University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1111 Ljubljana, Slovenia

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# COLORING THE CLIQUE HYPERGRAPH OF GRAPHS WITHOUT FORBIDDEN STRUCTURE

Sylvain Gravier Riste Škrekovski

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# Coloring the clique hypergraph of graphs without forbidden structure *


Sylvain Gravier[a], and Riste Škrekovski[†b,c]


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[a] CNRS - GeoD research group, Laboratoire Leibniz,
46 avenue Félix Viallet,
38031 Grenoble Cedex, France
Sylvain.Gravier@imag.fr

[b] Department of Mathematics, University of Ljubljana,
Jadranska 19, 1111 Ljubljana, Slovenia
skreko@fmf.uni-lj.si

[c] Department of Applied Mathematics, Charles University,
Malostranské nám. 2/25, 118 00, Prague, Czech Republic.



**Abstract**

For a given simple connected graph $G$ with at least one edge, the clique hypergraph is defined as the one with the same vertex set as $G$ but whose hyperedges are the maximal cliques of $G$. We characterize that $C_5$ is the only graph without induced $P_3 + P_1$, whose clique hypergraph is not 2-colorable. We prove also that the clique hypergraph is 2-colorable providing that the underlined graph is without induced $P_5$ and $C_5$. The later result is best possible in the sense that if we omit some of the forbidden graphs, then the claim is not true.


## 1 Introduction

All graphs considered in this paper are simple, connected and with at least one edge. Let $G$ be such a graph. The hypergraph $\mathcal{H} = \mathcal{H}(G)$ with the same vertex set as $G$, whose hyperedges are the vertex sets of the maximal cliques of $G$, is


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called the *clique hypergraph* (or the *hypergraph of maximal cliques*) of $G$. For a given $\mathcal{H}(G)$, we use to say that $G$ is its *underlined* graph; notice that $G$ is uniquely determined. A *$k$-coloring* of $\mathcal{H}$ is a function $c : V(\mathcal{H}) \to \{1, \ldots, k\}$ such that no hyperedge of $\mathcal{H}$ is monochromatic, i.e., $|c(e)| \geq 2$ for every $e \in E(\mathcal{H})$. If there exists such a function, then we say that $\mathcal{H}$ is *$k$-colorable*, or alternatively, we use to say $G$ is *$k$-clique-colorable*. Otherwise, we say that $G$ is *non-$k$-clique-colorable*. The minimal $k$ such that $\mathcal{H}$ admits a $k$-coloring is called the *chromatic number* of $\mathcal{H}$, and is denoted by $\chi(\mathcal{H})$.

The hypergraph of maximal cliques was introduced by T. Gallai by posing a question about clique-transversals of chordal graphs (see [3]). A pioneer work on the clique-transversals one can find in Erdös, Gallai, and Tuza [4]. Duffus, Sands, Sauer, and Woodrow [2] proved that the elements of every partially ordered set can be 2-colored so that no maximal chain is monochromatic. In other words, the hypergraph $\mathcal{H}(G)$ is 2-colorable if $G$ is a comparability graph. It is well known that comparability graphs are perfect, and hence naturally arise the problem from [2] (see also [8, Problem 15.15]):

**Problem 1.1** *Does there exists a constant $k$ such that every perfect graph is $k$-clique-colorable?*

Recall that a graph is *perfect* if, for every induced subgraph $H$, $\chi(H) = \omega(H)$, that is the chromatic number of $H$ is equal to its maximum clique size.

Above question still remains the central problem in this area, and to its support Duffus, Kierstead, and Trotter [3] proved that the complement of every comparability graph, which is known that is a perfect graph, is 3-clique-colorable. Bacsó, Gravier, Gyárfás, Preissmann, and Sebö [1] proved that that every claw-free perfect graph is 2-clique-colorable. They also proved that every diamond-free perfect graph is 3-clique-colorable provading that each edge of the graph is incident with some triangle, i.e. the graph has no maximal cliques of size 2. Moreover, they proved that almost all perfect graphs are 3-clique-colorable. The later result arises certain believe that the constant $k$ from the above problem is a small integer – perhaps 3! Notice that it is not known the existence of a perfect graph which is not 3-clique-colorable.

The clique hypergraph of planar graphs were also colored and list-colored [11] (see also [10]). In [11] it was shown that every planar graph is 3-clique-colorable. Notice that if $G$ is a triangle-free graph, then $G$ and $\mathcal{H}(G)$ coincide. Thus, the later result extends the well known Grötzsch's theorem, which claims that every triangle-free planar graph is 3-colorable. Moreover, Kratochvíl and Tuza [9] gave an algorithm which decides in a polynomial time whether the clique hypergraph of a planar graph is 2-colorable. Thus, these two results imply that the chromatic number of the clique hypergraph for planar graphs can be determined in a polynomial time.

The recent land-markable result of M. Chudnovsky, N. Robertson. P. Seymour, and R. Thomas that Berge's Strong Perfect Graph Conjecture is true implies that Problem 1.1 deals with the graphs without forbidden induced odd holes and induced odd anti-holes. Recall that a *hole* is an induced chordless cycle with at least four vertices, and *anti-holes* are their complements. Another widely published conjecture about colorings of graphs without forbidden certain structure is Gyárfás' Forbidden Subgraph Conjecture [6] (see also [8, Problem 8.11]), given here in its question form:

**Problem 1.2** *Let $F$ be a forest. Does there exists a function $f_F$ such that $\chi(G) \leq f_F(\omega(G))$ for every graph $G$, which does not contain $F$ as an induced subgraph?*

The above question in the concept of the clique hypergraphs was resolved by Gravier, Hoáng, and Maffray [5]. In particular they proved that for a given graph $F$, there exists a function $f_F$ such that $\chi(\mathcal{H}(G)) \leq f_F(\omega(G))$ for every $F$-free graph $G$ if and only if $F$ is a disjoint union of paths. Using this result and an application of the clique-coloring problem to the usual graph coloring from [7], it follows that every graph $G$ is $(\mu(G)-1)^{\omega(G)-1}$-colorable, where $\mu(G)$ is the number of vertices of the largest induced path in $G$. This improves the bound $\mu(G)^{\omega(G)-1}$ of Gyárfás [6]. Moreover, it is an interesting feedback of the clique-colorings to the proper colorings.

