University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1111 Ljubljana, Slovenia

Preprint series, Vol. 41 (2003), 888

CUBIC INFLATION, MIRROR GRAPHS, REGULAR MAPS, AND PARTIAL CUBES

Boštjan Brešar Sandi Klavžar
Alenka Lipovec Bojan Mohar

ISSN 1318-4865

July 24, 2003

Ljubljana, July 24, 2003

Cubic inflation, mirror graphs, regular maps, and partial cubes

Boštjan Brešar*
University of Maribor, FEECS,
Smetanova 17, 2000 Maribor, Slovenia
bostjan.bresar@uni-mb.si

Sandi Klavžar*
Department of Mathematics, University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
sandi.klavzar@uni-mb.si

Alenka Lipovec*
Department of Education, University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
alenka.lipovec@uni-mb.si

Bojan Mohar*
Department of Mathematics, University of Ljubljana
1000 Ljubljana, Slovenia
bojan.mohar@uni-lj.si

July 23, 2003

*Supported by the Ministry of Education, Science and Sport of Slovenia under the grants Z1-3073-0101-01, J1-0504-0101-01, J1-0504-0101-01, and J1–0502–0101–01, respectively.

Abstract

Cubic inflation is an operation that transforms a 2-cell embedded graph G into a cubic graph embedded in the same surface; its result can be described as the dual of the barycentric subdivision of G. New concepts of mirror and pre-mirror graphs are also introduced. They give rise to a characterization of Platonic graphs (i) as pre-mirror graphs and (ii) as planar graphs of minimum degree at least three whose cubic inflation is a mirror graph. As an application, the inflated Platonic graphs yield two new prime cubic partial cubes. Five more sporadic examples of such graphs are also constructed.

Key words: graph embeddings, barycentric subdivision, Platonic graphs, isometric subgraphs, hypercubes, graph automorphisms

AMS subject classification (2000): 05C10, 05C75, 05C12

1 Cubic inflation and mirror graphs

An *embedded graph* or a *map* is a connected graph together with a 2-cell embedding in some closed surface. Let G be a map without vertices of degree one. Then we define the map $\mathcal{CI}(G)$ as follows. First, we replace each vertex $v \in V(G)$ by a cycle Q_v of length $2\deg_G(v)$, and then replace every edge uv of G by two edges joining Q_u and Q_v in such a way that a cubic map on the same surface is obtained in which all cycles Q_v are facial and all edges of G give rise to 4-faces in that map. The result of such a change is shown locally in Figure 1. The resulting map $\mathcal{CI}(G)$ is called the *cubic inflation* of G. The map $\mathcal{CI}(K_4)$ is illustrated on Figure 2; it is interesting to note that $\mathcal{CI}(K_4)$ is isomorphic to the permutahedron Π_3, cf. [18, p.16].

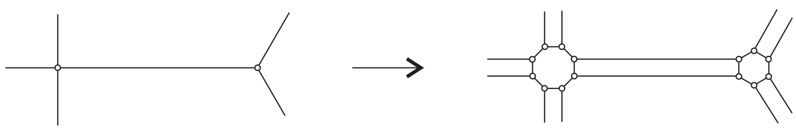

Figure 1: Cubic inflation locally

There is an alternative way to describe the cubic inflation. Let G be an embedded graph. Recall that the *barycentric subdivision* $B(G)$ of G is a triangulation obtained as follows [11]. Subdivide each edge of G by one vertex, and in the interior of each face add a vertex which is joined to all

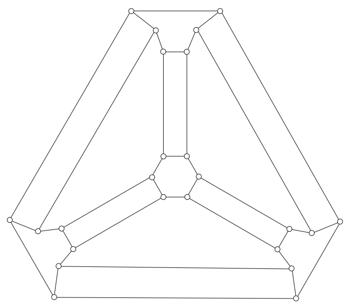

Figure 2: Inflated tetrahedron

vertices (including the new subdivision vertices) on the corresponding face boundary. Denote by G^* the dual map of the map G. The following result follows easily from the fact that $B(G) = B(G^*)$ for every embedded graph G.

Proposition 1 *For every embedded graph G without vertices of degree one, we have*

$$\mathcal{CI}(G) = B(G)^* = \mathcal{CI}(G^*)\,.$$

Our second central concept are mirror graphs. Let $G = (V, E)$ be a connected graph. Call a partition $\mathcal{P} = \{E_1, E_2, \ldots, E_k\}$ of E a *mirror partition* if for every $i \in \{1, \ldots, k\}$, there is an automorphism α_i of G such that

(M1) for every edge $uv \in E_i$, $\alpha_i(u) = v$ and $\alpha_i(v) = u$, and

(M2) $G - E_i$ consists of two connected components G_1^i and G_2^i, and α_i maps G_1^i isomorphically onto G_2^i.

