University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1111 Ljubljana, Slovenia

Preprint series, Vol. 41 (2003), 883

TRIANGULATIONS OF SEIFERT FIBRED MANIFOLDS

Aleksandar Mijatović

ISSN 1318-4865

June 19, 2003

Ljubljana, June 19, 2003

TRIANGULATIONS OF SEIFERT FIBRED MANIFOLDS

ALEKSANDAR MIJATOVIĆ

Abstract. It is not completely unreasonable to expect that a computable function bounding the number of Pachner moves needed to change any triangulation of a given 3-manifold into any other triangulation of the same 3-manifold exists. In this paper we describe a procedure yielding an explicit formula for such a function if the 3-manifold in question is a Seifert fibred space.

1 INTRODUCTION

There is a natural way of modifying a triangulation T of an n-manifold. Suppose D is a combinatorial n-disc which is a subcomplex both in this triangulation and in the boundary of the $(n+1)$-simplex Δ^{n+1}. We can change T by removing D and inserting $\Delta^{n+1} - \mathrm{int}(D)$. What we've just described is called a *Pachner move*. In dimension 3 there are four possible moves (see figure 1). Note that we can define Pachner moves even if the triangulation T is non-combinatorial (i.e. simplices of T are not uniquely determined by their vertices).

Since our aim is to deal with the triangulations of the manifolds that are not necessarily closed, we need to allow for some additional moves that will modify the simplicial structure on the boundary (throughout this paper we will be using the term *simplicial structure* as a synonym for a possibly non-combinatorial triangulation). The definition of a Pachner move readily generalises to this setting. Changing the triangulation of the boundary by an $(n-1)$-dimensional Pachner move amounts to gluing onto (or removing form) our manifold an n-simplex Δ^n, which must exist by the definition of the move. So in dimension 3 we have to use the three two-dimensional moves (usually referred to as $(2-2)$, $(1-3)$ and $(3-1)$) that can be implemented by gluing on or shelling a tetrahedron.

It was proved by Pachner (see [9]) that any two triangulations of the same PL n-manifold are related by a finite sequence of Pachner moves and simplicial isomorphisms. It is well known (see proposition 1.3 in [8]) that a computable function bounding the length of the sequence from Pachner's theorem in terms of the number of tetrahedra for a fixed 3-manifold M, gives an algorithm for recognising M among all 3-manifolds. The following theorem gives an explicit formula for such a bound in case M is a Seifert fibred space with a fixed triangulation on its boundary.

Theorem 1.1 *Let $M \to B$ be a Seifert fibred space with non-empty boundary. Let P and Q be two triangulations of M that coincide on ∂M and contain p and q tetrahedra respectively. Then there exists a sequence of Pachner moves of length at most $e^6(10p) + e^6(10q)$ which transforms P into a triangulation isomorphic to Q. The homeomorphism of M that realizes the simplicial isomorphism is, when restricted to ∂M, equal to the identity on the boundary of M.*

The exponent in the above expression containing the exponential function $e(x) = 2^x$ stands for the composition of the function with itself rather than for multiplication. The shameful enormity of the bound can be curbed by a more careful choice of subdivisions. The height of the tower of exponents can be reduced from 6 to 3, but the complexity of the constructions involved grows tenfold. Since we are mainly interested in the existence of an explicit formula, we shall not strive to get the best numbers possible.

The bound in theorem 1.1 is clearly computable and it hence gives a conceptually simple recognition algorithm for every bounded Seifert fibred space. It is true that the topology of

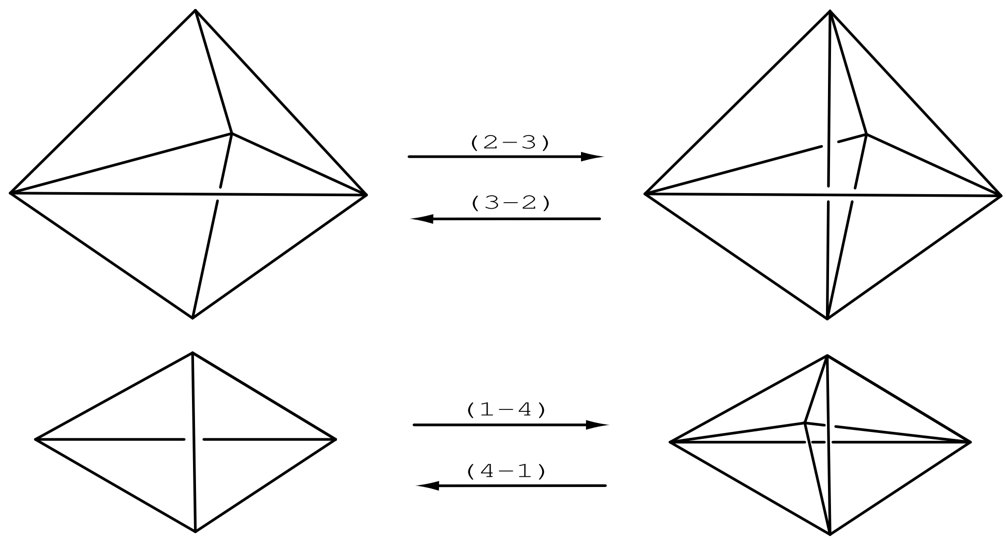

Figure 1: Three dimensional Pachner moves.

the Seifert fibred manifolds is not terribly exciting. They are however ubiquitous pieces in JSJ-decompositions of more interesting 3-manifolds. It is there that our theorem will find its best application.

2 SOME NORMAL SURFACE THEORY AND SOME TOPOLOGY

The pivotal tool for probing the triangulations of our 3-manifolds is normal surface theory. Many good accounts of it appeared in the literature (see for example [2], [6] or [1]). In this section we will state the basic definitions that will allow us to give some known properties of normal surface. We will then go on to explore how normal surfaces interact with boundary patterns. The section will be concluded with a discussion of some well known features of incompressible surfaces in Seifert fibred manifolds.

We are assuming throughout that all 3-manifolds we are dealing with are orientable. Let T be a triangulation of a 3-manifold M. An arc in a 2-simplex of T is *normal* if its ends lie in different sides of a 2-simplex. A simple closed curve in the 2-skeleton of T is a *normal curve* if it intersects each 2-simplex of T in normal arcs. A properly embedded surface F in M is in *normal form* with respect to T if it intersects each tetrahedron in T in a collection of discs all of whose boundaries are normal curves consisting of 3 or 4 normal arcs, i.e. triangles and quadrilaterals. A *normal discs* is a triangle or a quadrilateral. There are precisely seven normal disc types in any tetrahedron of T. An isotopy of M is called a *normal isotopy* with respect to T if it leaves all simplices of T invariant. In particular this implies that it is fixed on the vertices of T.

A normal surface is determined, up to normal isotopy, by the number of normal disc types in which it meets the tetrahedra of T. It therefore defines a vector with $7t$ coordinates. Each coordinate represents the number of copies of a normal disc type that are contained in the surface with t being the number of tetrahedra in T. It turns out that there is a certain restricted linear system that such a vector is a solution of. Moreover there is a one to one correspondence between the solutions of that restricted linear system and normal surface in M. If the sum of two vector solutions of this system satisfies the restrictions on the system, then it represents a normal surface in M. On the other hand there is a geometric process called *regular alteration* (see figure 2 in [1]) which can be carried out on the normal surfaces representing the summands and which yields the normal surface corresponding to the sum. It follows directly from the definition of regular alteration that the Euler characteristic is additive over normal addition. We can define the *weight* $w(F)$ of a surface F, which is transverse to the 1-skeleton of T, to be the number of points of intersection between the surface and the 1-skeleton. Since regular alteration only changes the surfaces involved away from the 1-skeleton, the weight too is additive over normal addition.

A normal surface is called *fundamental* if the vector corresponding to it is not a sum of two integral solutions of the linear system. The solution space of the linear system projects down to a compact convex linear cell which is called the *projective solution space*. A *vertex surface* is a connected two-sided normal surface that projects onto a vertex of the projective solution space (see [6] for a more detailed description).

The next proposition is proved in [2]. It will be important for us that its proof does not depend on the number of equations in the linear system arising from the triangulation of the manifold.

Proposition 2.1 *Let M be a compact triangulated 3-manifold containing t tetrahedra. Then each normal coordinate of a vertex surface in M is bounded above by 2^{7t}. If the normal surface is fundamental, then $7t2^{7t}$ puts an upper bound on all of its normal coordinates.*

Recall that a properly embedded surface F in a 3-manifold M is *injective* if the homomorphism $\pi_1(F) \to \pi_1(M)$, induced by the inclusion of F into M, is a monomorphism. A surface F is said to be *incompressible* if it satisfies the following conditions:

- The surface F does not contain 2-spheres that bound 3-balls nor does it contain discs which are isotopic rel boundary to discs in ∂M and
- for every disc D in M with $D \cap S = \partial D$ there is a disc D' in S with $\partial D = \partial D'$.

A *horizontal boundary* of an I-bundle over a surface is a part of the boundary corresponding to the ∂I-bundle. The *vertical boundary* is a complement of the horizontal boundary and consists of annuli that fibre over the bounding circles of the base surface. It is a well-known fact that a properly embedded one-sided surface in M is injective if and only if the horizontal boundary of its regular neighbourhood is incompressible. An embedded torus in M that is incompressible and is not boundary parallel is sometimes referred to as an *essential torus*.

There is also a relative notion of incompressibility which we will need to consider. A surface F is *∂-incompressible* if for each disc D in M such that ∂D splits into two arcs α and β meeting only at their common endpoints, with $D \cap F = \alpha$ and $D \cap \partial M = \beta$ there is a disc D' in F with $\alpha \subset \partial D'$ and $\partial D' - \alpha \subset \partial F$. Such a disc D is called a *∂-compression disc* for F. Note that if the manifold M is irreducible and has incompressible boundary, then we can isotope F rel α so that the disc D' becomes the ∂-compression disc D. A properly embedded annulus in M that is both incompressible and ∂-incompressible is called an *essential annulus*.

Before we can state the main technical results from normal surface theory, we need to define a very useful concept.

Definition. A *boundary pattern* P in a compact 3-manifold M is a (possibly empty) collection of disjoint simple closed curves and trivalent graphs embedded in ∂M such that the surface $\partial M - P$ is incompressible in M.

Boundary patterns usually appear in the context of hierarchies. In this paper however we will mainly be concerned with a special case when the pattern P consists of simple closed curves only. Therefore we will not investigate the correspondence between hierarchies and patterns any further. Notice also that it follows from the definition of the pattern that any simple closed curve component of P is homotopically non-trivial in ∂M.

Let M be a 3-manifold with non-empty boundary that contains a boundary pattern P. It follows from the definition that the pattern P can be empty if and only if the manifold M has incompressible boundary. Assume now that the pattern P is not empty. A subset of M is *pure* if it has empty intersection with the pattern P. Most concepts from general 3-manifolds carry over to 3-manifolds with pattern in a very natural way. For example, a properly embedded surface F in 3-manifold M with pattern P is *P-boundary incompressible* if for any pure disc D in M, such that $D \cap (\partial M \cup F) = \partial D$ and $D \cap F$ is a single arc in F, the arc $D \cap F$ cuts off a pure disc from F.

Notice that for $P = \emptyset$ this notion reduces to ∂-incompressibility. Also this notion is well defined only up to an isotopy of M that is invariant on the pattern P. In other words we can have two isotopic surfaces in M, out of which only one is P-boundary incompressible.

Let T be some triangulation of a 3-manifold M with pattern P. Throughout this paper we will be assuming that the pattern P lies in the 1-skeleton of T. This assumption immediately implies that any incompressible P-boundary incompressible surface in M can be isotoped into normal form. Also any normal surface F in M has a well defined intersection number $\iota(F)$, equal to the number of points in $\partial F \cap P$. Moreover this intersection number is additive over geometric sums of normal surfaces.

We say that a surface F in a 3-manifold M with pattern P has *minimal weight* if it can not be isotoped to a surface with lower weight by an isotopy that is invariant on the pattern. In case of $P = \emptyset$ this reduces to the usual definition of a minimal weight.

An incompressible P-boundary incompressible surface in M of minimal weight has to intersect each triangle in the 2-skeleton of T in normal arcs and possibly some simple closed curves. If M is irreducible we can remove these simple closed curves by isotopies in the usual way. The isotoped surface is then in normal form. The sum $F = F_1 + F_2$ is in *reduced form* if the number of components of $F_1 \cap F_2$ is minimal among all normal surfaces F_1' and F_2' isotopic rel P to F_1 and F_2 respectively such that $F = F_1' + F_2'$. If the pattern P is empty we get the familiar notion of reduced form which was used in [1]. We are now going to define a concept which will be of some significance to all that follows.

Definition. A *patch* for the normal sum $F = F_1 + F_2$ is a component of $F_1 - \text{int}(\mathcal{N}(F_1 \cap F_2))$ or $F_2 - \text{int}(\mathcal{N}(F_1 \cap F_2))$. A *trivial patch* is a pure patch which is topologically a disc and whose boundary intersects only one component of the 1-manifold $F_1 \cap F_2$.

This means that a boundary of a trivial patch is either a single simple closed curve in $F_1 \cap F_2$ or it consists of two arcs: one in $(\partial M) - P$ and the other in $F_1 \cap F_2$. The first possibility coincides with what was called a disc-patch in [1]. In the absence of pattern the notion of the trivial patch coincides with what is called a disc-patch in [6]. The reason for our slightly non-standard terminology is to avoid the confusion arising from the patches which are discs but are not disc-patches. There are several ways in which a disc patch (i.e. a patch which is a disc) can fail to be trivial, if ∂M is not empty. Clearly a disc patch which intersects the pattern P is non-trivial. A pure disc patch will also be non-trivial if it intersects ∂M in more than one arc.