The paper [5] concludes with a table, which expose for certain small graphs $F$, the upper bound $f(F)$ of the clique-chromatic number of the $F$-free graphs. The known $F$-free graphs for which the bound $f(F)$ is attained are also presented. In our work, we characterize that $C_5$ is the only graph without induced $P_3+P_1$, whose clique hypergraph is not 2-colorable. In this way, we purify that for $F := P_3 + P_1$ the bound $f(F) = 3$ is attained only for $C_5$. We prove also that the clique hypergraph is 2-colorable providing that the underlined graph is without induced $P_5$ and $C_5$. The later result is best possible in the sense that if we omit some of the forbidden graphs, the claim is not true. The later result implies that for every graph $G$ without induced $P_5$ and $C_5$, it holds $\chi(G) \leq 2^{\omega(G)} - 1$.

In the paper we use the following notation. Denote by $C_k$ the cycle of length $k$ and denote by $P_k$ the path on $k$ vertices. For a given graph $F$, a graph $G$ is a *$F$-free* if it does not contain an induced subgraph isomorphic to $F$. We say that $G$ is a *$(F_1, \ldots, F_k)$-free* graph, if it is a $F_i$-free graph for each $F_i$ with $i \in \{1, \ldots, k\}$. Denote by $G + H$ the union of vertex-disjoint graphs $G$ and $H$. Denote by $N(v)$ the set of neighbors of a vertex $v$ in a graph $G$. For a set $D$ of vertices of $G$ extend this notion by $N(D) = \cup_{v \in D} N(v)$. We say that a vertex $u$ *dominates* a vertex $v$ if $N(v) \setminus \{u\} \subseteq N(u) \setminus \{v\}$.

# 2 Coloring $P_3 + P_1$-free graphs

In this section, we show that the clique hypergraph of a $P_3 + P_1$-free graph has chromatic number 3 only if the underlined graph is the 5-cycle $C_5$. Notice that $P_3 + P_1$ is also called the *co-paw* graph, as it is the complement of the paw graph.

For the sake of simplicity, the cliques will be considered also as sets of their vertices in the both proofs of this paper.

**Theorem 2.1** *Every $P_3 + P_1$-free graph different from $C_5$ is 2-clique-colorable.*

**Proof.** Suppose that the theorem is false and $G$ is a counterexample with $|V(G)|$ minimum. Thus, $G$ is a $P_3 + P_1$-free graph, distinct from $C_5$, and moreover $\mathcal{H}(G)$ is not 2-colorable.

Let us fix the notation by assuming that $x_1, x_2$ are two adjacent vertices of $G$. Now, denote by $S_3$ the common neighbors of $x_1$ and $x_2$ in $G$. For $i = 1, 2$, let $S_i$ be the set of vertices which are adjacent to $x_i$ and which are not elements of $S_3 \cup \{x_{3-i}\}$. Let $\mathcal{S} = S_1 \cup S_2 \cup S_3$, $\mathcal{A} = \mathcal{S} \cup \{x_1, x_2\}$, and $\mathcal{F} = V(G) \setminus \mathcal{A}$. Denote by $F_0$ the subset of $\mathcal{F}$ which vertices have no neighbor in $S_1 \cup S_2 \cup S_3$. Denote by $F_{1,2,3}$ the set of vertices of $\mathcal{F}$ which have a neighbor in each of $S_1, S_2, S_3$. Let $F_1$ be the subset of $\mathcal{F}$ which vertices has a neighbor in $S_1$ and not in $S_2 \cup S_3$, and let $F_{1,2}$ be the subset of $\mathcal{F}$ which vertices have a neighbor in each of $S_1, S_2$ and no neighbor in $S_3$. We similarly define sets $F_2, F_3, F_{2,3}, F_{1,3}$.

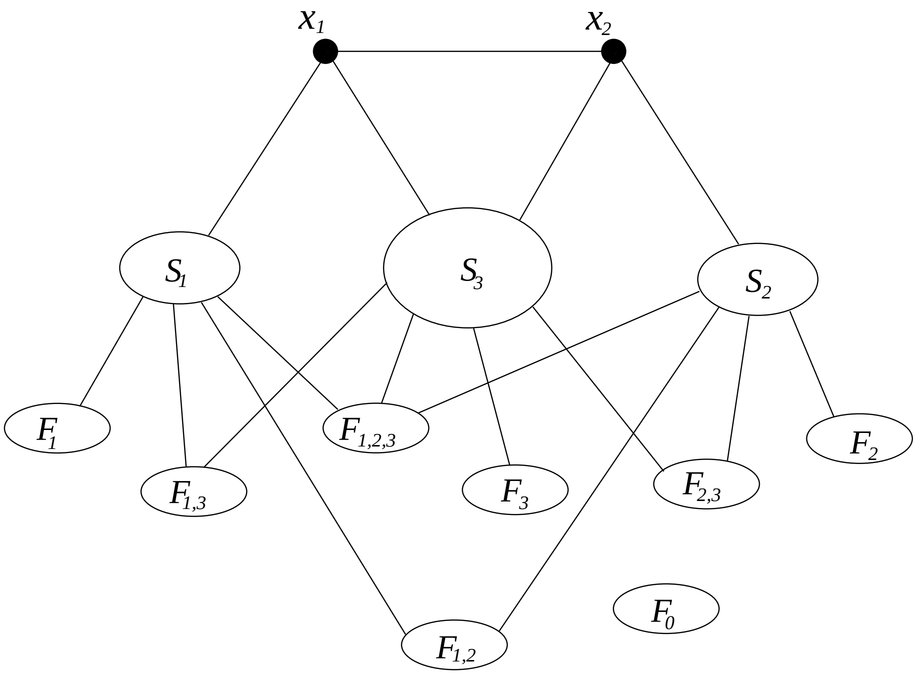


Figure 1: The local structure of $G$

In Fig. 1, the local structure due to above notion is depicted. In this figure, the subsets (or the vertices) of $\{x_1, x_2\}$, $\mathcal{S}$, and $\mathcal{F}$ are connected between if it is assumed the presence of edges in $G$ with end-vertices in the corresponding sets (or vertices). This hierarchy will be also used in the proof of the later result.