Since α_i is an automorphism of G, E_i is a matching in G joining G_1^i and G_2^i.

A connected graph is a *mirror graph* if it admits a mirror partition. Note that hypercubes and even cycles are mirror graphs. Also, if G_1 and G_2 are mirror graphs, then their Cartesian product $G_1 \Box G_2$ (defined in Section 3) is also a mirror graph. However, as the mirror partition condition is quite strong, mirror graphs that cannot be written as Cartesian products of other graphs are rather specific.

Note that (M2) implies that there is an isomorphism $p_i : G_1^i \to G_2^i$ such that the restriction of p_i to G_{10}^i is an isomorphism of G_{10}^i to G_{20}^i, where G_{10}^i (resp. G_{20}^i) is the subgraph of vertices of G_1^i (resp. G_2^i) that are incident with some edge of E_i.

2 Inflated graphs with mirror partitions

In this section we characterize mirror graphs that can be obtained by the cubic inflation from some planar map.

Let B be an Eulerian graph embedded in some surface. A *straight-ahead walk* in B is a closed walk such that every pair of consecutive edges (including the transition from the last edge back to the initial edge of the walk) passes through the corresponding vertex straight-ahead with respect to the local rotation at that vertex. Two straight-ahead walks are considered the same if one is a cyclic shift or the inverse of a cyclic shift of the other. Then every edge of B determines precisely one straight-ahead walk containing that edge.

Let $B = B(G)$ be the barycentric subdivision of a map G, and let $W = \nu_1\nu_2\ldots\nu_r\nu_1$ be a straight-ahead walk in B. The vertices $\nu_i \in V(B)$ appearing in W correspond to vertices, edges, and faces of G. We say that ν_i *appears essentially* in W if ν_i is either a vertex of G, or ν_i is an edge of G and ν_{i-1} and ν_{i+1} (indices considered modulo r) are faces of G. Then W determines a cyclic sequence of vertices and edges of G that is obtained by taking all essential appearances in W. Every such sequence of vertices and edges of G is said to be an *SA-walk* in G. Note that the collection of those SA-walks in G that contain at least one edge of G induces a partition of $E(G)$.

We are interested in graphs with special SA-walks that are somehow similar to mirror partition condition. If S is an SA-walk in G, let $G - S$ be the subgraph of G obtained by removing all edges and vertices that occur in S. Let us call a plane graph G a *pre-mirror graph* if for every SA-walk S of G:

(PM1) $G - S$ consists of two connected components G_1^S, G_2^S, and

(PM2) there is an automorphism α_S of G that maps G_1^S isomorphically onto G_2^S, where any element of S is invariant under α_S.

The main question in our investigations is which mirror graphs are cubic inflations. Recall that the Platonic maps are K_4, Q_3, octahedron, icosahedron, and dodecahedron.

Theorem 2 *Let G be a map in the plane with minimum vertex degree at least three. Then the following assertions are equivalent.*

(i) *$\mathcal{CI}(G)$ is a mirror graph.*

(ii) *G is a pre-mirror graph.*

(iii) *G is a Platonic graph.*

Proof. (i) $\Rightarrow$ (ii). Let G be a map in the plane with minimum vertex degree at least three such that $\mathcal{CI}(G)$ is a mirror graph.

We first observe that mirror graphs are vertex-transitive. First of all, it is clear by (M2) that every mirror graph H is connected. Let x and y be vertices of H, and let P be a path of length r from x to y. Let $i_1, \ldots, i_r$ be integers in $\{1, \ldots, k\}$ such that the jth edge on P belongs to the part E_{i_j} of the mirror partition, $j = 1, \ldots, r$. Then $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_r}$ is an automorphism of H that maps x to y.