It is not hard to prove that a trivial patch can not have zero weight. The argument of lemma 3.3 in [1] gives it to us for all trivial patches bounded by simple closed curve components of $F_1 \cap F_2$. In case when our disc patch is bounded by two arcs, we can use a simple doubling trick and then apply lemma 3.3 from [1] to obtain the desired conclusion. However there are patches that are topologically discs and have zero weight. But they must contain more than one component of $F_1 \cap F_2$ in their boundaries. Now we are finally in the position to state the following lemma.

Lemma 2.2 *Let M be an irreducible 3-manifold with a (possibly empty) boundary pattern P. Let F be a minimal weight incompressible P-boundary incompressible normal surface. If the sum $F = F_1 + F_2$ is in reduced form then each patch is both incompressible and P-boundary incompressible and no patch is trivial. Furthermore if F is injective, then each patch has to be injective.*

This lemma is a mild generalisation of both lemma 3.6 in [1] and lemma 6.6 in [6]. Even though patches are not properly embedded surfaces in M the notions of P-boundary incompressibility and injectivity can be naturally extended to this setting. Note also that if $P = \emptyset$, the manifold M has incompressible boundary, the surface F is boundary incompressible and so are the patches of $F = F_1 + F_2$.

Proof. We start by reducing the lemma to the statement that no patch of $F_1 + F_2$ is trivial. In case we have a patch which is either compressible or not injective, we can argue in precisely the same way as in the proof of lemma 3.6 of [1] to obtain a disc patch whose boundary is a single simple closed curve from $F_1 \cap F_2$ and is therefore trivial.

If there is a patch R of $F_1 + F_2$ which has a pure boundary compressing disc D, then we can assume without loss of generality that D is also a boundary compressing disc for the surface F.

Then the unique arc $D \cap R = D \cap F$ cuts off a pure disc D' in F. If D' contains a simple closed curve of $F_1 \cap F_2$, then we can find a compressible patch of $F_1 + F_2$ and we are in the previous case. If there are no simple closed curves of $F_1 \cap F_2$ in D', then the edge most arc from $F_1 \cap F_2$ in D' cuts off a pure disc patch which is clearly trivial. So it is enough to prove that no trivial patch exists.

If there exists a disc patch in $F_1 + F_2$ bounded by a single simple closed curve from $F_1 \cap F_2$, we can use an identical argument to the one in the proof of lemma 3.6 in [1] to construct two normal surfaces F' and S such that $F = F' + S$. F' is isotopic to F and S is a closed normal surface with $\chi(S) = w(S) = 0$. This is clearly a contradiction because no normal surface can miss the 1-skeleton. Also by our hypothesis the surface F might be only incompressible and not necessarily injective. So in order to use the argument from [1] which shows that every simple closed curve in $F_1 \cap F_2$, bounding a trivial patch, has to be two-sided in both F_1 and F_2, we need to note that when M equals $\mathbb{R}P^3$, every embedded projective plane in M is actually injective.

So now we can assume that $F_1 + F_2$ contains no disc patch disjoint from ∂M. Let D be a trivial patch, lying in F_1 say, which has least weight among all trivial patches in $F_1 + F_2$. The intersection $D \cap F_2$ consists of a unique arc α that is a component of $F_1 \cap F_2$. After regular alteration α produces two properly embedded arcs in F one of which cuts off a pure disc D' from F. This is because F is P-boundary incompressible. Disc D' is distinct from but might contain the trivial patch D.

We must have $w(D') = w(D)$. Otherwise we could isotope F, by an isotopy invariant on the pattern, so that its weight is decreased. Since D minimises the weight of all trivial patches (which is strictly positive) and D' must contain at least one such, there is exactly one trivial patch in D' and its weight is equal to $w(D)$. Every other patch in D' is topologically a disc whose boundary consists of 4 arcs. Two of them are in ∂M and the other two are components of $F_1 \cap F_2$.

If D' was itself a patch then we could isotope F_1 and F_2, using the definition of the pattern P and the irreducibility of M, to obtain normal surfaces F_1' and F_2' still summing up to F but having fewer components of intersection. This contradicts our assumption on the reduced form of $F_1 + F_2$.

So D' is not itself a trivial patch, but it has to contain one. Now we can imitate the argument in the proof of lemma 3.6 from [1] to obtain a normal sum $F = A + F'$ where F' is a normal surface isotopic to F and A is a pure normal annulus of zero weight. Since no normal surface can live in the complement of the 1-skeleton, this gives a final contradiction. □

Another crucial fact from normal surface theory, tying up normal addition with the topological properties of surfaces involved, is contained in the next theorem. It appeared several times in the literature in slightly different forms. The version that is of interest to us is the following.

Theorem 2.3 *Let M be an irreducible 3-manifold with a possibly empty boundary pattern P. Let F be a least weight normal surface properly embedded in M. Assume also that F is two-sided incompressible P-boundary incompressible and $F = F_1 + F_2$. Then F_1 and F_2 are incompressible and P-boundary incompressible.*

The proof of this theorem can be obtained by using lemma 2.2 and following (verbatim) the proof of theorem 6.5 in [6]. Before we state a further consequence of theorem 2.3, we need the following simple fact from topology.

Lemma 2.4 *Let M be an irreducible 3-manifold with incompressible boundary. Assume also that M is neither homeomorphic to the product $S^1 \times S^1 \times I$, nor to an I-bundle over a Klein bottle. Let S be a toral boundary component of M. If A and B are two properly embedded incompressible ∂-incompressible annuli in M such that at least one boundary component of both annuli lies in S, then these bounding simple closed curves must be isotopic in S, i.e. they determine the same slope in S.*

Proof. We start by isotoping the annuli A and B so that their intersection is minimal. If $A \cap B$ is either empty or it consists only of essential simple closed curves, then the boundary curves must be parallel in S. So we can assume that $\partial A \cap \partial B$ is non-empty. This implies that the boundary slopes of ∂A and ∂B in S are distinct. Therefore the complement $S - (\partial A \cup \partial B)$ is a disjoint union of disc.

Since both A and B are incompressible and ∂-incompressible, the intersection contains neither contractible simple closed curves nor boundary parallel arcs in either of the two annuli. In other words $A \cap B$ consists of spanning arcs in both annuli. So an I-bundle structure extends from $A \cup B$ to the the regular neighbourhood $\mathcal{N}(A \cup B)$.

If the bounding circles of A lie in distinct components of ∂M, then the manifold M has to be homeomorphic to $S^1 \times S^1 \times I$. This is because each annulus V in the vertical boundary of the I-bundle $\mathcal{N}(A \cup B)$ cuts off from M a 3-ball of the form $D \times I$ where D is one of the discs in $S - (\partial A \cup \partial B)$. This 3-ball can not contain $A \cup B$ since both annuli are incompressible. We can therefore extend the product structure over this 3-ball, thus obtaining an I-bundle over the torus S.

If both components of ∂A live in S, then there are two possibilities for the compressible annulus V. The dichotomy comes from the discs D_1 and D_2 in the surface S, bounded by the circles of ∂V. They can either be nested, say $D_1 \subset \text{int}(D_2)$, or disjoint. It is clear that the annulus A is disjoint from $\partial V = \partial D_1 \cup \partial D_2$. The horizontal boundary of the regular neighbourhood $\mathcal{N}(A \cup B)$ contains an embedded arc, running from ∂D_1 to ∂A, which is disjoint from ∂D_2. If the first of the two cases were true, this would imply that at least one of the boundary components of A is contained in D_2, which is clearly a contradiction. So we must have an embedded 2-sphere $D_1 \cup D_2 \cup V$ which bounds a 3-ball $D_1 \times I$, disjoint from $A \cup B$, like before. Adjoining all these solid cylinders to $\mathcal{N}(A \cup B)$ makes our manifold M into an I-bundle with a single toral boundary component. But this has to be an I-bundle over a closed non-orientable surface of Euler characteristic zero, i.e. a Klein bottle. This concludes the proof. □

A very useful consequence of theorem 2.3 that deals with orientable surfaces in M with zero Euler characteristic is contained in proposition 2.5. Most of it is a direct consequence of corollary 6.8 in [6].

Proposition 2.5 *Let M be an irreducible 3-manifold with incompressible boundary that contains either an essential torus or an essential annulus. Let T be the triangulation of M. Then the following holds:*

(a) If A is a least weight normal representative (with respect to T) in an isotopy class of an essential torus in M, then every vertex surface in the face of the projective solution space that carries A is an essential torus.

(b) Let A be a least weight essential annulus in M that is normal with respect to the triangulation T. Then there exists a vertex surface in M that is an essential annulus. If the boundary of M consists of tori only and if M is neither $S^1 \times S^1 \times I$ nor an I-bundle over a Klein bottle, then the boundary of the vertex annulus is parallel to the boundary of A in ∂M.

Proof. Since vertex surfaces are two-sided by definition, (a) is just restating corollary 6.8 in [6]. In (b) at least one of the vertex surfaces in the face of projective solution space has to be an annulus. In fact, by theorem 2.3, it has to be an essential annulus.

Assume now that M has toral boundary. All vertex annuli in the face of the projective solution space that carries A must have their respective boundaries lying in precisely the components of ∂M that contain ∂A. Since all these vertex annuli are essential we can apply lemma 2.4 and finish the proof. □

We will conclude this section by a short discussion of incompressible surfaces in Seifert fibred manifolds. A surface in a Seifert fibred space $M \to B$ is called *vertical* if it can be expressed as a union of regular fibres. So the only possibilities are a torus, a Klein bottle or an annulus. Moebius band does not come into the picture because the generator of its fundamental group can not be a regular fibre in M. On the other hand the surface that is transverse to all fibres in M is called *horizontal*. The following proposition says roughly that every essential surface in $M \to B$ has to be either vertical or horizontal.

Proposition 2.6 *Let S be an incompressible ∂-incompressible surface in a 3-manifold M. Then the following holds:*

(a) Assume further that $M \to B$ is an irreducible Seifert fibred space, possibly containing no singular fibres, over a (perhaps non-orientable) compact surface B. If S is two-sided and if it contains no singular fibres of M, then it is isotopic to either a vertical surface or a horizontal surface.

(b) Let $M \to B$ be an S^1-bundle over a (perhaps non-orientable) compact bounded surface B. If S is a one-sided, i.e. non-orientable, surface in M, then it is isotopic to either a vertical or a horizontal surface.

It should be noted that (b) of proposition 2.6 actually fails if the base surface B is closed. It is not hard to see that the lens space $L(2n, 1)$, which fibres as an S^1-bundle over S^2 with Euler number $2n$, contains an incompressible surface homeomorphic to a connected sum of n projective planes. Such a surface can not be vertical because of its Euler characteristic, but it can also not be horizontal since the Euler number of our S^1-bundle does not vanish. Also notice that the statement (b) for two-sided surfaces is already contained in (a).

Proof. If the surface S contained a disc component, then, by the definition of incompressibility, ∂M would have to compress. Since M is irreducible, this would make it into a solid torus. The surface S is then a disjoint collection of compression discs which is horizontal in any fibration of the solid torus. So from now on we can assume that S contains no disc components.

Part (a) of the proposition is a well-known fact about two-sided incompressible ∂-incompressible surfaces (which contain no disc components) in Seifert fibred spaces. Its proof can be found in [5] (theorem VI.34). We shall now prove part (b).

Let $M \to B$ be an S^1-bundle over a compact bounded surface B. Choose disjoint arcs in B whose union decomposes B into a single disc. Let A be a union of disjoint vertical annuli in M that are the preimages of the collection of arcs under the bundle projection.

Without loss of generality we can assume that the surface S was isotoped in M so that the number of components of $S \cap A$ is minimal. Then there are no simple closed curves in $S \cap A$ that are homotopically trivial in either S or A, because such curves can be used to reduce the number of components in $S \cap A$ (here we are using the fact that every bounded S^1-bundle is irreducible). Similarly arcs in $S \cap A$ that are ∂-parallel in either A or S do not occur because both surfaces are ∂-incompressible and ∂M is incompressible in M (otherwise M would be a solid torus and S would have to be a disc, which is not a one-sided surface).

It follows now that $S \cap A$ consists only of vertical circles or horizontal arcs, i.e. spanning arcs in the annular components of A. Let M_1 be the solid torus $M - \text{int}(\mathcal{N}(A))$ and let S_1 be the surface $M_1 \cap S$. There can be no trivial simple closed curves in $(\partial M) \cap M_1$ coming from ∂S_1, because S has no disc components. So we can conclude that ∂S_1 consists either of horizontal or vertical circles in the torus ∂M_1 (a horizontal simple closed curve is the one that intersects each fibre in ∂M_1 transversely).

The surface S_1 has to be incompressible in M_1. Every compression disc D for S_1 in M_1 yields a disc D' in S. Using irreducibility of M we could isotope S (rel ∂D) so that D' becomes D. If D' were not contained in S_1, this move would reduce the number of pieces in $S \cap A$.

Claim. If ∂S_1 consists of horizontal simple closed curves, then S_1 is a disjoint union of meridional discs in M_1.

It is enough to show that under the hypothesis of the claim the surface S_1 has to be ∂-incompressible in M_1. This is because the only connected incompressible ∂-incompressible surface in a solid torus is its meridian disc.

Assume to the contrary that S_1 is ∂-compressible. Let D be a ∂-compression disc for S_1 in M_1. We will modify D, in a thin collar of ∂M_1, so that the arc $D \cap \partial M_1$ lies in an annulus from $\partial M \cap \partial M_1$. This isotoped disc therefore lives in M and is a ∂-compression disc for the surface S. Like before this leads to a contradiction, because we can use the isotoped disc to construct an isotopy in M that will reduce the number of components in $S \cap A$.