In this proof, we follow several simple claims:

**Claim 1.** *The graph $G$ has no two adjacent vertices $u, v$ such that $u$ dominates $v$.*

Suppose that the claim is false, and $u, v$ is such a pair of vertices of $G$. Then, $G - v$ is a $P_3 + P_1$-free graph, and by the minimality of $|V(G)|$, the graph $G - v$ has a 2-clique-coloring $c$ with colors $\{1, 2\}$. Since $u$ dominates $v$, by setting $c(v) := 3 - c(u)$, we extend $c$ to a 2-clique-coloring of $G$, a contradiction.

By Claim 1, we may assume that $S_1$ and $S_2$ are non-empty sets.

**Claim 2.** $F_0 = F_1 = F_2 = F_3 = F_{2,3} = F_{1,3} = \emptyset$.

Suppose that the claim is false and $x$ is a vertex contained in one of these sets. By symmetry, we may assume that $x \in F_0 \cup F_1 \cup F_3 \cup F_{1,3}$. Since, $S_2$ is non-empty, let $s \in S_2$. Now, observe that $x_1, x_2, s, x$ induce a copy of $P_3 + P_1$, a contradiction.

**Claim 3.** *Each vertex from $F_{1,2} \cup F_{1,2,3}$ is adjacent to each vertex from $S_1 \cup S_2$.*

Suppose that the claim is false. Let $v$ be a vertex from $F_{1,2} \cup F_{1,2,3}$ which is non-adjacent to a vertex $u$ from $S_1 \cup S_2$. By the symmetry, we may assume that $u \in S_1$. Then, vertices $v, u, x_1, x_2$ induce a copy of $P_3 + P_1$ in $G$, a contradiction.

**Claim 4.** *$F_{1,2} \cup F_{1,2,3}$ induces a clique in $G$.*

Suppose that $u, v$ are non-adjacent vertices in this set. Let $w$ be a vertex from $S_1$. Then, it is adjacent two both $u$ and $v$ due to Claim 3. But then $x_2, u, v, w$ induce a copy of $P_3 + P_1$ in $G$, a contradiction.

**Claim 5.** $|F_{1,2} \cup F_{1,2,3}| = 1$.

Suppose that $u, v$ are two different vertices from $F_{1,2} \cup F_{1,2,3}$. Denote by $S_{3,2}$ the neighbors of $u$ in $S_3$, and set $S_{3,1} := S_3 \setminus S_{3,2}$. Now, we construct a 2-coloring $c$ of $G$ in the following way: let $c(x_1) = c(u) = 1$, color also each vertex from $S_{3,1} \cup S_2$ by 1, and all the remaining vertices of $G$ by color 2. Thus, $v$ and $x_2$ are colored by 2. We show that $c$ is a 2-clique-coloring of $G$.

Suppose that we have a maximal monochromatic clique $M$. Assume first that it is colored by 1. If this clique does not contain $u$, then all its vertices are contained in $\{x_1\} \cup S_3 \cup S_2$. Since all vertices from this set are adjacent to $x_2$, we obtain a contradiction to the maximality of $M$. If $M$ contains $u$, then $M$ is contained in $\{u\} \cup S_2$ due to the choice of $S_{3,2}$. But in this case, by Claims 3 and 4, vertex $v$ is adjacent to all vertices of $M$, a contradiction.

Suppose now that the vertices of $M$ are colored by 2. Since $x_1$ is colored by 1 and it is adjacent to every vertex from $S_1 \cup S_3 \cup \{x_2\}$, it follows that $M$ is contained in $S_1 \cup S_{3,2} \cup \mathcal{F}$. But in this case, we obtain that $u$ is adjacent to every vertex of $M$ due to Claims 3 and 4. This establish Claim 5.

By Claim 5, we may assume that $F_{1,2} \cup F_{1,2,3} = \{z\}$.

**Claim 6.** *Each of $S_1, S_2$ induces a clique.*

Suppose that $S_1$ or $S_2$ does not induce a clique, say $S_2$. Let $u, v$ be two non-adjacent vertices in $S_2$. Then, vertices $u, v, z, x_1$ induce a copy of $P_3 + P_1$ in $G$, a contradiction.

**Claim 7.** *Each of $S_1$, $S_2$ contains only one vertex.*

Suppose that the claim is false and suppose that $|S_2| \geq 2$. Let $u$ be a vertex in $S_2$. Here, we argue similarly as in Claim 5. Denote by $S_{3,2}$ the neighbors of $u$ in $S_3$, and set $S_{3,1} := S_3 \setminus S_{3,2}$. Color each vertex from $S_{3,1} \cup S_1 \cup \{x_2, u\}$ by 1, and all the remaining vertices of $G$ by color 2. We claim that this coloring is a 2-clique-coloring of $G$.

If there is a maximal clique $M$ whose vertices are colored by 2, then its vertices are contained either in $\{x_1\} \cup S_{3,2}$ or in $\{z\} \cup S_2 \cup S_{3,2}$. In the first case all vertices of $M$ are adjacent to $x_2$, and in the second case, they are adjacent to $u$ due to Claims 3 and 6 and the choice of $S_{3,2}$, a contradiction.

Suppose now that there is a maximal clique $M$ whose vertices are colored by 1. We consider several cases regarding whether $M$ contains $x_2$ and $u$. Assume for a moment that $x_2$ and $u$ belong to $M$. Then, $V(M) = \{x_2, u\}$ due to the choice of $S_{3,2}$. Since $|S_2| \geq 2$, there is a vertex in $S_2$ colored by 2, and adjacent to $x_2$ and $u$, a contradiction. If $x_2$ is in $M$ and $u$ is not in $M$, then vertices of $M$ are contained in $\{x_2\} \cup S_{3,1}$. But then $x_1$ is adjacent to all vertices of $M$. If $u \in V(M)$ and $x_2 \notin V(M)$, then $M$ is contained in $\{u\} \cup S_1$. But in this case, Claim 3 imples that $z$ is adjacent to all vertices of $M$. Finally, if none of $x_2, u$ is contained in $M$, then we infer that $M$ belongs to $S_1 \cup S_{3,1}$. But then all vertices of $M$ are adjacent to $x_1$, a contradiction.