Next we show that each part of the mirror partition of $\mathcal{CI}(G)$ corresponds to an SA-walk of G. So let E be an arbitrary part of the mirror partition of $\mathcal{CI}(G)$, let G_1, G_2 be the components of $\mathcal{CI}(G) - E$, and let α be an automorphism of $\mathcal{CI}(G)$ that maps G_1 isomorphically onto G_2 and interchanges the ends of every edge of E. Let C be an arbitrary facial cycle of $\mathcal{CI}(G)$ that contains an edge e of E, and denote by e' the antipodal edge of e in C. We claim that e' is the only other edge of C which belongs to E. Suppose that $f \in E$ is an edge of C different from e. Note that C is the unique facial cycle of $\mathcal{CI}(G)$ that contains both edges e and f. Mader [10] and Watkins [14] (cf. also [4]) proved that vertex connectivity of a vertex-transitive graph of degree k is at least $2(k+1)/3$. Since $\mathcal{CI}(G)$ is a mirror graph, it is vertex-transitive. Moreover, it is cubic and hence 3-connected. By a theorem of Whitney [16] (cf. also [11]), every automorphism of a 3-connected planar graph maps facial cycles onto facial cycles. Since e and f are preserved by automorphism α, we infer that C is invariant under α. But this is possible only if f is antipodal to e, that is $f = e'$.

Hence we can obtain all edges of E by starting at an arbitrary edge $e \in E$, choosing a facial cycle C that contains e, and then successively picking the antipodal edge in the unique facial cycle in which an edge lies beside the facial cycle we are in. At some point, since $\mathcal{CI}(G)$ is mirror, we arrive at e again. This gives the natural correspondence between the part E and an SA-walk of G. Now, note that only the 4-cycles in $\mathcal{CI}(G)$ are those that correspond to edges of G. As 4-cycles are preserved under α, this implies that there is a mapping in G which maps edges in the same way as α maps corresponding 4-cycles in $\mathcal{CI}(G)$. This mapping is clearly an automorphism of G which preserves elements of the SA-walk. Hence G is pre-mirror.

(ii) $\Rightarrow$ (iii). Let G be a pre-mirror graph and uv an arbitrary edge of G. Note that the existence of automorphisms for any SA-walk implies that G is vertex-transitive. Indeed, the SA-walk starting at an edge uv implies the existence of an automorphism that maps u to v; connectedness of G, and the fact that composition preserves automorphisms, imply the assertion. Since G is vertex-transitive of degree at least 3, we again invoke the result of

Mader and Watkins, deducing that G is 3-connected.

Next, consider the SA-walk S whose consecutive elements are adjacent vertices u and v. As G is pre-mirror, there is an automorphism α of G satisfying (PM2) that fixes u and v. Let F_1, F_2 be the two faces with which uv is incident, and let C_1, C_2 be the corresponding cycles in G. Using the theorem of Whitney again, we derive that only two possibilities for α remain. If α fixes C_1, then it is easy to see that it is the identity, but then α does not satisfy (PM2). Consequently, α maps C_1 onto C_2. Therefore, C_1 and C_2 have the same length.

The above argument holds for any incident facial cycles of the plane embedding of G. Thus the corresponding plane embedding is face-regular, that is, all facial cycles have the same length. It is well-known that the only graphs that are regular of degree at least 3, and have face-regular planar embedding, are the five Platonic graphs.

To complete the proof it is straightforward to verify that if G is any of the five Platonic graphs then $\mathcal{CI}(G)$ is a mirror graph. Thus (iii) implies (i). $\square$

Theorem 2 characterizes plane maps of minimum degree ≥ 3 whose cubic inflations are mirror graphs. They are precisely the Platonic maps. If G is a plane map with minimum degree 2 and its cubic inflation is a mirror graph, then it is easy to see that G is a cycle C_n, $n \geq 3$. Conversely, $\mathcal{CI}(C_n)$ is isomorphic to the Cartesian product of C_{2n} and K_2, and hence it is a mirror graph. However, C_n is not pre-mirror.

A map G is *regular* if its automorphism group acts transitively on the triples $(v, e, F) \in V(G) \times E(G) \times F(G)$ whose vertex v is incident with the edge e, and e is incident with the face F. It is known that regular maps in the sphere are precisely the Platonic maps and all cycles. If we allow graphs with multiple edges and loops, then the set of all regular spherical maps extends with cycles of length 1 and 2 and with *bonds* — dual maps of the cycles.

Corollary 3 *The cubic inflation of a spherical map G with minimum degree at least 2 is a mirror graph if and only if G is a regular spherical map.*

3 Applications to partial cubes

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ where the vertex (a, x) is adjacent to the vertex (b, y) whenever

$ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product of k copies of K_2 is a (*k-dimensional*) *hypercube* or *k-cube* Q_k. A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, where $d_G(u, v)$ denotes the distance between vertices u and v (defined as the length of a shortest (u, v)-path). Isometric subgraphs of hypercubes are called *partial cubes*.