So to prove the claim we need to isotope the disc D. Let α be the embedded arc $D \cap \partial M_1$, running between two points from ∂S_1. Notice that ∂M_1 is an alternating union of annuli coming from two families: one is $M_1 \cap \partial M$ and the other one is $M_1 \cap \mathcal{N}(A)$. We will now isotope α into the interior of $M_1 \cap \partial M$. Assume first that α is contained in the interior of an annulus from $M_1 \cap \mathcal{N}(A)$. Then, since ∂S_1 is horizontal in ∂M_1, the segments of ∂S_1 in this annulus have to be spanning arcs. α can either run between two distinct spanning arcs, or it can run around the annulus to hit the single spanning arc from two different sides. In either situation we can push α into the interior of an adjacent annulus from $M_1 \cap \partial M$. This isotopy can clearly be extended to the collar of ∂M_1, thus producing the desired ∂-compression disc. If, on the other hand, α is not contained in the interior of an annulus from ∂M_1, then we must somewhere have the situation as described by figure 2.

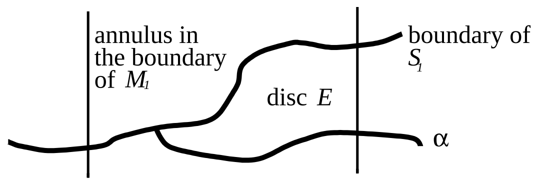

Figure 2: An annulus in ∂M_1 containing the disc E with the following property: the intersection $E \cap \alpha$ is a subarc of α containing precisely one point from $\partial\alpha$.

We can construct an isotopy of α, and hence of D, using the disc E, that will reduce the number of points in the intersection $\alpha \cap \partial(M_1 \cap \partial M)$. By repeating this move we arrive at a contradiction and the claim follows.

It follows directly from the claim that S_1 with the horizontal boundary can be isotoped (rel ∂S_1) so that it is horizontal in the fibration of M_1. This isotopy induces an isotopy of S in M that makes it horizontal.

If ∂S_1 is vertical, S_1 can only consist of annuli bounded by fibres of M_1. We can therefore isotope S_1 (rel ∂S_1) so that it becomes vertical. This concludes the proof. □

3 THE MAIN THEOREM

Let's start by defining precisely the class of Seifert fibred 3-manifolds that we shall consider. The manifold M has to be compact and orientable. It has to fibre into circles over a possibly non-orientable compact surface B. The case when M contains no singular fibres will also be considered.

Since we are only interested in general Haken 3-manifolds, i.e. the ones that are irreducible and that contain an injective surface different from a 2-sphere, we need to exclude some Seifert fibred spaces. The ones to go first are $S^1 \times S^2$ and the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$ since they are not irreducible (the latter manifold fibres as an S^1-bundle over a projective plane). Another obvious family that we have to disqualify are the lens spaces including the 3-sphere, because they contain no injective surfaces. Those are all Seifert fibred spaces over a 2-sphere with at most 2 singular fibres.

Among the spaces that contain no vertical essential tori are also manifolds that fibre over a projective plane with at most one singular fibre. Since the Klein bottle over the orientation reversing curve in $\mathbb{R}P^2$ is not injective, they contain no injective vertical surfaces. There are no horizontal surfaces either. The existence of such a surface would imply that the only singular fibre is of index $\frac{0}{1}$. In other words we would be dealing with the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$ that we've excluded already. So we need to eliminate all Seifert fibred manifolds with a base surface $\mathbb{R}P^2$ that have at most one exceptional fibre because they are not Haken.

The last exceptional class we need to consider are the small Seifert fibred spaces. They fibre over S^2 with three exceptional fibres. It is known that they contain no separating incompressible surfaces and are therefore Haken if and only if $H_1(M;\mathbb{Z})$ is infinite (see for example [5], theorem VI.15). If this condition is satisfied then every incompressible surface in our manifold induces a surface bundle structure over a circle. There are only three small Seifert fibred spaces that contain horizontal tori and are therefore homeomorphic to torus bundles over a circle. We shall exclude however all small Seifert fibred spaces from further consideration, because they contain no essential vertical tori.

So the 3-manifolds we are going to consider are precisely all Seifert fibred spaces, with a possibly empty collection of singular fibres, that either contain an essential vertical torus or have non-empty boundary. We can now state the main theorem of this paper.

Theorem 3.1 *Let $M \to B$ be a compact and orientable Seifert fibred 3-manifold that either contains an essential vertical torus or has non-empty boundary. The base space B might be non-orientable and the manifold M may be an S^1-bundle containing no singular fibres. Let Y be a possibly empty collection of boundary components of M. Let P and Q be two triangulations of M that contain p and q tetrahedra respectively. Assume also that these two triangulations induce isotopic simplicial structures in Y. Then there exists a sequence of Pachner moves of length at most $e^6(10p) + e^6(10q)$ which transforms P into a triangulation which is isomorphic to Q. The homeomorphism of M that realizes this simplicial isomorphism maps fibres onto fibres, does not permute the components of ∂M and, when restricted to Y, it is isotopic to the identity on Y.*

The exponent in the expression containing the exponential function $e(x) = 2^x$ stands for the composition of the function with itself and not for multiplication.

There are several Seifert fibred manifolds which satisfy the hypothesis of the theorem but have more symmetry than the "generic" case (see section 7). What we mean by symmetry is that such manifolds contain both vertical and horizontal essential annuli or both vertical and horizontal essential tori. The existence of such surfaces makes it possible to construct homeomorphisms that don't preserve the fibres. Most of these manifolds are S^1-bundles with no exceptional fibres. In this case the Euler characteristic of the base surface has to vanish because every horizontal surface covers it. If the base space B has boundary, then our manifold is either homeomorphic to $S^1 \times S^1 \times I$ or to the unique orientable S^1-bundle over a Moebius band. If, on the other hand, the base surface has no boundary, the manifold M has to be an S^1-bundle over a torus or over a Klein bottle. Even though these are infinite classes of manifolds, there is a unique manifold in each of them that contains a horizontal torus. This is because the Euler number of the these bundles is an integer which has to vanish if the corresponding Seifert fibred space is to contain a horizontal surface (see proposition 4.2 in [3]). The two manifolds in question are $S^1 \times S^1 \times S^1$ and the S^1-bundle over a Klein bottle that is obtained by identifying two copies of the unique orientable S^1-bundle over a Moebius band via an identity on the boundary. The triangulations of these manifolds require a slightly different technique and will be dealt with in section 7.

In the general case the proof of theorem 3.1 is divided into two parts. The first part is contained in section 5. In it we alter the triangulation of M so that it interacts well with the singular fibres of the fibration. Theorem 5.3 and proposition 5.4 subdivide the original triangulation of M so that a family of vertical solid tori, containing all singular fibres, are represented by a simplicial subcomplex of the subdivision. In the second part of the proof we deal with the triangulation of the S^1-bundle that is obtained by removing the nicely triangulated solid tori. This is described in section 6.

4 PACHNER MOVES AND NORMAL SURFACES

In this section we are going to subdivide, using Pachner moves, a triangulation T of a 3-manifold M that contains a normal surface F of bounded complexity. In other words we are going to alter

T so that its subdivision contains the surface F in its 2-skeleton. This procedure will be described by lemma 4.1. It will be applied several times throughout the construction of the "canonical" triangulation of our Seifert fibred space.

Lemma 4.1 *Let M be a 3-manifold with a triangulation T consisting of t tetrahedra. Assume further that F is a properly embedded normal surface in M with respect to T which contains n normal pieces. Than we can obtain a subdivision T_1 of T, using less than $200nt$ Pachner moves, with the following properties: T_1 contains the surface F in its 2-skeleton and it consists of not more than $20(n+t)$ tetrahedra.*

Proof. Let's start by describing the subdivision T_1. Notice that the complement $M - (T^2 \cup F)$ consists of 3-balls, where T^2 denotes the 2-skeleton of T. The 2-skeleton of T_1 will contain the polyhedron $T^2 \cup F$. Its faces are all discs of length at most 6, i.e. their boundaries consist of at most 6 arcs. The simplicial structure on this polyhedron is obtained by coning each face from one of the vertices in its boundary. Now we can define T_1 to be a union of cones on the boundaries of these 3-balls. By inspecting all possibilities we see that the number of triangles in the boundary of any of the above 3-balls is bounded by 20. Since there are no more than $(n + t)$ 3-balls, the triangulation T_1 has the desired properties.

The construction of T_1 using Pachner moves will be reminiscent of a similar process described in section 5 of [8]. The main difference here is that the manifold M can have a non-empty boundary. If ∂F is non-trivial, we will also have to change the simplicial structure on ∂M. The whole procedure is divided into six stages:

1. Make a 3-dimensional $(1-3)$ move on each triangle in ∂M and then make a 3-dimensional $(2-2)$ move for each 1-simplex of T in ∂M.
2. Add a vertex into each tetrahedron and each triangle of the triangulation T, that we need to subdivide, and then cone.
3. Subdivide the 1-skeleton of T so that it becomes a subcomplex of T_1, and keep the triangulation in the 3-simplices of T coned.
4. Subdivide the 2-skeleton of T to get a subcomplex of T_1, and keep the triangulation in the 3-simplices of T coned.
5. Chop up tetrahedra of T by the appropriate normal pieces coming from F, and triangulate the complementary regions by coning them from points in their interiors.
6. Shell all tetrahedra that are not contained in any of the 3-simplices of T.

In the first step we add some tetrahedra to the existing triangulation of M. We stop just short of constructing the whole collar on ∂M, but we do go far enough so that we can later change the original simplicial structure of ∂M just by using $(2-3)$ and $(1-4)$ moves. In step six we get rid of all redundant 3-simplices that we have created in the beginning.

The first step takes $10t$ Pachner moves, since there are at most $4t$ faces in ∂M and less than $6t$ edges. Adding a vertex into 3-simplices in T takes not more than t Pachner moves. Adding one into a triangle takes two Pachner moves. So the second step requires not more than $9t$ moves.

There are at most $4n$ vertices of T_1 that are contained in the interiors of 1-simplices of T. A single vertex can be created on an edge in T by at most $2t$ Pachner moves (see step 2 in section 5 of [8]). So step 3 can be accomplished by $8tn$ moves.

There are at most $4n$ normal arcs in all 2-simplices in T. Each of the complementary discs in each 2-simplex can be triangulated by at most 4 triangles. So the 2-skeleton of T will be subdivided by less than $16n$ triangles. By lemma 4.2 from [8] this configuration can be obtained by $16n$ two-dimensional Pachner moves. Suspending this process gives an upper bound of $32n$ Pachner moves used in step 4.

In order to glue in the normal discs of F into the relevant tetrahedra of T we will need to use the procedure called the changing of cones, which was described by lemma 5.1 from [8]. We will therefore need to know how many triangles there are in the disc we are gluing in and also how many of them there are in the discs we are changing. Since we are always gluing in normal pieces, the first number is bounded above by 2. The second one is bounded by 10, which can be seen by inspection. Lemma 5.1 from [8] now implies that step 5 can be accomplished by $4(2+10)n = 48n$ Pachner moves.

It is clear that the triangulation of the "collar" that we need to get rid of in step 6, is shellable. Since Pachner moves that change the simplicial structure of the boundary are equivalent to elementary shellings, we only need to count the number of tetrahedra in the "collar". We know that there are less than $16n$ triangles subdividing the original triangulation of ∂M. In the 3-simplices, that were created by step 1, above the 2-simplices of ∂M we get not more than $16n$ tetrahedra. Each vertex of the subdivision in an edge of T lying in ∂M gives rise to 2 tetrahedra in the "collar". Since there are not more than $4n$ vertices of T_1 contained in the 1-skeleton of T, the tetrahedra from the second part of step 1 can contain at most $8n$ 3-simplices that need to be shelled. In other words step 6 requires $24n$ Pachner moves. This completes the proof. □

If the boundary of the manifold M is empty, then lemma 4.1 still holds. In fact the procedure described in its proof is shortened because steps 1 and 6 become redundant. Also, if M is a submanifold and is triangulated as a subcomplex of the ambient triangulation, then the statement of lemma 4.1 remains true. A slight modification of step 1 is required, because in this setting we can alter the triangulation of the boundary of the submanifold using $(1-4)$ and $(2-3)$ moves only. This situation will arise several times in sections 5 and 6.

5 DETECTING SINGULAR FIBRES

Let $M \to B$ be a Seifert fibred space over a (possibly non-orientable) compact surface B and let T be its triangulation consisting of t tetrahedra. Our main goal in this section is to find a subdivision of T which will support in its 2-skeleton an embedded torus around each singular fibre and will also have some additional properties that will be made precise later. We will achieve this by decomposing M along normal essential tori. At the end of the section we will discuss the case when the surface B is closed and the manifold M contains no singular fibres. In that situation we will subdivide T so that its 2-skeleton contains a single embedded torus around some regular fibre. We are assuming throughout this section that the manifold M is not homeomorphic to one of the five exceptional cases which are described in section 7.

Let's assume that our manifold M contains an essential vertical torus. We shall now construct a maximal collection of such tori so that they are pairwise disjoint and are not topologically parallel. We will also make sure that these tori have bounded normal complexity. The whole procedure will be reminiscent of the strategy used in [8] to build a maximal collection of pairwise disjoint non-parallel normal 2-spheres. The only major difference in the recursive construction of the family of essential tori is that each time we need to produce a new surface in our collection, we have to invoke proposition 2.5, rather than using basic normal surface theory, as we did with the 2-spheres. The estimates for normal complexity of the surfaces involved here will, however, be identical to the ones used to bound the number of normal pieces in the 2-spheres.

