above line using $\eta = c_i - \tau$ we obtain

$$\tau = c_i \frac{(\theta_{i-1}^* - \theta_1^*)(\theta_1^* - \theta_i^*) + (\theta_0^* - \theta_i^*)(\theta_2^* - \theta_i^*)}{(\theta_0^* - \theta_i^*)(\theta_{i-1}^* - \theta_i^*)}.$$

Simplifying the above line using Lemma 3.5(iii) we get

$$\tau = c_i \frac{b^{i-2} - 1}{b^i - 1}.$$

(ii) By Lemma 4.4(ii) the vertex v is at distance i from every vertex in D_1^1. Hence there are exactly $a_1 c_i c_{i-1} \cdots c_1$ paths of length i from v to D_1^1. By Lemma 6.2(i), exactly $a_1 \frac{b^{i-2}-1}{b^i-1} c_i c_{i-1} \cdots c_1$ of these paths pass through $D_{i-1}^{i-1}(0)$. The remaining $a_1 \frac{b^i - b^{i-2}}{b^i-1} c_i \cdots c_1$ paths must pass through $D_i^i(1)$. Let $w \in D_i^i(1)$. Since there is a unique vertex $z \in D_1^1$, such that $\partial(z, w) = i-1$, there are exactly $c_{i-1} \cdots c_1$ paths of length $i-1$ between w and D_1^1. Hence v has exactly $a_1 c_i \frac{b^i - b^{i-2}}{b^i-1}$ neighbours in $D_i^i(1)$. ∎

Lemma 6.3 *With reference to Definition 5.1 the following (i), (ii) hold.*

(i) For each integer i $(2 \leq i \leq d)$ and for all $v \in D_i^i(1)$,

$$|\Gamma(v) \cap D_{i-1}^{i-1}(1)| = c_{i-1}.$$

(ii) For each integer i $(1 \leq i \leq d)$ and for all $v \in D_i^i(1)$,

$$|\Gamma(v) \cap D_i^i(1)| = (a_1 - 1)(c_i - c_{i-1}) + a_{i-1}.$$

Proof. (i) Let z be the unique vertex in D_1^1 such that $\partial(z, v) = i-1$. Then there are exactly $c_{i-1} \cdots c_1$ paths of length $i-1$ joining z and v, and all of these paths pass through $D_{i-1}^{i-1}(1)$. Since every neighbour of v in $D_{i-1}^{i-1}(1)$ is joined to z via $c_{i-2} \cdots c_1$ paths of length $i-2$, we find v has exactly c_{i-1} neighbours in $D_{i-1}^{i-1}(1)$.

(ii) First assume $i = 1$. Then the result holds by Lemma 4.4(i). Next assume $i \geq 2$. Let z denote the unique vertex in D_1^1 such that $\partial(z, v) = i-1$. If z' is another vertex from D_1^1, then $\partial(z', v) = i$. Therefore there are exactly $(a_1 - 1) c_i c_{i-1} \cdots c_1 + (a_{i-1} + \cdots + a_1) c_{i-1} \cdots c_1$ paths of length i from v to D_1^1. Since v has c_{i-1} neighbours in $D_{i-1}^{i-1}(1)$, there are exactly $c_{i-1}((a_1-1)c_{i-1} \cdots c_1 + (a_{i-2} + \cdots + a_1) c_{i-2} \cdots c_1)$ of these paths, for which v is the only vertex in $D_i^i(1)$. Recall there are exactly $c_{i-1} \cdots c_1$ paths of length $i-1$ from each vertex in $D_i^i(1)$ to D_1^1. From these comments we find v has exactly $(a_1 - 1)(c_i - c_{i-1}) + a_{i-1}$ neighbours in $D_i^i(1)$. ∎