Partial cubes were introduced by Graham and Pollak [5] and intensively studied afterward, see the classical references [2, 3, 17], the book [7], recent studies in [8], and references therein. Distance regular partial cubes are characterized in [15], while in [9] this result is extended to a certain broader metrical hierarchy. For the complexity issues on partial cubes we refer to [6, 7].

One of the most challenging open problems in the area is to classify regular partial cubes, in particular the cubic ones. For one of the most important subclasses of partial cubes—median graphs—Mulder [12] proved that hypercubes are the only regular examples. Besides hypercubes, the even cycles are also regular partial cubes. Observe that the Cartesian product of two (regular) partial cubes is a (regular) partial cube. We say that a regular partial cube is *prime* if it cannot be written as a Cartesian product of two (necessarily regular) partial cubes, each containing at least two vertices.

Restricting to the cubic case, until now only 6 prime examples were known: the generalized Petersen graph $P(10, 3)$ on 20 vertices; the permutahedron Π_3 from Fig. 2; and four more sporadic examples on 30, 36, 42, and 48 vertices, cf. [1]. Moreover, in the latter paper it was verified by a computer search that up to 30 vertices, there are only three (the abovementioned) prime cubic partial cubes.

For the connection between partial cubes and the concepts introduced in the previous section we need the following notion. Two edges $e = xy$ and $f = uv$ of a graph G are in the Djoković-Winkler [3, 17] relation Θ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. Winkler [17] proved that a connected graph is a partial cube if and only if it is bipartite and Θ is transitive (and hence an equivalence relation).

Proposition 4 *Every mirror graph is a partial cube. Moreover, its mirror partition coincides with its Θ-equivalence classes.*

Proof. We first show that a mirror graph G is bipartite. If not, let $C = u_1u_2 \dots u_{2s+1}u_1$ be a shortest odd cycle, and let $u_1u_2 \in E_i$, where E_i is a part of a mirror partition of G. Let $u_1 \in G_1^i$ and $u_2 \in G_2^i$. By (M2), there is another edge u_ru_{r+1} of C that belongs to E_i. Let us assume that vertices of C have been enumerated such that r is minimum possible. Then, clearly, $r \leq s+1$. Since C is a shortest odd cycle, it is isometric in G. Therefore, $d_G(u_1, u_{r+1}) \geq r-1$ and $d_G(u_2, u_r) = r-2$. But this contradicts the fact that $\alpha_i(u_1) = u_2$ and $\alpha_i(u_{r+1}) = u_r$.

Let G be a mirror graph with a mirror partition $\mathcal{P}$. Let uv be an edge of $E_i \in \mathcal{P}$, where $u \in G_1^i$ and $v \in G_2^i$. Let $z \in G_1^i$. We claim that $d(v,z) = d(u,z)+1$. Let P be a (v,z)-geodesic path and let ww' be the first edge of P with $w \in G_2^i$ and $w' \in G_1^i$. Then $d_G(u,w') = d_G(v,w)$ which implies that $d(v,z) > d(u,z)$. Clearly, $d(v,z) \leq d(u,z)+1$.

Let $uv, xy \in E_i$. We may assume that $u,x \in G_1^i, v,y \in G_2^i$. By the above, $d(u,y) = d(u,x)+1$ and $d(v,x) = d(v,y)+1$. Thus $uv\Theta xy$.

Assume that $uv\Theta xy$ where uv is an edge of E_i and $u \in G_1^i$, $v \in G_2^i$. We need to show that $xy \in E_i$ as well. Suppose not, and assume without loss of generality that $x,y \in G_1^i$. Then $d(v,x) = d(u,x)+1$ and $d(v,y) = d(u,y)+1$ thus $d(v,x)+d(u,y) = d(v,y)+d(u,x)$, a contradiction. □

By Theorem 2, $\mathcal{CI}(G)$ is a mirror graph if G is a Platonic graph, hence Proposition 4 implies:

Corollary 5 *Let G be any of the five Platonic graphs. Then $\mathcal{CI}(G)$ is a prime cubic partial cube.*

This corollary gives us two new examples of cubic prime partial cubes. As we already know, $\mathcal{CI}(K_4)$ is the permutahedron Π_3. Since octahedron O is the dual of the 3-cube, Proposition 1 implies that $\mathcal{CI}(Q_3)$ and $\mathcal{CI}(O)$ are isomorphic graphs on 48 vertices. However, this graph is not isomorphic to the graph on 48 vertices from [1] since both are 3-connected but the graph from [1] has a facial cycle of length 12. Clearly, $\mathcal{CI}(Q_3)$ is prime, so it is indeed a new cubic prime partial cube. Similarly, since the icosahedron I is the dual of the dodecahedron, $\mathcal{CI}(I)$ is a new cubic prime partial cube on 120 vertices. Some more examples of such graphs are listed in the next result.