We start by taking a vertical incompressible torus A' that is not boundary parallel in M and isotoping it so that it becomes both normal with respect to T and so that it has minimal weight in its isotopy class. Then proposition 2.5 gives us an essential torus A in M which is a vertex surface. By (a) of proposition 2.6 this torus has to be isotopic to either a vertical or a horizontal surface. In the former case we take the minimal weight representative in the isotopy class of A to be the first surface A_1 in our collection. The latter case can not arise if M for example has non-empty boundary, because every horizontal surface in M is a branched covering over the base surface B.

But if the manifold M is closed and it contains both a horizontal torus and a vertical one, then its topology is very limited. This hypothesis implies that the following formula for the Euler characteristic of the base surface must hold: $\chi(B) = \Sigma_i(1 - \frac{1}{q_i})$, where the sum is over all singular fibres (if there are no singular fibres in M, the sum on the left should be replaced with zero). The integers q_i are just multiplicities, i.e. denominators of the indices $\frac{p_i}{q_i}$, of singular fibres in M. Apart from certain small Seifert fibred spaces that we have already excluded, in the closed case we can also get one of the following 3-manifolds: $S^1 \times S^1 \times S^1$, the unique orientable S^1-bundle over a Klein bottle which contains an embedded section, a fibration over $\mathbb{R}P^2$ with two singular fibres with indices $\frac{1}{2}$ and $-\frac{1}{2}$, and a Seifert fibred space over S^2 with four singular fibres, two with index $\frac{1}{2}$ and the other two with index $-\frac{1}{2}$. Notice that the second manifold and the fourth manifold from the list are homeomorphic. We will deal with these exceptional cases in section 7.

So we can now assume that our vertex torus A_1 is vertical. Its normal complexity is bounded above by proposition 2.1. We cut M open along A_1 to obtain a manifold M_1 that inherits a polyhedral structure from T. In [8] it is described how to adapt normal surface equations so that their solutions describe closed normal surfaces in this polyhedral structure of M_1. In fact there can be at most 11 distinct disc types in any tetrahedron of T.

If the manifold M_1 contains an essential vertical torus, then we repeat the above procedure to obtain a normal torus A_2, which is again a minimal weight representative in an isotopy class of a vertex surface. Now A_2 is necessarily vertical because ∂M_1 is not empty. Its normal complexity in the initial triangulation T can be bounded using proposition 2.1 again. Just like in [8] it implies that the number of copies of any given normal disc type in the polyhedral structure of the complement M_1 is bounded above by 2^{11t}.

It is clear that these upper bounds remain true in the polyhedral structure of the manifold $M_2 = M_1 - \mathrm{int}(\mathcal{N}(A_2))$, as well as in all subsequent manifolds obtained by the recursive construction of vertical tori. The Kneser-Haken finiteness theorem (see [4]) implies that this recursion has to terminate. Moreover the number of tori in our collection can not exceed $20t$. This is because both Betti numbers in the formula $8t + \beta_1(M;\mathbb{Z}) + \beta_1(M;\mathbb{Z}_2)$, bounding the number of disjoint closed two-sided incompressible surfaces in M, are smaller than $6t$.

Now decompose M along this disjoint maximal collection of non-parallel incompressible tori. Since M either had an incompressible boundary or it contained an essential vertical torus, there are five possible bounded Seifert fibred spaces that can arise. The respective base surfaces are depicted in figure 3.

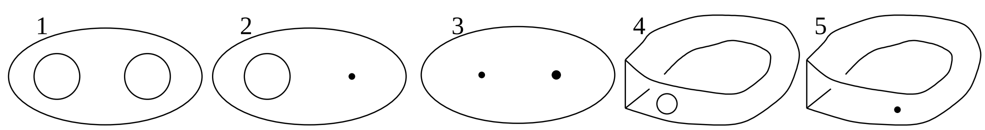

Figure 3: The base spaces for all possible bounded Seifert fibred 3-manifolds containing no essential tori. The dots in cases 2 and 3 represent singular fibres and in case 5 it represents a fibre that may or may not be singular.

Notice also that figure 3 together with the Kneser-Haken finiteness theorem implies that any Seifert fibred space, triangulated by t tetrahedra, can contain at most $40t$ singular fibres.

Now we form a sub-collection $\mathcal{A}$ that consists of some of the essential vertical tori we've constructed. We say that an essential torus belongs to $\mathcal{A}$ if an only if it bounds on at least one side a Seifert fibred piece that contains a singular fibre. In other words the boundaries of cases 2, 3 and possibly 5 from figure 3 are precisely the tori of $\mathcal{A}$. The number of copies of each normal disc type of the triangulation T in $\mathcal{A}$ is bounded above by $(2 \cdot 2^{11t})^{20t}$ because at any stage of the construction of $\mathcal{A}$ there are at most $11t$ normal variables (the factor of 2 is a consequence of the generalisation of normal surface theory to the polyhedral setting, see [8]).

We are still interested in constructing the family $\mathcal{B}$ of compressible tori, one around each singular fibre of M. Assume now that we have already triangulated pieces of types 2, 3 and 5 from figure 3 in the complement of $\mathcal{A}$. Those are the only complementary pieces of $\mathcal{A}$ that contain singular fibres. In particular the type 5 piece can not be an S^1-bundle over a Moebius band.

We proceed by looking for vertical annuli, that are not ∂-parallel, in the fibration of the pieces. If we manage to do that, by finding one such annulus among vertex surfaces, then it is clear how the horizontal boundary of the regular neighbourhood of this vertex annulus, together with the corresponding element of $\mathcal{A}$, can be used to construct the family $\mathcal{B}$ of compressible tori isolating all singular fibres.

Let A be a least weight vertical annulus that is not boundary parallel in a component X of $M - \text{int}(\mathcal{N}(\mathcal{A}))$ that is of type 2, 3 or 5. Unless X is homeomorphic to an I-bundle over a Klein bottle, i.e. an S^1-bundle over a Moebius band, then proposition 2.5 guarantees the existence of an essential vertex annulus whose boundary coincides with that of A. The vertex annulus is therefore vertical and isotopic to A (but not necessarily rel ∂A). In a type 5 piece an element of $\mathcal{B}$ consists of the horizontal boundary of the regular neighbourhood of our vertex annulus, together with an annulus in the boundary of X. In the pieces of type 2 and 3 a torus in $\mathcal{B}$ would consist of a single copy of our vertex annulus together with an annulus in ∂X.

The exceptional case, when X is equivalent to an I-bundle over a Klein bottle, can actually arise. A type 3 piece, with both exceptional fibres of index $\frac{1}{2}$, is homeomorphic to an S^1-bundle over a Moebius band. This S^1-bundle structure is, of course, different from the original Seifert fibration of X. Our vertical annulus A becomes a horizontal surface in the S^1-bundle structure. It is in fact equal to the horizontal boundary of a regular neighbourhood of some Moebius band section of this S^1-bundle. Up to isotopy however, there are only two possible Moebius band sections of this S^1-bundle, one for each isotopy class of orientation reversing simple closed curves in the vertical Klein bottle. Clearly, these two isotopy classes of Moebius band sections correspond to singular fibres in our original Seifert fibration of X. If we could make either of these two Moebius bands fundamental, then we could easily construct compressible tori in $\mathcal{B}$ around each of the singular fibres. One of them would just be the boundary of the regular neighbourhood of the fundamental Moebius band and the other one would be parallel to the boundary of its complement in X. Note also that any properly embedded Moebius band in X has to be both incompressible and ∂-incompressible. The latter follows because the surface is one-sided, ∂X is incompressible and X is irreducible. The former is true since the bounding simple closed curve can not be trivial in the incompressible boundary of X. Therefore, by (b) of proposition 2.6, any Moebius band in X is horizontal in the S^1-bundle structure on X.

Take A to be the smallest weight essential Moebius band in X. If we had $A = F + G$, we could assume that the sum was in reduced form. Then lemma 2.2 implies that no patch is trivial (in this case boundary pattern is empty). In other words F and G are different from a disc, a 2-sphere and, since X is irreducible, also from a projective plane. So we must have $\chi(F) = \chi(G) = 0$.

If G is closed, then F must have the same boundary as A which consists of a single simple closed curve. That makes F a Moebius band. But this is a contradiction since the weight of F is smaller than that of A. So both F and G must have boundary. The same proof tells us that neither of the surfaces can be a Moebius band. But if they are both annuli, then, since there are no trivial patches in A, the sum must have at least two boundary components which is again a contradiction. Therefore A has to be fundamental. We are now ready to prove the following proposition.

Proposition 5.1 *Let $M \to B$ be a Seifert fibred space over a possibly non-orientable compact surface B and let T be its triangulation consisting of t tetrahedra. Assume also that M contains at least one singular fibre and is either bounded or it contains an embedded essential vertical torus. Then there is a subdivision T_1 of T containing the family $\mathcal{B}$ of compressible tori, one around each singular fibre of M, in its 2-skeleton. Furthermore T_1 can be obtained from T by making less than $2^{(2^{400t^2})}$ Pachner moves. This also gives a bound on the number of 3-simplices in T_1.*

Proof. The construction of T_1 will follow the same lines as the construction of subdivision S in section 5 of [8]. The main technical tool for subdividing T with Pachner moves is the procedure described by lemma 4.1.

Now we can define the subdivision T_1 and then construct it using Pachner moves starting from the triangulation T. Since a single torus in $\mathcal{A}$ can be in the boundary of two adjacent pieces from figure 3, we need to take doubles of all normal surfaces contained in $\mathcal{A}$. We'll denote it by $2\mathcal{A}$. The subdivision T_1 will be obtained in two steps. In the first step we make sure that the 2-skeleton contains the normal surface $2\mathcal{A}$. If the family $\mathcal{A}$ is empty, then the manifold M has to be one of the spaces of types 2, 3 or 5 from figure 3. In that case we take $\mathcal{A}$ to be a single normal torus, parallel to some boundary component of M. Then we do the same as above.

We know that $2 \cdot 5t(2 \cdot 2^{11t})^{20t}$ is an upper bound on the number of normal pieces in $2\mathcal{A}$. Lemma 4.1 implies that this first subdivision can be obtained by 2^{300t^2} Pachner moves. This is also a bound on the number of tetrahedra in it.

In the second step we subdivide further, so that singular fibres are isolated in all relevant pieces in $M - 2\mathcal{A}$. Let $\mathcal{C}$ be the collection of two parallel copies of all vertical annuli constructed previously, in all components of $M - \mathrm{int}(\mathcal{N}(2\mathcal{A}))$ that are homeomorphic to type 2, 3 or 5 Seifert fibred pieces from figure 3. If a piece of the above complement is a special case of type 3, then the annulus we have in mind is the horizontal boundary of the regular neighbourhood of the fundamental Moebius band. The normal complexity of each of these surfaces is at most four times the normal complexity of a fundamental surface. Since the union of the relevant components of $M - 2\mathcal{A}$ is triangulated by less than 2^{300t^2} tetrahedra, lemma 4.1 and proposition 2.1 imply that

$$200 \cdot 2^{300t^2}(4(5 \cdot 2^{300t^2})(7 \cdot 2^{300t^2} 2^{7 \cdot 2^{300t^2}})) < 2^{2^{400t^2}}$$

bounds the number of Pachner moves required to construct the subdivision T_1. The same number also bounds the number of tetrahedra in T_1. □

Our next task is to construct a compression disc in each solid torus that is bounded by an element of $\mathcal{B}$. Notice that the triangulations of these solid tori support, in their 1-skeleta, a pattern P which consists of a single simple closed curve that is isotopic to a regular fibre (take for example a boundary component of a surface in $\mathcal{C}$).

Let D be a compression disc that has as few intersections with the pattern as possible. In other words $\iota(D) = q$ where $\frac{p}{q}$ is the index of the singular fibre in our solid torus. Assume also that D minimises the weight among all such discs. We will prove that under these circumstances D has to be fundamental.

Assume to the contrary that $D = F + G$ and that the sum is in reduced form. By lemma 2.2 there are no trivial patches. So neither F nor G can be a 2-sphere. They can also not be homeomorphic to a projective plane because we are living in a solid torus. The last possibility is that F is a disc. Again the surface G can not be closed, because there are no trivial patches. Since $\iota(D) = \iota(F) + \iota(G)$, we must have $\iota(D) \geq \iota(F)$. Since $w(F) < w(D)$ we can conclude that F must be parallel to a disc F' in the boundary of our solid torus. Look at the disc regions in F' that are bounded by the arcs coming from ∂G. By a nice argument, that is attributed to Haken (see claim 4.1.1 in [1]), we can conclude that one of the disc regions in F' must, after we do all normal alterations in $F + G$, either be bounded by the whole ∂D or by a single arc in ∂D. The first case can not occur because D is a compression disc. In the second case we get an arc in $F \cap G$ with one end point in the bad corner of the disc region. Using this arc and the disc region we can construct a pure boundary compression disc for D. One of the discs, obtained by compressing D, will have to be both pure and boundary parallel (this is because D has minimal intersection with the pattern). This means that the two subdiscs in D, chopped off by our arc from $F \cap G$, are either pure or are disjoint from the pure ∂-parallel disc we've created by compressing D. In the first case we get a pure disc patch in D. In the second case we can construct a weight reducing isotopy for D. Both of these possibilities lead to contradiction.

Now we know that we can find a fundamental normal disc in every solid torus component of the complement of $\mathcal{B}$. Assume that we've subdivided the triangulation T_1 from proposition 5.1, so that these discs are contained in the 2-skeleton. Their complements in the solid tori are now 3-balls. We want to change the triangulations in these 3-balls so that they become cones on their respective boundaries. A slight reformulation of the main result in [8] allows us to do precisely that.