From the above, we conclude that $|S_2| = 1$. Similarly, one can show that $|S_1| = 1$. This proves Claim 7.

By Claim 7, let $S_1 = \{s_1\}$ and $S_2 = \{s_2\}$. By the assumption of the theorem that $G \neq C_5$, we infer $S_3 \neq \emptyset$ or $s_1$ and $s_2$ are adjacent. To conclude the proof, we consider two possibilities regarding whether $s_1$ and $s_2$ are adjacent vertices.

If $s_1$ and $s_2$ are non-adjacent, then $S_3 \neq \emptyset$. Now, color $x_1, x_2, z$ by 1, and all the remaining vertices by 2. If there is maximal clique colored by 1, then its vertices are $x_1, x_2$ but then there is a vertex in $S_3$ adjacent to both of them. Suppose that there is a maximal clique colored by 2. Since $s_1$ and $s_2$ are non-adjacent, vertices of the clique are contained in either $\{s_1\} \cup S_3$ or $\{s_2\} \cup S_3$. But then $x_1$ or $x_2$ is adjacent to all its vertices.

So assume now that $s_1$ and $s_2$ are adjacent vertices. If none of $s_1, s_2$ has a neighbor in $S_3$, then use the same coloring as above and argue similarly that there is no monochromatic maximal clique. Finally, by the symmetry, we may assume that $s_1$ has a neighbor in $S_3$. In this case color $x_1, x_2, s_1, z$ by 1 and all other vertices by 2. A possible maximal clique colored by 2 has all its vertices in $\{s_2\} \cup S_3$ but $x_2$ is adjacent to all of them. And, any maximal clique colored by 1 is of size two. Observe that for any such a pair of vertices there exists a common neighbor colored by 2. This proves the theorem. $\square$

# 3 Coloring $(P_5, C_5)$-free graphs

A result in [5] claims that every $P_4$-free graph is 2-clique-colorable. Weaking the condition of $P_4$-freeness to $P_5$-freeness, it turns that the claim is false, since the 5-cycle $C_5$ is a $P_5$-free graph which is not 2-clique-colorable. In this section, we prove that if a $P_5$-graph is not 2-clique-colorable, then it contains an induced copy of $C_5$. Notice that odd cycles are examples of $C_5$-free graphs which are non-2-clique-colorable. Thus, if we omit some of the forbidden graphs from the theorem below, the claim does not hold. In this sense this theorem is best possible.

We remark here that beside $C_5$, there exist also infinitely many $P_5$-free graphs which are not 2-clique-colorable. Notice that if in a $P_5$-free graph $G$ is replaced a vertex $v$ by a $P_5$-free graph $H$ so that every vertex of $H$ is connected with every neighbor of $v$, then the resulting graph $G^*$ is also $P_5$-free graph. Moreover, if $G$ is not 2-clique-colorable, then also $G^*$ is not 2-clique-colorable. Thus, if we apply this operation repeatedly on $C_5$, we obtain arbitrary many graphs with desired properties. It is not clear, if every $P_5$-free graph which is non-2-clique-colorable can be constructed in this way.

**Theorem 3.1** *Every $(P_5, C_5)$-free graph is 2-clique-colorable.*

We prove the above theorem by contradiction. We suppose that the claim is false and that $G$ is a counterexample with $|V(G)|$ minimum. We use the same notion for vertices $x_1, x_2$, and the sets regarding them given in the second paragraph in the proof of Theorem 2.1. Additionally, let $S_1^*$ be those vertices of $S_1$, which have a neighbor in $\mathcal{F}$. Similarly define $S_2^*$. We use also the partition of $S_3$ into the following four sets $A, B, C$, and $D$:

$$\begin{aligned} A &= \{v \in S_3 \mid S_2^* \subseteq N(v) \text{ and } S_1^* \not\subseteq N(v)\}, \\ B &= \{v \in S_3 \mid S_2^* \not\subseteq N(v) \text{ and } S_1^* \not\subseteq N(v)\}, \\ C &= \{v \in S_3 \mid S_2^* \not\subseteq N(v) \text{ and } S_1^* \subseteq N(v)\}, \\ D &= \{v \in S_3 \mid S_2^* \subseteq N(v) \text{ and } S_1^* \subseteq N(v)\}. \end{aligned}$$

In the proof of Theorem 3.1, we apply the following two lemmas, which are results about the local structure of $P_5$-free graphs due to our notion:

**Lemma 3.2** *Every $P_5$-free connected graph $G$ has the following properties:*

(*a*) *Let $x$ and $y$ be two adjacent vertices of $\mathcal{F} \setminus (F_3 \cup F_0)$. Then,*

$$N(x) \cap (S_1 \cup S_2) = N(y) \cap (S_1 \cup S_2).$$

(*b*) *For every vertex $x \in F_0$, it holds $N(x) \subseteq F_0 \cup F_3$. Moreover, if $x$ and $y$ are two adjacent vertices of $F_0$, then $N(x) \cap F_3 = N(y) \cap F_3$.*

(*c*) *$G$ has no edge with one end-vertex in $F_0 \cup F_3$, and the other end-vertex in $\mathcal{F} \setminus (F_0 \cup F_3)$.*

(*d*) *Suppose that $N(D) \cap \mathcal{F} = \emptyset$. Then, for every two adjacent vertices $x \in F_3$ and $y \in F_3 \cup F_0$, it holds $N(x) \cap S_3 = N(y) \cap S_3$. In particular, $F_0 = \emptyset$.*

(*e*) *Let $x \in S_1 \cup S_2$ be adjacent to a vertex $y \in \mathcal{F}$. Then, $N(y) \cap \mathcal{F} \subseteq N(x) \cap \mathcal{F}$.*

**Proof.** Suppose that (*a*) does not hold for vertices $x$ and $y$ from $\mathcal{F} \setminus (F_3 \cup F_\emptyset)$. Without lose of generality, we may assume that there is a vertex $z \in S_1 \cup S_2$ adjacent to $x$ but non-adjacent to $y$. Now, notice that vertices $x_2, x_1, z, x, y$ induce a copy of $P_5$ in $G$, a contradiction.