Proposition 6 *Cubic inflations of planar maps shown in Figure 3 (a)–(e) are partial cubes.*

In order to prove Proposition 6, one has to verify that the relation Θ is transitive. This was checked by computer. The cubic inflation of the first graph in Fig. 3(a) has 48 vertices. It is neither isomorphic to $\mathcal{CI}(Q_3)$ (since it has two adjacent 8-faces) nor to the 48-vertex partial cube from [1] (which

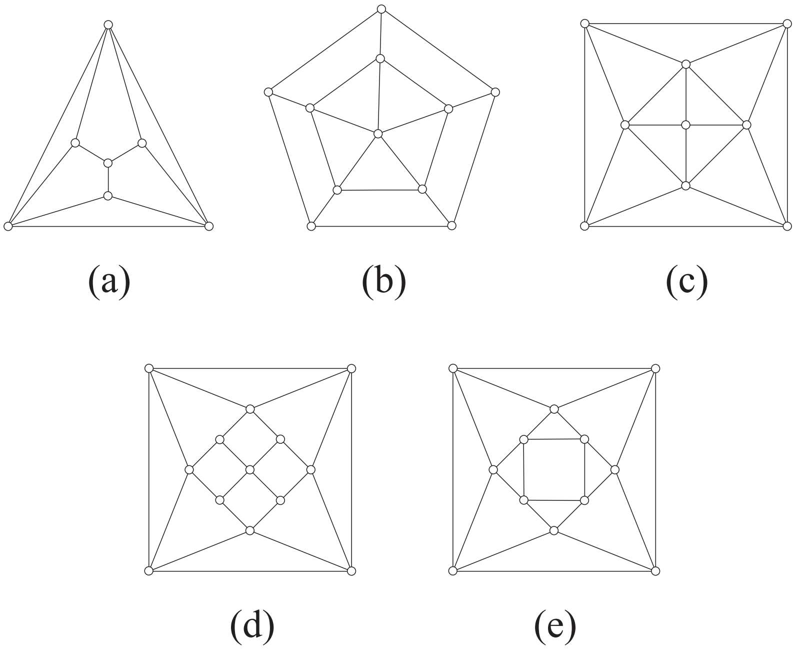

Figure 3: Planar maps yielding cubic partial cubes

has adjacent 4-faces). The graphs (b) and (c) inflate into cubic graphs on 80 vertices while the graphs (d) and (e) inflate to 96 vertices. Since the face lengths of these pairs of graphs are pairwise different, Proposition 6 gives rise to five new examples of prime cubic partial cubes.

We conclude this section by noting that the cubic inflation of every cycle C_n ($n \geq 2$) is also a cubic partial cube. However, $\mathcal{CI}(C_n) = C_{2n} \Box K_2$ is not prime.

4 Concluding remarks

1. We have only partly solved the question for which plane graphs their cubic inflation is a partial cube. Perhaps the following related question could be easier to attack: For which plane graphs their SA-walks are not self-crossing? An SA-walk is called *self-crossing* if there exist two elements of this walk that share a common face, but are not opposite on that face. Note that if G is embedded in the plane and $\mathcal{CI}(G)$ is a partial cube, then the SA-walks of G are not self-crossing. On the other hand, even if no SA-

walk in G is self-crossing, $\mathcal{CI}(G)$ is not necessarily a partial cube. However, answering this question would considerably reduce the class of graphs, for which the first question is relevant.

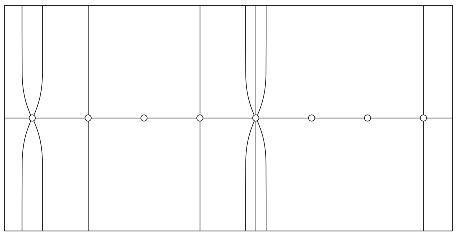

Figure 4: Toroidal examples

2. Cubic inflations on more general surfaces may also yield partial cubes. Such examples are shown by an example in Figure 4. Here we start with an n-cycle embedded as a horizontal "meridian" in the torus, and then add $k \geq 1$ loops embedded as shown in the figure. Each vertex becomes incident with zero or more loops. The graph of the cubic inflation is isomorphic to $C_{2n+2k} \square K_2$, hence it is a (nonprime) partial cube and also a mirror graph. By Proposition 1, the dual map has the same cubic inflation. However, this does not yield new examples, since the dual admits the same structure as exhibited in Figure 4. It would be of certain interest to find nontrivial examples of this kind.