Theorem 5.2 *Let T be a triangulation of a 3-ball with t tetrahedra. Then it can be changed to a* *cone on the bounding 2-sphere, without altering the induced triangulation of the boundary, by less than* $at^2 2^{at^2}$ *Pachner moves, where the constant a is bounded above by* $6 \cdot 10^6$.

What we are aiming for at this stage is a subdivision T_2 of the triangulation T_1 that will be "canonical" around singular fibres. What we mean by that is that a torus τ in $\mathcal{B}$, isolating a singular fibre of index $\frac{p}{q}$, is triangulated in the following way: the 1-skeleton in τ contains q edges coming from a regular fibre P_τ on τ and q edges coming from the boundary of the compression disc D_τ. This decomposes τ into q discs of length four. We triangulate each one of them by 2 triangles (just pick a diagonal). By doing so, we haven't introduced any new vertices in the boundary of D_τ. We can therefore triangulate it with q triangles by coning from a point in the disc's interior. The 3-ball region of $M - (\tau \cup D_\tau)$ we triangulate by coning as well. We have thus described the subdivision T_2. Now we need to construct it using Pachner moves.

The subdivision we have so far is a cone in the 3-ball component of $M - (\tau \cup D_\tau)$. But the triangulations of τ and D_τ are not correct. Let's assume that the subdivision contains two parallel copies of τ that have identical simplicial structures and that the product region $\tau \times I$ between them is triangulated by cones on the boundary in every 3-ball of the form $\Delta \times I$, where Δ is a triangle from τ. Assume further that the polyhedron $(\partial D_\tau \cup P_\tau) \times I$ is contained in the 2-skeleton of the subdivision. Therefore the arcs of $\partial D_\tau \cup P_\tau$ are already contained in the 1-skeleton of τ. But at the moment they contain many edges of the subdivision. Later we shall simplify their triangulation to obtain the "canonical" subdivision T_2. All assumptions we've made here can be implemented while building the compression discs corresponding to the singular fibres. The number of Pachner moves they require can be incorporated in the bounds we have so far.

First we will work on the triangulation of the product $\tau \times I$. Let α_τ denote one of the components of $P_\tau - \partial D_\tau$. The complement of $(\partial D_\tau \cup \alpha_\tau) \times I$ in $\tau \times I$ is a 3-ball which is triangulate by a shellable triangulation. We can therefore expand the cone structure of $\Delta \times I$, for some triangle Δ from τ, to the whole complement (in section 4 of [8] it is described how elementary shellings relate to Pachner moves). Further more we can assume that all three discs in $((\partial D_\tau \cup \alpha_\tau) - (\partial D_\tau \cap \alpha_\tau)) \times I$ are triangulated as cones on their boundaries. Since all this can be achieved by linearly many Pachner moves, we can assume that our subdivision had these properties all along.

The PL disc $\tau - (\partial D_\tau \cup \alpha_\tau)$ in the subdivision has now cones on both sides. We can therefore change its triangulation so that it matches the simplicial structure induced on it by the triangulation T_2, everywhere but along its boundary. This can again be accomplished by linearly many Pachner moves (see lemma 4.2 in [8]). Now we have to go back to simplifying the triangulation of the graph $\partial D_\tau \cup \alpha_\tau$. This boils down to amalgamating relevant pairs of consecutive edges to a single edge. The procedure is the same whether the pair of edges we want to get rid of is contained in ∂D_τ or in α_τ. This amalgamation is equivalent to crushing one of the edges in the pair, and thus flattening its star in the subdivision. This procedure is explicitly described in [8]. In our setting it will take $7 = 12 - 6 + 1$ (cf. lemma 4.1 in [8]) Pachner moves for one pair of edges. Now we can prove the main result of this section.

Theorem 5.3 *Let the 3-manifold M satisfy the same assumptions as in proposition 5.1. Then there is a subdivision* T_2 *of the triangulation T with the properties that are described above. Moreover* T_2 *can be obtained from T by making less than* $e^4(500t^2)$ *Pachner moves, where* $e(x) = 2^x$. *The number of tetrahedra in* T_2 *is bounded above by* $e^3(500t^2)$.

Proof. We start by applying lemma 4.1 to the fundamental compression discs in the disjoint union of solid tori that are bounded by the surfaces in $\mathcal{B}$. The number of tetrahedra in the solid tori is bounded above by $f(t) = e^2(400t^2)$. So the number of normal pieces in all these compression discs is bounded above by $n(t) = 40t \cdot 5f(t) \cdot 7f(t)2^{7f(t)}$. The factor $40t$ bounds the number of singular fibres in M. So by lemma 4.1 we need to make no more than $200n(t)f(t)$ Pachner moves to get the compression discs into the 2-skeleton of the subdivision.

Coning the complementary 3-balls in the solid tori will, by theorem 5.2, take not more than $a(20(n(t)+f(t))^2 e(a(20(n(t)+f(t))^2)$ Pachner moves, where the constant a is bounded above by $6 \cdot 10^6$. Notice that after this step, the number of tetrahedra in the subdivision is still in the order of magnitude of $e^3(400t^2)$. In particular the number of triangles in the compression discs and toral components of $\mathcal{B}$ are of the similar size. Since changing the simplicial structure of these surfaces, as was described above, can be done by linearly many (in the number of tetrahedra) Pachner moves, the number $e^4(500t^2)$ is surely an upper bound on the total number of Pachner moves required to build the subdivision T_2. The inequality $20(n(t)+f(t)) < e^3(500t^2)$ is also obvious. This completes the proof. □

We shall conclude this section by briefly discussing the case when M is a closed 3-manifold containing no singular fibres. Since we are dealing with Haken 3-manifolds, this implies that the base surface B has non-zero genus. We would like to construct a subdivision T_2 of the triangulation T of M so that it contains in its 2-skeleton a compressible vertical torus. Moreover we want the compression disc of this torus to be a part of the 2-skeleton as well. The 1-skeleton in the torus will contain both two disjoint parallel regular fibres and the boundary of this disc. The compression disc itself will be triangulated by two 2-simplices, while the torus will consist of four triangles. The complementary region of this two-dimensional polyhedron that is a 3-ball, will be triangulated as a cone on its boundary. This completely describes the subdivision T_2.

We can assume that M is neither $S^1 \times S^1 \times S^1$ nor an S^1-bundle over a Klein bottle which is a double of the unique orientable S^1-bundle over a Moebius band. Then proposition 2.5 gives us a normal vertical torus that is a vertex surface. Now we can apply lemma 4.1 to two parallel normal copies of this torus. This requires not more than $200 \cdot 5t2^{7t}t = 1000t^2 2^{7t}$ Pachner moves and produces a subdivision with less than $20(5t2^{7t} + t) < 100t^2 2^{7t}$ tetrahedra.

Now we look at the complement of the regular neighbourhood of our vertical torus in M. Since M is a Seifert fibred space and is not homeomorphic to either of the exceptional manifolds, at least one component of this complement is not homeomorphic to either $S^1 \times S^1 \times I$ or to an I-bundle over a Klein bottle. By proposition 2.5 we can find an essential vertical annulus in one of the components of the complement. Furthermore this annulus is a vertex surface. We take two normally parallel copies of it and subdivide further so that they become part of the 2-skeleton. This can be done by lemma 4.1. The compressible torus, that we want to have in the 2-skeleton of T_2, will be the boundary of the regular neighbourhood of our vertical annulus. Since the triangulation of this regular neighbourhood, which is a solid torus, is shellable, we do not need to apply theorem 5.2 in order to get the cone. Also, since the triangulation of the normal vertical annulus contains an embedded spanning arc in its 1-skeleton, we do not need to look for a compression disc, because it is already there (just take the disc above that arc in the product structure of the regular neighbourhood). All the manipulations on the above subdivision, needed to produce the triangulation T_2, are very similar to what we did before and also require linearly many Pachner moves. Thus the following proposition follows.

Proposition 5.4 *Let $M \to B$ be an S^1-bundle over a closed (possibly non-orientable) surface B that is not homeomorphic to any of the exceptional manifolds from section 7. Let T be its triangulation containing t tetrahedra. Then we can construct a subdivision T_2 of T, described above, by making less than $2^{(2^{400t^2})}$ Pachner moves. This also bounds the number of tetrahedra in T_2.*

6 TRIANGULATIONS OF S^1-BUNDLES

Let B be a bounded, possibly non-orientable, compact surface with negative Euler characteristic and let $M \to B$ be an S^1-bundle over it that is triangulated by T. Since ∂B is not empty, there exists a *section* of the S^1-bundle M. That is to say that M contains a properly embedded surface which is horizontal and it intersects each fibre precisely once. This is because M can be obtained from a solid torus $S^1 \times D^2$ by identifying pairs of annuli in the boundary $S^1 \times \partial D^2$. Each identification can be chosen to be either an identity or a fixed reflection in the S^1 factor. If $x \in S^1$ is invariant under this reflection, then the disc $\{x\} \times D^2$ yields our section. Notice that we can construct a connected horizontal surface in M, meeting each fibre in precisely n points, for any natural number n, in a similar way.

Let Y denote a non-empty collection of boundary components in M. In this section we are going to construct a triangulation of M that is going to depend on the triangulation $T|_Y$ of Y (T is the original triangulation of M). It is however going to be unique up to homeomorphisms of M that do not permute components of ∂M and that are fixed (up to isotopy) on the components of Y. For example the simplicial structure in the toral components of Y naturally arises when our S^1-bundle is a submanifold of a Seifert fibred space and the tori in Y bound singular fibres. We can make sure that Y is not empty by removing a neighbourhood of a regular fibre and triangulating the complement as was described at the end of section 5. Roughly speaking we are going to look for a "canonical" section of M and a family of disjoint vertical annuli that are homologically non-trivial in $H_2(M, \partial M; \mathbb{Z}_2)$ and whose complement is a solid torus. The union of the "canonical" section together with these annuli contains all topological information about M, since its complement is just a 3-ball. The triangulation we are looking for is going to be a cone on the boundary of this 3-ball.

6.1 NORMAL SECTIONS

The "canonical" section of $M \to B$ will depend on the simplicial structure of the boundary components in Y. Before we define it and prove that such a surface is always fundamental (see theorem 6.2), we need a way of relating any two sections whose boundaries coincide in Y, by a homeomorphisms of M. It turns out that twists suffice. A *twist* along a properly embedded annulus or torus in a 3-manifold M is any homeomorphism of M which is an identity outside a regular neighbourhood of the surface. We can now prove the following lemma.

Lemma 6.1 *Let $M \to B$ be an S^1-bundle over a possibly non-orientable bounded surface B with negative Euler characteristic. Let Y be a subset of ∂M as above and let S_1 and S_2 be two sections of the bundle M, such that the simple closed curves from $Y \cap S_1$ and $Y \cap S_2$ are isotopic in Y. Then there exists a sequence of twists along vertical annuli in M that transforms S_1 into a section which is isotopic to S_2. Furthermore this composition of twists restricted to Y is isotopic to the identity on Y.*

Proof. Take a collection of disjoint vertical annuli that decompose our S^1-bundle into a fibred solid torus. Any section intersects each of these annuli in a single arc. Now it is clear that any section can be isotoped so that it coincides with any other section outside a regular neighbourhood of the vertical annuli. The isotopy can be chosen so that the surface we are isotoping intersects every fibre of the bundle precisely once, throughout the process. After twisting an appropriate number of times in a neighbourhood of a vertical annulus, we can obviously make the two sections coincide.

This implies that the sections S_1 and S_2 are related by a sequence of twists along vertical annuli. We can not guarantee that the annuli we have to twist along are disjoint from the boundary components in Y. But each such twist is an orientation preserving homeomorphism of M which is invariant on the fibres of M. Also their composition has to preserve the homotopy classes of the meridians $S_1 \cap Y$ of the tori in Y. We can therefore conclude that the restriction of the homeomorphism we get in the end is isotopic to the identity on Y. $\square$

Let T_Y denote the triangulation of Y obtained by restricting the triangulation T of the S^1-bundle M to the boundary components in Y. We shall now fix a collection $\mathcal{P}$ of simple closed curves in Y with the following properties:

- $\mathcal{P}$ contains precisely one simple closed curve on each torus in Y and each such curve is a union of normal arcs in T_Y,
- $\mathcal{P}$ bounds some normal section S of the S^1-bundle M, in other words $S \cap Y = \mathcal{P}$, and

- no collection of simple closed curves satisfying the above properties has fewer normal arcs in T_Y.

A collection $\mathcal{P}$ as described here always exists, because our S^1-bundle M always contains a section. Having chosen the collection $\mathcal{P}$ in this way, we will now prove (theorem 6.2) that we can find a fundamental normal section S of M with the property $S \cap Y = \mathcal{P}$. This fundamental surface will therefore interact with the simplicial structure of Y in a way which is independent of the triangulation of the submanifold $M - Y$. It is also obvious that lemma 6.1 is precisely what's needed to relate any two such sections.

Before we embark on the proof of theorem 6.2, we need to introduce some notation. For any surface F that is embedded in the 3-manifold M and is transverse to the 1-skeleton of T, we define the "Y-weight", $w_Y(F)$, to be the number of points in the intersection of F with the 1-skeleton of T_Y. In other words it follows directly from the above definition that the normal section S that realizes our chosen collection $\mathcal{P}$, minimises the "Y-weight" among all normal sections of the bundle M.

Theorem 6.2 *Let $M \to B$ be an S^1-bundle over a possibly non-orientable bounded surface B that has negative Euler characteristic. Let Y be a non-empty collection of boundary components of M that contains the family $\mathcal{P}$ as in the above definition. Let S be a normal section of M, with respect to the triangulation T, which minimises the weight among all normal sections that satisfy the following identity: $S \cap Y = \mathcal{P}$. Then S has to be fundamental.*

Proof. Assume that S can be obtained as a normal sum $S = F + G$ of two connected surfaces F and G. Without loss of generality we can take the sum to be in reduced form. We shall see that this assumption leads into contradiction.