Consider now the claim (*b*). If there exists a vertex $x \in F_0$ with a neighbor $z \notin F_0 \cup F_3$, then $z \in \mathcal{F} \setminus (F_3 \cup F_0)$. In this case, $z$ has a neighbor $u \in S_1 \cup S_2$. Observe that $x, z, u, x_1, x_2$ induce a copy of $P_5$. This proves the first part of the claim (*b*).

For the second part of the same claim, assume that $x, y \in F_0$ and $z \in F_3$ are such vertices that $x$ is adjacent to $y$, vertex $z$ is adjacent to $x$, and $z$ is non-adjacent to $y$. Since $z \in F_3$, there exists a vertex $u \in S_3$ adjacent to $z$. In order to establish claim (*b*), observe that vertices $y, x, z, u, x_1$ induce a copy of $P_5$ in $G$.

For the claim (*c*), suppose that $x \in F_0 \cup F_3$ and $y \in \mathcal{F} \setminus (F_0 \cup F_3)$ are adjacent. Then, $y$ has a neighbor $z \in S_1 \cup S_2$. Since $x$ and $z$ are non-adjacent, we conclude that $x, y, z, x_1, x_2$ induce a copy of $P_5$, a contradiction.

Consider the claim (*d*). Suppose that $x \in F_3$ and $y \in F_3 \cup F_0$ are vertices that contradict the claim. Without lose of generality, we may assume that there exists a vertex $z \in S_3$ adjacent to $x$ and non-adjacent to $y$. Since $N(D) \cap \mathcal{F} = \emptyset$, we infer that $z \in A \cup B \cup C$. Hence, there exists a vertex $\bar{z} \in S_1 \cup S_2$, which is non-adjacent to $z$. Assume that $\bar{z} \in S_1$. Since $x, y \in F_0 \cup F_3$, none of these two vertices is adjacent to $\bar{z}$. This implies that $y, x, z, x_1, \bar{z}$ induce a copy of $P_5$ in $G$, a contradiction. This establish the first part of the claim (*d*).

If $F_0 \neq \emptyset$, then there is a vertex $y \in F_0$ adjacent to a vertex $x \in F_3$ due to connectivity of $G$ and the claim (*c*). By the first part of this claim, we infer that $N(x) \cap S_3 = N(y) \cap S_3$. Notice that the right side of this equality is an empty set since $y \in F_0$. From other side, $x \in F_3$ has a neighbor in $S_3$, which implies that $N(x) \cap S_3$ is a non-empty set. This contradiction proves the second part of the claim (*d*).

Finally, for the claim (*e*), suppose that $u \in \mathcal{F}$ is adjacent to $y \in \mathcal{F}$ and non-adjacent to $x \in S_1 \cup S_2$, and suppose that $x$ and $y$ are adjacent. Note that $x$ is adjacent to $x_1$ or $x_2$ (but not both). Also note that neither $u$ nor $y$ is adjacent to $x_1$ or $x_2$. Thus, the following vertices $u, y, x, x_1, x_2$ induce a copy of $P_5$, a contradiction. □

**Lemma 3.3** *Let $M$ be a maximal clique of a $P_5$-free connected graph $G$. Then, $M$ satisfies precisely one of the following conditions:*

(1) $M \subseteq \mathcal{A}$;

(2) $M \cap \mathcal{S} \neq \emptyset$, $M \cap \mathcal{F} \neq \emptyset$ *and* $M \cap (F_3 \cup F_0) = \emptyset$;

(3) $M \cap S_3 \neq \emptyset$, $M \cap F_3 \neq \emptyset$ *and* $M \subseteq S_3 \cup F_3$;

(4) $M \subseteq F_3$;

(5) $M \cap F_0 \neq \emptyset$, $M \cap F_3 \neq \emptyset$, *and* $M \subseteq F_3 \cup F_0$.

*Moreover, if one of the conditions* (4) *and* (5) *holds, then* $N(D) \cap \mathcal{F} \neq \emptyset$.

**Proof.** Suppose that (4) holds and $N(D) \cap \mathcal{F} = \emptyset$. Then, $M$ is contained in a component induced by the vertices of $F_3$. By the connectivity of $G$ and by Lemma 3.2(c), we obtain a vertex $z \in S_3$ adjacent to a vertex of this component. Now, Lemma 3.2(d) implies that $z$ is adjacent to all vertices of $M$, which is a contradiction to the maximality of $M$. Similarly, if (5) holds and $N(D) \cap \mathcal{F} = \emptyset$, then Lemma 3.2(d) implies that $F_0 = \emptyset$. But this is a contradiction to the assumption that $M \cap F_0 \neq \emptyset$. This proves the second part of the lemma.

Now, we prove that $M$ always satisfies precisely one of the five conditions. Since $\mathcal{A} \cap \mathcal{F} = \emptyset$, if (1) holds then none of the other conditions holds. Similarly, since $\mathcal{A} \cap \mathcal{F} = \emptyset$ and $F_0 \cap F_3 = \emptyset$, if (4) holds then all others are excluded. In the sequel, we assume that $M \not\subseteq \mathcal{A}$ and $M \not\subseteq F_3$.

Suppose first that $M$ has a vertex in $F_0$. Then, by Lemma 3.2(b), all its vertices belong to $F_3 \cup F_0$. If $M \subseteq F_0$, then $M$ is contained in a component of $F_0$. Since $G$ is a connected, there is a vertex of this component adjacent to a vertex $z \in F_3$. Now, the second part of Lemma 3.2(b) implies that all vertices of $M$ are adjacent to $z$, a contradiction. So, we conclude that $M$ must have a vertex in $F_3$, and in this way, we encounter condition (5). Notice than neither (2) nor (3) can appear simultaneously with (5). In what follows, assume that $M$ has no vertex in $F_0$.