Lemma 6.4 *With reference to Definition 5.1 the following (i), (ii) hold.*

(i) For each integer i $(1 \leq i \leq d)$,

$$|D_i^i(1)| = a_1 \frac{b_1 \cdots b_{i-1}}{c_1 \cdots c_{i-1}}.$$

(ii) For each integer i $(1 \le i \le d)$,

$$|D_i^i(0)| = \frac{b_1 \cdots b_{i-1}}{c_1 \cdots c_i}(a_i - a_1 c_i).$$

PROOF. (i) Assume $i \ge 2$; otherwise the result is clear. By Lemma 5.3 every vertex in $D_{i-1}^{i-1}(1)$ has b_{i-1} neighbours in $D_i^i(1)$. Moreover, by Lemma 6.3(i) every vertex in $D_i^i(1)$ is adjacent to exactly c_{i-1} vertices in $D_{i-1}^{i-1}(1)$. Counting the edges between $D_{i-1}^{i-1}(1)$ and $D_i^i(1)$ in two different ways, we obtain

$$|D_{i-1}^{i-1}(1)|\, b_{i-1} = |D_i^i(1)|\, c_{i-1}.$$

Evaluating this equation using induction we obtain $|D_i^i(1)| = |D_1^1(1)| \dfrac{b_1 \cdots b_{i-1}}{c_1 \cdots c_{i-1}}$. Recall $D_1^1(1) = D_1^1$ and $|D_1^1| = a_1$, so $|D_i^i(1)| = a_1 \dfrac{b_1 \cdots b_{i-1}}{c_1 \cdots c_{i-1}}$.

(ii) Observe $|D_i^i(0)| = |D_i^i| - |D_i^i(1)|$ and that $|D_i^i| = p_{ii}^1$. To finish the proof, evaluate p_{ii}^1 using Lemma 2.2 and evaluate $|D_i^i(1)|$ using part (i) above. ∎

Lemma 6.5 *With reference to Definition 5.1 the following (i), (ii) hold.*

(i) $D_i^i(1) \ne \emptyset$ $(1 \le i \le d)$.

(ii) $D_i^i(0) \ne \emptyset$ $(2 \le i \le d)$.

PROOF. (i) This follows from Lemma 6.4(i).

(ii) Since Γ is not a near polygon, we obtain $b \ne -a_1 - 1$ from [1, Theorem 6.2.1, Theorem 6.4.1]. The assertion now follows from Lemma 6.4(ii) and Lemma 2.3. ∎

Theorem 6.6 *Let Γ denote a Q-polynomial kite-free distance-regular graph with diameter $d \ge 3$. Assume the intersection number $a_1 \ne 0$, and that Γ is not a near polygon. Let x, y denote adjacent vertices of Γ. Then with reference to Definition 3.1 and Definition 4.3, the partition of $V\Gamma$ into the sets D_i^{i-1}, D_{i-1}^i, $D_i^i(1)$ $(1 \le i \le d)$ and $D_i^i(0)$ $(2 \le i \le d)$ is equitable. Moreover, the corresponding parameters are independent of the choice of x, y.*

PROOF. Immediate from Lemma 5.3, Lemma 6.1, Lemma 6.2, Lemma 6.3 and Lemma 6.5. ∎

We end the paper with some comments on Theorem 6.6.

Corollary 6.7 *Let Γ denote a Q-polynomial kite-free distance-regular graph with diameter $d \geq 3$. Assume the intersection number $a_1 \neq 0$. Then the following (i),(ii) are equivalent.*

(i) Γ *is 1-homogeneous,*

(ii) Γ *is a near polygon.*

PROOF. If Γ is a near polygon, then it is 1-homogeneous by [4, Theorem 1]. On the other hand, if Γ is not a near polygon, then by Lemma 6.5 we have $D_2^2(0) \neq \emptyset$ and $D_2^2(1) \neq \emptyset$; using this we find Γ is not 1-homogeneous. ∎

Let Γ be as in Theorem 6.6. By Theorem 4.1 Γ has classical parameters (d, b, α, β) with $b < -1$. Now by a result of Weng [7, Thm. 10.3] at least one of the following (i)–(iv) holds:

(i) $d = 3$,

(ii) $c_2 = 1$,

(iii) Γ is the Hermitean form graph $Her_{-b}(d)$,

(iv) $\alpha = (b-1)/2$, $\beta = -(1+b^d)/2$, and $-b$ is a power of an odd prime.

See [1, Section 9.5] for the definition of the Hermitean form graph.

We suspect that Theorem 6.6 implies new feasibility conditions which will enable us to classify cases (i), (ii) and (iv) above. We will pursue this in a future paper.

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