Clearly the homology class $[S]$ is non-trivial in $H_2(M, \partial M; \mathbb{Z}_2)$, since it intersects a fibre of M in a single point. Because we are working over the field $\mathbb{Z}_2$, we have the following identity $[F + G] = [F] + [G]$, where the first sum is a normal sum and the second is just addition in the vector space $H_2(M, \partial M; \mathbb{Z}_2)$. This implies that at least one of the summands, say $[F]$, is different from zero. Now compress and ∂-compress the surface F as much as possible. Let F' be the surface obtained by this process, where we've discarded all homologically trivial components. So F' must be homologous to F and both incompressible and ∂-incompressible. Proposition 2.6 implies that F' is either vertical or horizontal in the bundle structure of M.

If F' is vertical, then the homology class $[G]$ can not be zero. This is because the boundary homomorphism $\delta : H_2(M, \partial M; \mathbb{Z}_2) \to H_1(\partial M; \mathbb{Z}_2)$ would in this case give $[\partial S] = \delta([S]) = \delta([F] + [G]) = \delta([F']) = [\partial F']$. This is a contradiction because on some boundary torus of M we would have a homology class of the meridian equalling a homology class of a fibre.

If, on the other hand, our surface G carries non-trivial homology, then we can effectuate the same process as before, thus obtaining an incompressible ∂-incompressible surface G' which is homologous to G. So G' is either horizontal or vertical. If we had both F' and G' vertical, we would get a contradiction like before, because on some boundary torus of M we would have a sum of fibres being homologous to a meridian. In other words, we can assume that one of the two surfaces, say F', is horizontal.

So F' has to cover the base surface B and their respective Euler characteristics are related by $\chi(F') = k_{F'}\chi(B)$, for some integer $k_{F'}$. Since the inequality $\chi(F) \leq \chi(F')$ holds, we get the following:

$$\chi(F) \leq \chi(F') = k_{F'}\chi(B) = k_{F'}\chi(S) = k_{F'}(\chi(F) + \chi(G)) \leq k_{F'}\chi(F).$$

The last inequality follows from the next claim and from the fact that M contains no embedded projective planes.

Claim. The surface G is neither a disc nor a 2-sphere.

Since $\chi(B)$ is strictly smaller than zero by assumption, the same must be true for $\chi(F')$ and thus for $\chi(F)$. So the above inequality is in fact an equality with $k_{F'} = 1$, $\chi(G) = 0$, and $\chi(F) = \chi(F')$. The last equality implies that we neither compressed nor ∂-compressed F while creating F'. So F' is a subsurface of F. Since the surface F is connected, it must be equal to F' and is therefore a normal section of M.

On the other hand we have $w_Y(S) = w_Y(F) + w_Y(G)$ and $w_Y(S) \leq w_Y(F)$. The inequality follows from the definition of the family $\mathcal{P}$. This gives us $w_Y(G) = 0$, or in other words $G \cap Y = \emptyset$. This implies that the normal section F also realizes the family $\mathcal{P}$. But since G is a normal surface, its weight $w(G)$ is strictly positive. This yields the inequality $w(F) < w(S)$ which contradicts our choice of the normal section S.

So all that is left now is to prove the claim. It is enough to show that the sum $S = F+G$ contains no trivial patches (we take boundary pattern to be empty). Even though the sum $S = F + G$ is in reduced form, we can not directly apply lemma 2.2 because we couldn't isotope S freely to its minimal weight representative with respect to the triangulation T (the boundary of S in Y was fixed). But away from Y the normal surface S was chosen so that it minimises the weight. So the same argument as in the proof of lemma 2.2 tells us that any potential trivial patch has to have one arc from its boundary contained in Y.

Chose now a trivial patch D with $w_Y(D)$ minimal among all trivial patches in S. Like in the proof of lemma 2.2, using the fact that the surface S is boundary incompressible, we get a disc D' in S bounded by two arcs: one is in Y and the other is "parallel" to the arc in the boundary of the trivial patch D. From the definition of the family $\mathcal{P}$ it follows that $w_Y(D') = w_Y(D)$. If $w_Y(D)$ was equal to zero, then we would have both arcs $D \cap Y$ and $D' \cap Y$ contained in a single triangle of T_Y. This would imply that there exist two normal arcs, one in each summand of the sum $F + G$, that intersect in more than one point. This contradicts lemma 2.1 in [1]. So we can assume that $w_Y(D)$ is strictly positive.

If D' is a disc patch itself and is hence trivial, then we can isotope F and G respectively so that they still sum up to the normal surface S, but have fewer components of intersection. This contradicts the reduced form assumption on the sum $F + G$ and can therefore not occur. If D' is not a disc patch, then it has to contain precisely one trivial patch with its Y-weight equal to $w_Y(D)$. Like in lemma 2.2, we can construct a normal annulus A, with its boundary contained in Y, such that $S = A + F'$ and $w_Y(A) = 0$. But such an annulus does not exist. This proves the claim and the theorem. □

6.2 VERTICAL ANNULI

So far we having found a "canonical" section of M. Now we need to build a collection of $(1 - \chi(B))$ vertical annuli that are going to decompose our S^1-bundle into a single solid torus. Clearly this decomposition is going to be highly non-unique because the mapping class group of the base surface is full of non-trivial elements. So what we want is to define this collection in such a way, so that any two possible choices are related by a homeomorphism of M which is an identity on the boundary ∂M. Also we would like our process to yield annuli that interact with the triangulation of Y in a prescribed way.

We start by ordering the toral components of ∂M. We make sure that in this ordering we enumerate all the components of Y before the components of $\partial M - Y$. Let Y_1 in Y be the first torus in this ordering and let $A_1, \ldots, A_n$ be a collection of disjoint annuli we are trying to describe (the subscript n equals $1 - \chi(B)$). Each annulus A_i will contain at least one boundary component in Y_1 and every other torus in ∂M will contain precisely one boundary curve of the surface $A_1 \cup \ldots \cup A_n$. We also stipulate that the first $2g$ annuli (where g is the genus of the orientable base surface B) have both of their boundary components contained in the torus Y_1. In case the surface B is not orientable, its genus g is the maximal number of projective plane summands it contains when expressed as a connected sum. In this case the first g annuli from our collection have both of their boundary components contained in the torus Y_1. And finally, again if our base B is not orientable, we want to make sure that the complement of the arc $p(A_1)$ in the surface B is an orientable surface. The map $p\colon M \to B$ denotes our usual bundle projection. Notice that all of the above conditions on the family of vertical annuli $A_1, \ldots, A_n$ could be described in terms of their projections onto the base surface.

Now we need to discuss how these vertical annuli interact with the triangulation T_Y of the surface Y. On each component of Y we already have a normal curve from the collection $\mathcal{P}$. The boundaries of the vertical annuli are fibres of M. So the algebraic intersection of a curve γ in $\mathcal{P}$ and one such boundary component, both lying in the same torus of Y, is $+1$ (if the orientations are chosen judiciously). We pick a fibre (of the bundle M) λ in each component of Y with the following property: λ consists of the smallest number of normal arcs in T_Y in the isotopy class of the fibre (lying in the corresponding component of Y). We can assume that simple closed curves γ and λ intersect transversely. If they intersect in more than one point, we will modify λ, without increasing the number of normal arcs in it, so that in the end the set $\gamma \cap \lambda$ contains one point only. This is done in the following way. The complement of γ, in a component of Y it lies in, is an annulus. If λ intersects this annulus in single spanning arc, then we are done. If not, there must exist an outermost inessential subarc (of λ) in the annulus, that is parallel to a subarc of γ (this is because the intersection number of λ and γ is $+1$). It follows from the definition of $\mathcal{P}$ and the choice of λ that these two subarcs must contain the same number of normal arcs. We can thus modify λ, without increasing its weight, so that the number of points in $\gamma \cap \lambda$ goes down by 2. Repeating this procedure as long as possible does the job.

So far we have constructed a normal fibre λ in each component of Y which consists of the smallest number of normal arcs in its isotopy class, with respect to T_Y. Furthermore these simple closed curves intersect the corresponding elements of $\mathcal{P}$ in a single point. Now we can define the family $\mathcal{F}$ of $2n$ fibres, lying in the boundary of M, which are going to bound our family of vertical annuli $A_1, \dots, A_n$. There will be precisely one fibre on every torus in $\partial M - Y_1$. On each torus of $Y - Y_1$ we take the simple closed curve λ we constructed above. Since the triangulation of the boundary components in $\partial M - Y$ is not fixed, we can afford to define the simple closed curves of $\mathcal{F}$ on those tori, only up to isotopy. The distinguished isotopy class is, of course, that of a fibre.

The situation on the torus Y_1 is as follows. If the base surface B is orientable (respectively non-orientable), then we take $2g+1-\chi(B)$ (respectively $g+1-\chi(B)$) parallel copies of the simple closed curve λ defined above. It is clear that the family $\mathcal{F}$ of normal fibres of M that we've just defined, bounds a collection of vertical annuli $A_1, \dots, A_n$ with all the required properties.

Now we need to find a way of constructing these annuli using normal surface theory. The procedure will be recursive. So assuming that we have already created a subcollection $A_1, \dots, A_k$ (for some k smaller than n) of vertical normal annuli whose boundaries lie in $\mathcal{F}$ and which satisfy all other requirements, we look at the manifold $M_k = M - (A_1 \cup \ldots \cup A_k)$ which comes naturally equipped with the polyhedral structure from the original triangulation T of M. Because the annulus A_{k+1} we are looking for lives in the manifold M_k and because its boundary is contained in $\mathcal{F}$, i.e. it lies in $M_k \cap \partial M$, we can set up the normal surface theory using this polyhedral structure on M_k. How to get the equations and how to bound the complexity of fundamental surfaces in this normal surface theory is explained in [8]. From the combinatorial point of view, the procedure of getting vertical annuli is completely analogous to the process of building the maximal collection of non-parallel normal 2-spheres in [8] and is also very similar to the process of defining topologically non-parallel vertical tori that was used in section 5. All we need now is a way of finding a fundamental vertical annulus in M_k that will satisfy all the necessary conditions.

Proposition 6.3 *Let $M \to B$ be a triangulated S^1-bundle over a possibly non-orientable bounded surface B with negative Euler characteristic and let $A_1, \dots, A_k$, for $1 \leq k < n$, be normal vertical annuli as described above. Fix the appropriate simple close curves e and f from $\mathcal{F} \cap M_k$, that are not contained in the union $A_1 \cup \ldots \cup A_k$ and are supposed to bound the next annulus in our collection. Let A_{k+1} be the normal annulus whose boundary consists of the fibres $e \cup f$ and which minimises the weight, with respect to the polyhedral structure on M_k, among all normal annuli that carry non-trivial elements of $H_2(M_k, \partial M_k; \mathbb{Z}_2)$ and which are bounded by $e \cup f$. Then A_{k+1} is fundamental in M_k.*

Proof. Assume to the contrary that A_{k+1} can be expressed as a sum $A_{k+1} = A + F$, where A and F are connected normal surfaces. Without loss of generality we can take it to be in reduced form.
Claim. There are no trivial patches in A_{k+1}.
Just like in the claim we used to prove theorem 6.2, there can obviously be no trivial patches in A_{k+1} that are disjoint from the boundary tori in $Y \cap M_k$. Note also that the boundary curves

∂A_{k+1} in $Y \cap M_k$ have, by construction, the same minimal property the simple closed curves of $\mathcal{P}$ had in the previous subsection. So we can use an identical argument, to the one that yielded the claim in the proof of theorem 6.2, to finish the proof of this claim.

This implies that both surfaces A and F are different from a disc or a 2-sphere. Since our manifold M_k contains no projective planes their Euler characteristics are not positive. Since we have $0 = \chi(A_{k+1}) = \chi(A) + \chi(F)$, both Euler characteristics are in fact equal to zero.

The claim also implies that the patches on the annulus A_{k+1} are either discs of length four or annuli. In the former case each of the discs is bounded by two spanning arcs in A_{k+1} and by two subarcs in the boundary ∂A_{k+1}. This means that both surfaces A and F have non-empty boundary. The equation $[A_{k+1}] = [A] + [F]$ in $H_2(M_k, \partial M_k; \mathbb{Z}_2)$ implies that at least one of them, say F, is homologically non-trivial. Since F is connected and its Euler characteristic is zero, it has to be both incompressible and boundary incompressible (here we are using irreducibility of M_k and incompressibility of its boundary). By proposition 2.6 it is therefore either vertical or horizontal in the S^1-bundle structure of $M_k \to B_k$. But it can not be horizontal if $k \geq 2$ because it is disjoint from the annuli $A_1 \cup \ldots \cup A_{k-1}$. If k equals one, then it would have to cover the base surface B and would thus imply $\chi(B) = 0$. Since this too is a contradiction, the surface F must be vertical and therefore an annulus. But on the other hand we must have at least one boundary component of both F and A contained in Y. Since F is a vertical annulus, its boundary consists of fibres that satisfy $w_Y(F) < w_Y(A_{k+1})$. It is true that the boundary of F can intersect a component of $\mathcal{P}$ more than once, but we could then improve that simple closed curve in Y (by the same process we used on λ when defining $\mathcal{F}$), without increasing the number of normal arcs, so that the intersection contains a single point, and thus arrive at a contradiction.