Suppose now that $M$ has no vertex in $F_3$. Since $M \not\subseteq \mathcal{A}$, we may assume that $M$ has a vertex $\mathcal{F} \setminus (F_0 \cup F_3)$. This implies that none of $x_1, x_2$ is in $M$. If $M \cap \mathcal{S} = \emptyset$, then, by Lemma 3.2(a), all vertices of $M$ are adjacent to a same vertex from $S_1 \cup S_2$. But this contradicts the maximality of $M$. So, we infer that $M \cap \mathcal{S} \neq \emptyset$. In particular, we infer that condition (2) is satisfied. Notice that condition (3) does not hold in the same time.

Finally suppose that $M$ has a vertex in $F_3$. Then, by Lemma 3.2(c), we infer that $M$ has all its vertices in $S_3 \cup F_3 \cup F_0$. Since, we assumed that $M$ is vertex-disjoint from $F_0$ and since $M$ is not contained in $F_3$, we obtain the condition (3). This establish the lemma. □

**Proof of Theorem 3.1.** As we said at the begining of this section, we suppose that the theorem is false, and $G$ is a counterexample with $|V(G)|$ minimum. Then, $G$ is a connected graph on at least three vertices. We us the notation for vertices $x_1, x_2$ and the sets regarding them given in the second paragraph in the proof of Theorem 2.1 and depicted in Fig 1.

**Claim 1.** *Every vertex of $S_1^*$ is adjacent to every vertex of $S_2^*$.*

Suppose that the claim is false, and that vertices $u_1 \in S_1^*$ and $u_2 \in S_2^*$ are non-adjacent. Let $w$ be a neighbor of $u_1$ in $\mathcal{F}$. If $w$ and $u_2$ are adjacent, then vertices $w, u_1, x_1, x_2, u_2$ induce a copy of $C_5$. Otherwise, these vertices induce a copy of $P_5$. This establish Claim 1.

In order to prove the theorem, we distinguish the following two cases:

**Case 1:** $N(D) \cap \mathcal{F} = \emptyset$.

In this case, if $D$ is not an empty set, then each of its elements has no neighbor in $\mathcal{F}$. Color vertices of $\mathcal{A}$ so that each vertex of $\{x_1\} \cup B \cup C \cup D \cup S_2$ is colored by 2, and each vertex of $\{x_2\} \cup A \cup S_1$ is colored by 1. Denote this coloring by $c$. Since each vertex of $\mathcal{S}$ with assigned color $i \in \{1, 2\}$ is adjacent to $x_{3-i}$, it follows that there is no monochromatic clique of $G$, whose all vertices are in $\mathcal{A}$.

Now, we extend $c$ to $\mathcal{F}$. First color every vertex of $F_2 \cup F_{1,2} \cup F_{1,2,3}$ by color 1. Next, color every vertex of $F_1 \cup F_{1,3}$ by 2. Finally, color each $v \in F_3$ by color 2 if $N(v) \cap S_3 \subseteq A$, and otherwise color $v$ by 1. By Lemma 3.2($d$), $F_0 = \emptyset$, and so we have colored all the vertices of $G$.

Let $M$ be an arbitrary maximal clique of $G$ which is monochromatic. By above, $M$ is not contained in $\mathcal{A}$. Since $N(D) \cap \mathcal{F} = \emptyset$, the clique $M$ satisfies one of conditions (2) and (3) of Lemma 3.3. Consequently, we consider these two possibilities:

**Subcase 1.1:** *$M$ satisfies condition* (2) *of Lemma 3.3.*

Consider first the case that all the vertices of $M$ are colored by 1. Then, $M \cap \mathcal{F} \subseteq F_2 \cup F_{1,2} \cup F_{1,2,3}$ and $M \cap \mathcal{S} \subseteq A \cup S_1$. Since $M \cap \mathcal{F} \neq \emptyset$, it follows that $M \cap \mathcal{S} \subseteq A \cup S_1^*$. Let $f$ be a vertex from $M \cap \mathcal{F}$. Then, there is a vertex $s \in S_2^*$ adjacent to $f$. Note that $s \notin M$. By Lemma 3.2($e$), vertex $s$ is adjacent to all vertices of $M \cap \mathcal{F}$, and by Claim 1, it is adjacent to all vertices of $M \cap S_1^*$. Moreover, by the definition of $A$, vertex $s$ is also adjacent to all vertices of $A$. We conclude that $s$ is adjacent to all vertices of $M$. Since $s \notin M$, we obtain a contradiction.

Suppose now that all vertices of $M$ are colored by 2. Then, $M \cap \mathcal{F} \subseteq F_1 \cup F_{1,3}$ and $M \cap \mathcal{S} \subseteq B \cup C \cup D \cup S_2$. Since $M$ has a vertex in $f \in F_1 \cup F_{1,3}$ and since $N(D) \cap \mathcal{F} = \emptyset$, we infer that $M \cap \mathcal{S} \subseteq B \cup C$. By Lemma 3.2($e$), there exists a vertex $s \in S_1^*$ adjacent to all vertices of $M \cap (F_1 \cup F_{1,3})$. By the definition of the set $C$, vertex $s$ is adjacent also to all vertices of $M \cap C$. Now, if $M \cap B = \emptyset$, we conclude that $s$ is adjacent to all vertices of $M$, which is a contradiction. So, assume that there exists a vertex $b \in M \cap B$ which is non-adjacent to $s$. By the definition of $B$, set $S_2^*$ contains a vertex $\bar{b}$ non-adjacent to $b$. By Claim 1, $\bar{b}$ is adjacent to $s$, and by the definitions of $F_1$ and $F_{1,3}$, it is non-adjacent to $f$. Therefore, vertices $s, f, b, x_2, \bar{b}$ induce a copy of $C_5$ in $G$, a contradiction.