3. Planar pre-mirror graphs (and all cycles) correspond bijectively to mirror graphs which are cubic inflations. The natural question is, are there any nontrivial prime mirror graphs that are not cubic inflations of regular maps? Secondly, are there any prime mirror graphs that are not planar? Is there a similar characterization of those maps on some other surface whose cubic inflation is a mirror graph? Perhaps this could be done by using some kind of SA-walks or their unions.

4. All nontrivial examples of cubic partial cubes that we have obtained so far, have the property that by removing any edge, graphs are no longer partial cubes. Partial cubes with this property are said to be *edge-critical* [1]. Therefore, we also ask: Is every cubic partial cube edge-critical?

5. The following simple result shows that cubic inflations of arbitrary maps are hamiltonian.

Proposition 7 *Let H be the cubic inflation of a graph G embedded in some surface. Then H contains a Hamilton cycle.*

Proof. Let $\mathcal{C}_1$ be the collection of all cycles of H that correspond to vertices of G, and let T be a spanning tree of G. Let $\mathcal{C}_2$ be the set of all 4-cycles of H that correspond to the edges of T. Then the symmetric difference $\mathcal{C}_1 + \mathcal{C}_2$ is a Hamilton cycle of H. □

This result led us to the following

Conjecture 8 *Every cubic partial cube is hamiltonian.*

It is possible that every regular partial cube is hamiltonian. We do not dare to conjecture this since a much weaker well-known conjecture is far from being understood. Namely, the middle level graphs (which are regular partial cubes) are conjectured to be hamiltonian, and no real progress has been made towards a proof of this conjecture. See [13] for more details.

References

[1] C.P. Bonnington, S. Klavžar and A. Lipovec, On cubic and edge-critical isometric subgraphs of hypercubes, *Australas. J. Combin.* **28** (2003), 217–224.

[2] V. Chepoi, *d*-Convexity and isometric subgraphs of Hamming graphs, *Cybernetics* **1** (1988), 6–9.

[3] D. Djoković, Distance preserving subgraphs of hypercubes, *J. Combin. Theory Ser. B* **14** (1973), 263–267.

[4] C. Godsil and G. Royle, *Algebraic Graph Theory*, *Springer*, New York, 2001.

[5] R.L. Graham and H. Pollak, On the addressing problem for loop switching, *Bell System Tech. J.* **50** (1971), 2495–2519.

[6] W. Imrich and S. Klavžar, On the complexity of recognizing Hamming graphs and related classes of graphs, *European J. Combin.* **17** (1996), 209–221.

[7] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, *Wiley*, New York, 2000.

[8] S. Klavžar and H.M. Mulder, Partial cubes and crossing graphs, *SIAM J. Discrete Math.* **15** (2002), 235–251.

[9] J.H. Koolen and S.V. Shpectorov, Distance-regular graphs the distance matrix of which has only one positive eigenvalue, *European J. Combin.* **15** (1994), 269–275.

[10] W. Mader, Über den Zusammenhang symmetrischer Graphen, *Arch. Math. (Basel)* **21** (1970), 331–336.

[11] B. Mohar and C. Thomassen, *Graphs on Surfaces Johns Hopkins University Press*, Baltimore, 2001.

[12] H.M. Mulder, n-Cubes and median graphs, *J. Graph Theory* **4** (1980), 107–110.

[13] C.D. Savage, Long cycles in the middle two levels of the Boolean lattice, *Ars Combin.* **35A** (1993), 97–108.

[14] M.E. Watkins, Connectivity of transitive graphs, *J. Combin. Theory* **8** (1970), 23–29.

[15] P.M. Weichsel, Distance regular subgraphs of a cube, *Discrete Math.* **109** (1992), 297–306.

[16] H. Whitney, 2-isomorphic graphs, *Amer. J. Math.* **55** (1933), 245–254.

[17] P. Winkler, Isometric embeddings in products of complete graphs, *Discrete Appl. Math.* **7** (1984), 221–225.

[18] G.M. Ziegler, *Lectures on Polytopes*, *Springer*, New York, 1998.