From now on we can assume that all the patches are annuli. This, together with the fact that A_{k+1} is vertical, implies that both surfaces A and F are vertical because they are just unions of vertical patches of A_{k+1}. If both of them were annuli, then their boundaries would have to be disjoint. This can not be, because our annulus A_{k+1} has only two boundary components. So we can safely assume that A is an annulus and that F is a closed vertical surface. This means that ∂A equals ∂A_{k+1}. Our choice of the annulus A_{k+1} then gives that $[A]$ is trivial in $H_2(M_k, \partial M_k; \mathbb{Z}_2)$ and that $[F]$ carries non-zero homology. In other words F is not separating in M_k.

Let $p : M_k \to B_k$ be the bundle projection, where B_k is the base surface obtained from B by cutting it along the arcs $p(A_1) \cup \ldots \cup p(A_k)$. Then the arc $p(A)$ separates the base space B_k into two components B' and B''. Also there must exist an arc α in the family $p(F - (F \cap A))$, lying in say B', which is non-trivial in $H_1(B', \partial B'; \mathbb{Z}_2)$. We can now construct a vertical homologically non-trivial annulus in M_k, with the boundary equal to ∂A_{k+1}, whose weight is strictly smaller than that of A_{k+1}. Let β be a simple closed curve in B' that intersects α in a single point (see figure 4). Now discarding the subarc of $p(A)$, lying between the points of $\partial\alpha$, and replacing it with

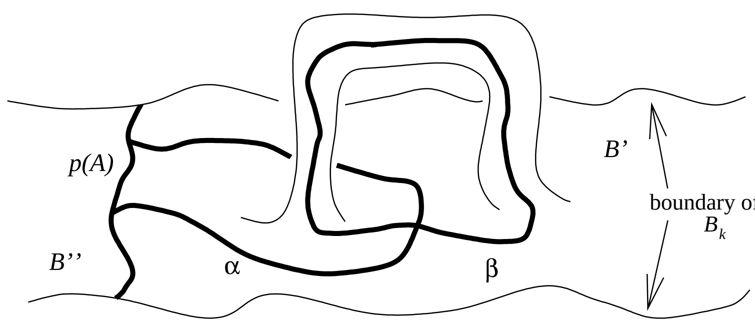

Figure 4: The subarc α of the simple closed curve $p(F)$ is not homologically trivial in B'.

α yields a properly embedded arc in B_k which is homologically non-trivial since it intersects the simple closed curve β in a single point. The vertical annulus A' above it represents a non-trivial element in $H_2(M, \partial M; \mathbb{Z}_2)$ and the equation $\partial A' = \partial A_{k+1}$ holds. If the weights of the two annuli were the same, then there would exist a normal isotopy making A and F disjoint. Since this can not occur, A' must have smaller weight than A, which is again a contradiction. □

The number of vertical annuli in our collection is bounded above by the dimension of the vector space $H_2(M, \partial M; \mathbb{Q})$, which, by Poincaré duality, equals the first Betti number $\beta_1(M; \mathbb{Q})$. This is just a dimension of a vector space $H_1(M; \mathbb{Q})$ that is obtained as a quotient of a space which is at most $6t$ dimensional, where t is the number of tetrahedra in the S^1-bundle M. We therefore have $n \leq 6t$.

If the base surface of the bundle $M \to B$ is orientable, then, using proposition 6.3, we can construct our maximal collection $A_1, \ldots, A_n$ of vertical normal annuli that has all the required properties. Furthermore we have control over its normal complexity. Just like, when bounding the normal complexity of the family $\mathcal{A}$ of essential vertical tori in section 5, we can conclude that the number of copies of each normal disc type of the triangulation of M, that are contained in $A_1 \cup \ldots \cup A_n$, is bounded above by $(2 \cdot 2^{11t})^{6t}$.

In case the base surface B is non-orientable, we need to fulfill one more requirement. Namely the complement in M of the first annulus A_1 in our collection has to be an S^1-bundle over an orientable surface. If we are unlucky, none of the annuli that are recursively constructed by proposition 6.3 will have this property. If we nevertheless construct the first g (where g denotes the "non-orientable" genus of the base surface B) vertical annuli $A_1, \ldots, A_g$ in the way described above, using proposition 6.3, then their boundaries are contained in the torus Y_1. But more to the point, the 3-manifold $M_g = M - (A_1 \cup \ldots \cup A_g)$ fibres over a planar, and therefore orientable, connected surface B_g. Now look at the g arcs $p(A_1), \ldots, p(A_g)$ in the surface B, where $p\colon M \to B$ is the bundle projection. Their end points are contained in the circle $p(Y_1)$. By concatenating some subcollection of these arcs, we can obviously construct a single arc α such that the complement $B - \text{int}(\mathcal{N}(\alpha))$ is a compact orientable surface. Also, this construction gives us control over the weight of the vertical annulus $p^{-1}(\alpha)$ above the arc α. This is because the concatenation of arcs can be implemented in the S^1-bundle M by using some of the annuli in $Y_1 - (A_1 \cup \ldots \cup A_g)$ that are pushed slightly into the interior of M. Such annuli will have zero weight.

So in case when M fibres over a non-orientable base space B, we choose the first annulus A_1 to be the smallest weight annulus that satisfies all our requirements. It follows from what was said above that there exists a normal annulus, containing less than $(2 \cdot 2^{11t})^{6t}$ copies of each normal disc type in the triangulation of M, that does the job. This is because $g < n \leq 6t$, where t is the number of tetrahedra in M. The rest of the vertical annuli in the collection $A_1, \ldots, A_n$ can be obtained as before, using proposition 6.3. The number of copies of a given disc type in the surface $A_1 \cup \ldots \cup A_n$, is certainly bounded above by $(2 \cdot 2^{11t})^{12t}$.

Let S be a minimal weight normal section of the triangulated S^1-bundle M, as described by theorem 6.2. Without loss of generality we can assume that the intersection between the surface S and our normal vertical annuli, is transversal and that it misses the 1-skeleton. It is clear that the horizontal surface S can be isotoped so that it intersects the vertical surface $A_1 \cup \ldots \cup A_n$ in precisely n arcs. But we want this isotopy not to increase the weight of either of the two surfaces.

If there exists a simple closed curve of intersection, then it has to be inessential in the annulus it lies in. This is because the generic intersection contains a spanning arc on the annulus. Since S injects on the π_1 level, this simple closed curve has to be homotopically trivial in our section as well. It therefore bounds a disc in S. So the innermost simple closed curve of intersection that bounds the lightest disc in one of the surfaces, has to bound a disc of the same weight in the other surface. This is because otherwise we would have a weight reducing isotopy of one of the surfaces. Moreover, after we do the exchange, i.e. isotope over the 3-ball bounded by the two discs, the surfaces we get are in normal form. Repeating this procedure eliminates all simple closed curves of intersection between S and $A_1 \cup \ldots \cup A_n$. If there are any inessential arcs of intersection in our vertical annuli, then their boundaries have to be contained in $\partial M - Y$. This is because each fibre in $\mathcal{F}$ intersects every simple closed curve in $\mathcal{P}$ at most once. So using the fact that S is ∂-incompressible, we can isotope it like before, so that the intersection $S \cap (A_1 \cup \ldots \cup A_n)$ consists of spanning arcs only. This isotopy will not increase the weight of S. The surface we obtain in the end is still in normal form (otherwise we could decrease its weight further).

All we need now is a way of transforming one possible collection of vertical annuli, that we've constructed, to another. For achieving this we will need the following lemma.

Lemma 6.4 *Let B be a possibly non-orientable bounded surface and let α and β be two properly embedded arcs in B with the properties: the elements of $H_1(B, \partial B; \mathbb{Z}_2)$ carried by arcs α and β are non-zero, $\partial\alpha = \partial\beta$, and both surfaces $B_\alpha = B - \mathrm{int}(\mathcal{N}(\alpha))$ and $B_\beta = B - \mathrm{int}(\mathcal{N}(\beta))$ are orientable. Then there exists a homeomorphism of B which is an identity on the boundary and which maps α onto β.*

Proof. Let B'_α and B'_β denote the surfaces B_α and B_β with discs glued to all of their boundary components. This means that both B'_α and B'_β are closed orientable surfaces. The number of boundary components of B_α and B_β can differ by at most 1. But their Euler characteristics have to be equal: $\chi(B_\alpha) = \chi(B) + 1 = \chi(B_\beta)$. On the other hand the Euler characteristics of B'_α and B'_β are both even, so we must have $\chi(B'_\alpha) = \chi(B'_\beta)$. So B'_α is homeomorphic to B'_β. Moreover B_α and B_β are connected orientable surfaces with the same number of boundary components. They are therefore also homeomorphic.

Let a be one of the two arcs in $\partial B - (\partial\alpha)$. Choose an orientation for it. Notice that a is contained in ∂B_α as well as in ∂B_β. This induces an orientation on one boundary component of B_α and on a boundary component of B_β. Since both of these surfaces are orientable, we can extend this orientation over them. This therefore induces orientations of closed surfaces B'_α and B'_β, and of the simple closed curves ∂B_α and ∂B_β lying in them. By the classification of closed orientable surfaces we can find an orientation-preserving homeomorphism $f' : B'_\alpha \to B'_\beta$ with the following properties: f' maps homotopically trivial simple closed curves ∂B_α in B'_α onto the family $\partial B_\beta \subset B'_\beta$, it preserves the orientations of the curves, and acts as an "identity" both on all components of ∂B_α that are disjoint from α and on the arcs from $\partial B_\alpha - (\partial\alpha)$. The two arcs $\partial B_\alpha \cap \mathcal{N}(\alpha)$ are mapped onto $\partial B_\beta \cap \mathcal{N}(\beta)$ by f', with their induced orientations preserved. All this is possible because we can always isotope a finite disjoint collection of discs in any surface onto any other such collection. Once this is established, everything else follows from the fact that our homeomorphism is orientation preserving.

It is clear that the homeomorphism f' induces a homeomorphism $f : B_\alpha \to B_\beta$ between the two bounded surfaces. It also follows that the map f extends to a homeomorphism from $B = B_\alpha \cup \mathcal{N}(\alpha)$ to the surface $B = B_\beta \cup \mathcal{N}(\beta)$. This extension is an identity on the boundary of B. □

Using lemma 6.4 and a well-known fact that any homeomorphism of the base surface B, that is an identity on the boundary of B, extends to a homeomorphism of the whole S^1-bundle $M \to B$ which is fixed on ∂M, tells us that any collection of our vertical annuli can be obtained from any other collection by such a homeomorphism.

6.3 THE SIMPLIFIED TRIANGULATION

Let T be a triangulation of the S^1-bundle M and let Y be a non-empty collection of boundary components of M. Let S be a normal section of M and let $A_1, \ldots, A_n$ be the normal vertical annuli in M, that intersect S in a disjoint collection of arcs. We also have: $S \cap Y = \mathcal{P}$ and $(A_1 \cup \ldots \cup A_n) \cap Y = \mathcal{F} \cap Y$.

The simplified triangulation of the S^1-bundle M can be defined as follows. Each torus in $\partial M - Y$ contains precisely two circles, one from ∂S and the other from $\partial(A_1 \cup \ldots \cup A_n)$, that intersect in a single point. We can triangulate this torus by 2 triangles. Any component of Y inherits a simplicial structure from the triangulation T_Y and from the families of normal simple closed curves $\mathcal{P}$ and $\mathcal{F}$. We obtain a subdivision of T_Y by coning each of these PL discs from one of the vertices in their boundaries. What we defined so far specifies the triangulation of the boundaries of both our vertical annuli and of the section S. On each annulus there is precisely one spanning arc coming from its intersection with the surface S. The endpoints of this arc are vertices in the already defined triangulation of the boundary of the annulus. So we can take this arc to be an edge of the simplified triangulation. We can then triangulate the vertical annulus by coning from one of endpoints of the arc. The arcs from $S \cap (A_1 \cup \ldots \cup A_n)$ decompose the section S into a single disc. The boundary of this disc is already triangulated. So we can triangulate S by coning this disc from some vertex in its boundary. Finally, the complement of the polyhedron $S \cup A_1 \cup \ldots \cup A_n$ in the S^1-bundle M, is a 3-ball. Since the triangulation of its boundary is already defined, we can take the simplified triangulation of M to be the cone in this 3-ball.

Using the simplified triangulation of the manifold M we can now establish the following key step in the proof of our main theorem.

Theorem 6.5 *Let $M \to B$ be an S^1-bundle over a possibly non-orientable bounded surface B with negative Euler characteristic. Let Y be a non-empty collection of boundary components of the manifold M. Let P and Q be two triangulations of M containing p and q tetrahedra respectively. Assume also that the triangulation of Y induced by P is isotopic (in Y) to the triangulation obtained by restricting the triangulation Q to the components of Y. Then there exists a sequence of Pachner moves of length less than $e^2(400p^2) + e^2(400q^2)$ that transforms the triangulation P into a triangulation which is isomorphic to the triangulation Q. The homeomorphism of M that realizes this simplicial isomorphism maps fibres onto fibres, does not permute the components of ∂M and, when restricted to Y, it is isotopic to the identity on Y.*

Proof. Instead of transforming the triangulation P to Q, we shall change each one to a simplified triangulation. It follows directly from lemma 6.1 and lemma 6.4 that the two simplified triangulations obtained in this way, can be related by a homeomorphism that has the required properties. So what we need to find is a bounded sequence of Pachner moves that will change the triangulation P (and Q) to a simplified triangulation.

We start by fixing a "minimal" family $\mathcal{P}$ of simple closed curves in the components of Y that bounds a normal section in M. Theorem 6.2 implies that there exists a section S with the prescribed boundary which is normal with respect to P, and is also fundamental. We now choose a family $\mathcal{F}$ of fibres in the boundary of M, that consist of the smallest possible number of normal arcs. Using proposition 6.3 we have already constructed a family of vertical normal annuli $A_1, \ldots, A_n$ with bounded complexity, whose boundary equals $\mathcal{F}$. We also know that we can assume that the intersection between the annuli and the section consists of spanning arcs only.