**Subcase 1.2:** *$M$ satisfies condition* (3) *of Lemma 3.3.*

Suppose first that all vertices of $M$ are colored by 2. In this case, $M$ contains a vertex $s \in S_3$ and a vertex $f \in F_3$. Since $s$ is colored by 2 and since $N(D) \cap \mathcal{F} = \emptyset$, it follows that $s \in B \cup C$. From other side, as the coloring $c$ is defined, we know that all neighbors of $f$ from $S_3$ are contained in $A$. Hence, we obtain that $f$ and $s$ are non-adjacent vertices of $M$, a contradiction.

Suppose now that vertices of $M$ are colored by 1. Let $f \in F_3$ be a vertex of $M$. Since $f$ is colored by 1, it has a neighbor $v \in B \cup C$ due to the definition of the coloring $c$. By Lemma 3.2($d$), $v$ is adjacent to all vertices from $M \cap F_3$. By maximality of $M$, there exists a vertex $a \in M \cap A$, which is non-adjacent to $v$. By the definitions of $B$ and $C$, there exists a vertex $\bar{v} \in S_2^*$, which is non-adjacent to $v$. Since $a \in A$, vertex $\bar{v}$ is adjacent to $a$. Similarly, by the definition of $A$, there exists a vertex $\bar{a} \in S_1^*$, which is non-adjacent to $a$. Claim 1 implies that vertices $\bar{a}$ and $\bar{v}$ are adjacent. Since $f \in F_3$, it is neither adjacent to $\bar{a}$ nor $\bar{v}$. Finally, regarding whether $\bar{a}$ and $v$ are adjacent, we encounter a copy of $C_5$ or a copy of $P_5$ in $G$, a contradiction.

This establish the case $N(D) \cap \mathcal{F} = \emptyset$.

**Case 2:** $N(D) \cap \mathcal{F} \neq \emptyset$.

Let $d^* \in D$ be a vertex such that the set $N(d^*) \cap \mathcal{F}$ is as large as possible. Notice that $N(d^*) \cap \mathcal{F} \neq \emptyset$. We prove first the following claim:

**Claim 2.** *For every vertex $u \in S_3$ which is non-adjacent to $d^*$, it holds*

$$N(u) \cap \mathcal{F} \subseteq N(d^*) \cap \mathcal{F}.$$

Suppose that the claim is not true for the vertex $u \in S_3$. So, $u$ and $d^*$ are non-adjacent, and there exists a vertex $f \in \mathcal{F}$ adjacent to $u$ but non-adjacent to $d^*$. If $u \notin D$, then there exists a vertex $\bar{u} \in S_i$ with $i \in \{1, 2\}$ such that $\bar{u}$ is non-adjacent to $u$. Note that $d^*$ and $\bar{u}$ are adjacent. Now, observe that regarding whether $\bar{u}$ and $f$ are adjacent, we infer that vertices $f, u, x_{3-i}, d^*, \bar{u}$ induce a copy of $C_5$ or a copy of $P_5$ in $G$, respectively. And, if $u \in D$, then by the maximality of $|N(d^*) \cap \mathcal{F}|$, there exists a vertex $\bar{f} \in N(d^*) \cap \mathcal{F}$, which is non-adjacent to $u$. Again, regarding whether $f$ and $\bar{f}$ are adjacent we obtain that vertices $f, u, x_1, d^*, \bar{f}$ induce a copy of $C_5$ or a copy of $P_5$, respectively. This proves the claim.

Now, consider the coloring $c$ of $G - F_0$ which assigns color 1 to each neighbor of $d^*$, and which assigns color 2 to all other remaining vertices of $G - F_0$. Note that $c(d^*) = 2$. In order to extend $c$ to the vertices of $F_0$, we prove first the following claim:

**Claim 3.** *If there is a vertex $f \in F_0$ adjacent to two non-adjacent vertices $u$ and $v$ of $F_3$, then $c(u) = c(v)$.*

Suppose that the claim is false. We may assume that $f \in F_0$ is adjacent to two vertices $u, v \in F_3$, which are non-adjacent and for which $c(u) \neq c(v)$. Since $c(u) \neq c(v)$, precisely one of vertices $u$ and $v$ is adjacent to $d^*$. Then, vertices $x_1, d^*, u, f, v$ induce a copy of $P_5$ in $G$, a contradiction.

Now, extend the coloring $c$ to vertices of $F_0$ in the following way: if $f \in F_0$ is adjacent to some vertex from $F_3$ which is colored by 1, then color $f$ by 2, otherwise color $f$ by 1. By the connectivity of $G$ and by Lemma 3.2($b$), each vertex of $F_0$ has a neighbor in $F_3$. Thus, a vertex of $F_0$ is colored by 1 if and only if all its neighbors from $F_3$ are colored by 2.

In what follows, we will prove that no maximal clique of $G$ is monochromatic. This will establish the theorem. So, suppose that $M$ is a maximal clique of $G$, which is monochromatic regarding $c$. We consider several possibilities due to Lemma 3.3.

**Subcase 2.1:** *$M$ satisfies condition* (5) *of Lemma 3.3.*

Then, $M$ contains a vertex $f$ from $F_0$. If all vertices of $M$ are colored by 1, then it follows that all neighbors of $f$ in $F_3$ are colored by 2, so $M$ is not monochromatic, a contradiction. Suppose now that all vertices of $M$ are colored by 2. Then, $f$ has neighbor $f^* \in F_3$ colored by 1. Note that by Lemma 3.2($b$), all vertices of $M \cap F_0$ are adjacent to $f^*$. Thus, by the maximality of $M$, there exists a vertex $\bar{f} \in M \cap F_3$ which is non-adjacent to $f^*$. Since $\bar{f}$ and $f^*$ are colored differently, we obtain a contradiction to Claim 3.

Suppose now that $M$ satisfies one of the conditions (1)-(4) of Lemma 3.3. Then, $M \cap F_0 = \emptyset$. Note that if vertices of $M$ are colored by 1, then all of them are adjacent to $d^*$ but this contradicts the maximality of $M$. So we may assume that $M$ is colored by 2.