We can now apply lemma 4.1 to the triangulation P and the normal section S. By proposition 2.1 we have that the number of normal discs in S is bounded above by $5p(7p2^{7p})$. (The factor $5p$ comes in since S contains at most 5 distinct disc types in any tetrahedron of P.) So we can subdivide P by making less than $200 \cdot 35p^3 2^{7p}$ Pachner moves. The triangulation we get contains the section S in its 2-skeleton and consists of $20(35p^2 2^{7p} + p) < 20 \cdot 2^{11p}$ tetrahedra. The complementary regions of normal discs of S in the tetrahedra of P are, in this subdivision, triangulated as cones on their boundaries. So we can assume that the surface $A_1 \cup \ldots \cup A_n$ is still normal in the subdivision. Furthermore, since each normal disc in P intersects any 3-simplex of the subdivision in at most one disc, we can bound the normal complexity of our vertical annuli in this new triangulation.

From the discussion in the previous subsection we know that the number of copies of any disc type in the surface $A_1 \cup \ldots \cup A_n$, that is contained in a 3-simplex of P, is bounded above by $(2 \cdot 2^{11p})^{12p}$. This implies that each normal coordinate of the union of vertical annuli, in the subdivided triangulation, is certainly bounded above by $5(2 \cdot 2^{11p})^{12p}$. So the surface $A_1 \cup \ldots \cup A_n$ consists of less than $20 \cdot 2^{11p} 5(2 \cdot 2^{11p})^{12p}$ normal discs in the subdivision. Applying lemma 4.1 again we obtain a further subdivision of P that contains the polyhedron $S \cup A_1 \cup \ldots \cup A_n$ in its 2-skeleton. This subdivision can be constructed by $200(20 \cdot 2^{11p})(20 \cdot 2^{11p} 5(2 \cdot 2^{11p})^{12p})$ Pachner moves. The number of 3-simplices in the subdivision is, according to lemma 4.1, bounded above by $20(20 \cdot 2^{11p} + 20 \cdot 2^{11p} 5(2 \cdot 2^{11p})^{12p}) < 2^{150p^2}$.

We can now apply theorem 5.2 to the complement of the polyhedron $S \cup A_1 \cup \ldots \cup A_n$ in the 3-manifold M. It gives us that after $2^{2^{380p^2}}$ Pachner moves, we can assume that the triangulation of the complement is a cone on the boundary. The number of tetrahedra in this triangulation is bounded above by 2^{150p^2}.

Our next task is to modify the triangulation of both the vertical annuli and the section S. The arcs $S\cap(A_1\cup\ldots\cup A_n)$ decompose all vertical annuli into discs. Since these arcs are already contained in the 1-skeleton of the subdivision, we can apply lemma 4.2 from [8] to simplify the triangulations of these discs. Since the discs have a cone on both sides, each 2-dimensional Pachner moves can be implemented by the 3-dimensional ones in the usual way. This procedure requires linearly many (in the number of tetrahedra) Pachner moves. We can thus assume that our triangulation has this property already. In a similar way we get the required triangulation of the section S. We also need to amalgamate all 1-simplices contained in the arcs $S \cap (A_1 \cup \ldots \cup A_n)$. By lemma 4.1 in [8] we can crush one 1-simplex in an arc by less than a hundred Pachner moves. Since there are again linearly many (in the number of tetrahedra in the subdivision) 1-simplices, we can assume that in our subdivision the arcs from $S \cap (A_1 \cup \ldots \cup A_n)$ are 1-simplices.

The triangulation we have so far is almost isomorphic to the simplified triangulation. They differ only on the boundary components of M that are not contained in Y. Let A be one such torus. Then a boundary component of S and a boundary circle of some vertical annulus decompose A into a disc. We can change the triangulation of this disc into a cone on its boundary using lemma 4.2 in [8]. We can implement the 2-dimensional Pachner moves by the 3-dimensional ones. Here we have to use the 3-dimensional moves that alter the triangulation of the boundary. All this can be done by linearly many moves. Also, we have to change the triangulation of the boundary curves of S and the vertical annulus in A. This too can be achieved by linearly many Pachner moves. This finally brings us to the simplified triangulation.

Putting everything together we can conclude that the whole process that is described above takes not more than $e^2(400p^2)$ Pachner moves. This proves the theorem. □

7 EXCEPTIONAL MANIFOLDS

We are now going to consider the triangulations of the following 3-manifolds: $S^1 \times S^1 \times I$, the unique orientable S^1-bundle over the Moebius band, the doubles of these two manifolds, and a Seifert fibred space over a projective plane with two singular fibres with indices $\frac{1}{2}$ and $-\frac{1}{2}$. The reason why our usual techniques fail in this context is because these manifolds contain horizontal surfaces of zero Euler characteristic.

Let's first deal with the two product manifolds $S^1 \times S^1 \times I$ and $S^1 \times S^1 \times S^1$. Our original strategy was to find essential surfaces that are vertical in a given fibration of the manifold. In these cases the spaces fibre in infinitely many ways, but all of these fibrations are equivalent under a homeomorphism. Loosely speaking what we do now is precisely the opposite from before. We first choose an essential surface and then decide which fibration we are going to use.

Assume that at least one component of the boundary of $S^1 \times S^1 \times I$ has a prescribed simplicial structure. What this means is that we are not allowed to use homeomorphisms of the manifold which are not isotopic to the identity on the boundary. We fix an isotopy class which contains a simple closed curve in the boundary component of the product $S^1 \times S^1 \times I$, that consists of the smallest number of normal arcs among all homotopically non-trivial curves. If both components have a prescribed triangulation, we first choose a component and then fix the curve in it. Let S be a minimal weight normal annulus among all annuli with one boundary circle in the chosen isotopy class and the other circle in the opposite component of $\partial(S^1 \times S^1 \times I)$. If $S = F + G$, then by theorem 2.3 we can conclude that the surfaces F and G are both incompressible and ∂-incompressible. Since their Euler characteristics have to vanish (there are no trivial patches by lemma 2.2) and since $S^1 \times S^1 \times I$ contains no embedded Moebius bands, at least one of them, say F, has to be an annulus. It follows immediately that F has a possibly "shorter" boundary than S. But its weight is certainly smaller than that of S. This contradiction implies that S has to be fundamental.

Now we can use the vertical annulus S to construct a vertical solid torus in the interior of the manifold, without changing the triangulation of the boundary. This is achieved by taking a thin regular neighbourhood of our annulus that stops just before the boundary of the manifold and then using the same technique as in the proof of proposition 5.4. After removing this solid torus from the manifold, the Euler characteristic of the base surface becomes negative and we are back in the "generic" case.

If the triangulation of the boundary is not fixed, we can still find a fundamental vertical annulus as described above. In the product manifold $S^1 \times S^1 \times I$ we can always find a homeomorphism mapping one choice to the other. This proves theorem 3.1 in this exceptional case.

It is well known that any two incompressible tori in $S^1 \times S^1 \times S^1$ are related by a homeomorphism of the ambient manifold. So we are free to pick the one which has smallest weight. Such a torus is fundamental by theorem 2.3, because the manifold contains no embedded Klein bottles. We then look for an essential annulus in the complement of the regular neighbourhood of the torus. This can be done in the same way as above. Again we apply the techniques of proposition 5.4 to obtain a vertical solid torus neighbourhood of a regular fibre. After we remove it we can use the same strategy as in the "generic" case to further simplify the triangulation.

In case of the S^1-bundles over a Moebius band we are going to use the fact that this manifold is homeomorphic to a Seifert fibred space over a disc with two exceptional fibres of index $\frac{1}{2}$. This situation was dealt with in section 5 when we were isolating singular fibres of such Seifert fibred spaces (c.f. the exceptional case in the construction of family $\mathcal{B}$). In case our S^1-bundle has a prescribed triangulation of its boundary, a combination of the strategy for $S^1 \times S^1 \times I$ and the techniques mentioned above prove the main theorem.

The second homology group $H_2(M;\mathbb{Z})$, where M is our S^1-bundle over a Klein bottle, is isomorphic to $\mathbb{Z}$. The vertical torus above the non-separating orientation preserving simple closed curve in the Klein bottle is a generator of $H_2(M;\mathbb{Z})$. The results from [10] imply that the least weight normal representative in this homology class has to be a fundamental torus. Furthermore this fundamental torus is isotopic to the vertical one we started with. Its complement is therefore homeomorphic to the product $S^1 \times S^1 \times I$. So M supports a torus bundle structure with holonomy equal to $-\mathrm{id}$. Notice that any homeomorphism of the fibre $S^1 \times S^1$ extends to the whole of M. We can thus choose a spanning annulus of minimal weight in the complement $S^1 \times S^1 \times I$ which is vertical up to a homeomorphism and which is also fundamental. Removing its regular neighbourhood yields an S^1-bundle over a bounded surface and our usual techniques apply.

Let M be a Seifert fibred space over a projective plane with two singular fibres with indices $\frac{1}{2}$ and $-\frac{1}{2}$. There is only one essential vertical torus A in M up to homeomorphism. It is lying above the simple closed curve in $\mathbb{R}P^2$ that cuts off a disc containing both singular points. Since $H_2(M,\mathbb{Z})$ is trivial, any horizontal torus A' has to be separating. Both components X and Y of the complement are homeomorphic to an I-bundle over a Klein bottle. Any incompressible ∂-incompressible annulus in X and Y is vertical in some Seifert fibration of the pieces. The torus A gives us at least two such annuli, one in X and one in Y. Their respective boundary slopes are identified by the gluing map from ∂X to ∂Y. This means that these fibrations form a new Seifert fibration of M in which A' is vertical. But by the classification of Seifert fiberings it follows that M has a unique Seifert fibration up to homeomorphism. So there is a homeomorphism of M that maps A' onto A. In other words we have proved that there is only one incompressible torus in M up to homeomorphism. It also follows immediately that M contains precisely one injective Klein bottle up to homeomorphism (the boundary of its regular neighbourhood is the unique incompressible torus). We now pick a normal surface which has the smallest weight out of all essential tori and injective Klein bottles in M. By theorem 4.1 from [1] we can conclude that such a surface is fundamental. It is clear now how to construct an essential torus in M. We can then use the same method as in $S^1 \times S^1 \times S^1$ to finish the proof of theorem 3.1 for the exceptional manifolds.

8 CONCLUSION OF THE PROOF OF THEOREM 3.1

We have now developed all the necessary tools to prove theorem 3.1. Section 7 deals with the case when the space M is homeomorphic to one of the exceptional manifolds. The strategy in the generic case is as follows. We start by applying theorem 5.3 to both triangulations P and Q of the Seifert fibred space M. If we are in the situation where there are no singular fibres, we apply proposition 5.4 instead. This produces a subdivision P_2 of the triangulation P containing less than $e^3(500p^2)$ tetrahedra. The process requires less than $e^4(500p^2)$ Pachner moves. The subdivision Q_2, obtained in the similar way from triangulation Q, induces a triangulation in the components of ∂M that belong to Y, which is isotopic to the one induced by P_2. Also both triangulations P_2 and Q_2 contain subcomplexes that triangulate the same S^1-bundle with non-empty boundary. This bundle is obtained by removing the nicely triangulated regular neighbourhoods of the singular (or possibly one regular) fibres in M. It follows from the definition of the subdivisions P_2 and Q_2 that the simplicial structures in the newly acquired boundary components coincide. So we can add them to the collection Y.

Theorem 6.5 can now be applied. It implies that we can transform one of the subcomplexes, triangulating the S^1-bundle, into another by making at most $e^2(400e^3(500p^2)^2)+e^2(400e^3(500q^2)^2)$ Pachner moves. It is also clear that the homeomorphism of the bundle, described in theorem 6.5, extends to the whole 3-manifold M. It obviously has all the required properties that are stated in theorem 3.1. Putting all the above numbers together concludes the proof.

References

[1] Anneke Bart and Martin Scharlemann, *Least weight injective surfaces are fundamental*, Topology and its Applications **69** (1996), 251–264.

[2] Joel Hass, Jeffrey C. Lagarias, and Nicholas Pippenger, *The computational complexity of knot and link problems*, preprint (1998).

[3] Allen Hatcher, *Notes on basic 3-manifold topology*, available from http://www.math.cornell.edu/ hatcher/.

[4] John Hempel, *3-manifolds*, vol. 86, Ann. Math. Studies, Princeton University Press, Princeton, New Jersey, 1976.

[5] William Jaco, *Lectures on three-manifold topology*, vol. 43, American Mathematical Society, 1980.

[6] William Jaco and Jeffrey L. Tollefson, *Algorithms for the complete decomposition of a closed 3-manifold*, Illinois Journal of Mathematics **39** (1995), no. 3, 358–406.

[7] W.B.Raymond Lickorish, *Homeomorphisms of non-orientable two-manifolds*, Proc. Camb. Phil. Soc. **59** (1963), 307–317.

[8] Aleksandar Mijatović, *Simplifying triangulations of s^3*, to appear in Pacific Journal of Math.

[9] U. Pachner, *Pl homeomorphic manifolds are equivalent by elmentary shellings*, Europ. J. Combinatorics **12** (1991), 129–145.

[10] Jeffrey L. Tollefson and Ningyi Wang, *Taut normal surfaces*, Topology **35** (1996), no. 1, 55–75.

Department of Pure Mathematics and Mathematical Statistics, Center for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge, CB3 0WB, UK
E-mail address: a.mijatovic@dpmms.cam.ac.uk