**Subcase 2.2:** *$M$ satisfies condition* (1) *of Lemma 3.3.*

By the maximality, $M$ contains a vertex $a \in S_1$ and a vertex $b \in S_2$; otherwise $x_1$ or $x_2$ is adjacent to all vertices of $M$. Since $a$ and $b$ are colored 2, each of them is non-adjacent to $d^*$. By the definition of $D$, vertex $d^*$ is adjacent to all vertices of $S_1^*$ and $S_2^*$. So, we conclude that $a \in S_1 \setminus S_1^*$ and $b \in S_2 \setminus S_2^*$. Note that we have assumed $N(D) \cap \mathcal{F} \neq \emptyset$. Hence, $d^*$ has a neighbor in $f \in \mathcal{F}$. Notice that $f$ is non-adjacent each of $a, b$. Thus, $f, d^*, x_2, b, a$ induce a copy of $P_5$, a contradiction.

**Subcase 2.3:** *$M$ satisfies condition* (2) *or* (3) *of Lemma 3.3.*

In this case, $M$ meets both $\mathcal{S}$ and $\mathcal{F}$. Since all vertices of $M$ are colored by 2, each of them is non-adjacent to $d^*$. Recall that $d^*$ is adjacent to all vertices of $S_1^* \cup S_2^*$. This implies that $M \cap \mathcal{S} \subseteq S_3$. Thus, $M$ has a vertex $s \in S_3$, which is non-adjacent to $d^*$. By Claim 2, it holds $N(s) \cap \mathcal{F} \subseteq N(d^*) \cap \mathcal{F}$. In particular, each vertex of the non-empty set $M \cap \mathcal{F}$ is adjacent to $d^*$. But then we obtain a contradiction since each such vertex must be colored by 1.

**Subcase 2.4:** *$M$ satisfies condition* (4) *of Lemma 3.3.*

In this case, $M \subseteq F_3$. Since $M$ is colored by 2, no vertex of $M$ is adjacent to $d^*$. Suppose for a moment that a neighbor $f$ of $d^*$ from $\mathcal{F}$ is adjacent to some vertex $m$ belonging to $M$. By the maximality of $M$, there exists a vertex $m'$ of $M$ which is non-adjacent to $f$. In this case, we obtain that vertices $m', m, f, d^*, x_1$ induce a copy of $P_5$, a contradiction.

We may assume now that every vertex of $N(d^*) \cap \mathcal{F}$ has no neighbor in $M$. Let $m$ be a vertex of $M$. Then, it has a neighbor in $F_3$, say the vertex $s$. By its maximality, the clique $M$ contains a vertex $m'$ non-adjacent to $s$. By Claim 2, vertices $s$ and $d^*$ are adjacent. If $s \notin D$, then there exists a vertex $\bar{s} \in S_1^* \cup S_2^*$, which is non-adjacent to $s$. We may assume that $\bar{s} \in S_1^*$. Notice that $d^*$ is adjacent to $\bar{s}$. Since $m, m' \in F_3$, vertex $\bar{s}$ is non-adjacent to each of $m, m'$. Thus, we infer that $\bar{s}, d^*, s, m, m'$ induce a copy of $P_5$. So, assume now that $s \in D$. Then, the maximality of $N(d^*) \cap \mathcal{F}$ assures us that $d^*$ has a neighbor $f$ in $\mathcal{F}$, which is non-adjacent to $s$. Note that by the assumption at the beginning of this paragraph, $f$ is non-adjacent to $m$ and $m'$. Thus, $m', m, s, d^*, f$ induce a copy of $P_5$.

We conclude that $c$ is a 2-clique-coloring of $G$. This establish Case 2 and also the theorem. □

Let $G$ be a $(P_5, C_5)$-free graph. By a result from [5], it follows that $\chi(G) \leq 3^{\omega(G)-1}$. If $G = K_1$, then $\chi(G) = 1$. Otherwise, $G \neq K_1$, let $G_1, G_2$ be the subgraphs of $G$ induced by the color classes of the 2-clique-coloring of $G$ given by Theorem 3.1. Notice that for each $i \in \{1, 2\}$, it holds $\omega(G_i) < \omega(G)$. Thus, by an inductional hypothesis that $\chi(G_i) \leq 2^{\omega(G_i)-1}$ for each $i = \{1, 2\}$, it follows that

$$\chi(G) \leq \chi(G_1) + \chi(G_2) \leq 2^{\omega(G_1)-1} + 2^{\omega(G_2)-1} \leq 2^{\omega(G)-1}.$$

The above argument proves the following consequence. Perhaps this bound is far from optimal for large $\omega(G)$. But for $\omega(G) = 2$ and 3 it is tight just conisder some bipartite graphs or the anti-hole on seven vertices, respectively.

**Corollary 3.4** *Suppose that $G$ is a $(P_5, C_5)$-free graph. Then, $\chi(G) \leq 2^{\omega(G)-1}$.*

We conclude the paper with a discusion of a related concept to the coloring of the clique hypergraph, which is introduced by Hoáng and McDiarmid [7]. A graph $G$ is called *strongly $k$-divisible*, if every induced connected subgraph of $G$ with at least one edge is $k$-clique-colorable. A *$k$-division* of a graph $G$ is a $k$-coloring so that no clique of maximum size, i.e. of size $\omega(G)$, is monochromatic. And, a graph is *$k$-divisible*, if every induced subgraph with at least one edge has a $k$-division. Obviously, every strongly $k$-divisible graph is $k$-divisible. Specially, of great interest are 2-divisible graphs since the Strong Perfect Graph Conjecture can be restate as: A graph is perfect if and only if it and its complement are 2-divisible graphs (see [7]).

Since every induced graph of a $F$-free graph is also $F$-free graph, Theorem 3.1 implies that that every $(P_5, C_5)$-free graph is (strongly) 2-divisible. And, Theorem 2.1 implies a result of [7] that every $(C_5, P_3+P_1)$-graph is strongly 2-divisible.

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