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ACYCLIC COLORINGS OF LOCALLY PLANAR GRAPHS

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Acyclic colorings of locally planar graphs

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Abstract

It is proved that graphs embedded in a surface with large nonseparating edge-width can be acyclically 8-colored. The condition on large nonseparating edge-width is expressed in terms of non-null-homologous circuits and is a weaker requirement than asking for large edge-width (which is based on homotopy).

1 Introduction

All graphs in this paper are simple. We follow standard terminology. For terms related to graphs embedded in surfaces we refer to [11]. All embeddings of graphs in surfaces are assumed to be 2-cell embeddings. A surface is assumed to be closed (i.e., compact and without boundary) unless stated otherwise. The *Euler genus* of a surface S is the nonnegative integer $g = 2 - \chi(S)$, where $\chi(S)$ is the Euler characteristic of S.

Let c be a coloring of vertices of G. If C is a cycle in G on which only two colors a and b appear, then we say that C is *bi-colored* or (a, b)-*colored* if we need a specific reference to its colors. A coloring of a graph G is *acyclic* if there are no bi-colored cycles. The *acyclic chromatic number* $\chi_{\rm ac}(G)$ of G is the minimum integer k such that G admits an acyclic k-coloring.

Grünbaum [6] proved that every planar graph has an acyclic 9-coloring and conjectured that all planar graphs have acyclic 5-colorings, mentioning that this would imply several known results in point-arboricity. This result was improved, little by little, in a series of papers (Mitchem [10], Albertson

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and Berman [1], Kostochka [8], Borodin [3]), culminating with the following ultimate result:

Theorem 1.1 (Borodin) *Every planar graph has an acyclic 5-coloring.*

This result is best possible since double wheels with at least six vertices (the simplest of which is the octahedron) require 5 colors in any acyclic coloring, as observed already in [6]. Wegner [13] constructed a planar graph in which for every 4-coloring, the union of any two color classes induces a subgraph with a cycle. Kostochka and Melnikov [9] constructed a planar graph G that is triangle-free (and hence $\chi(G) \leq 3$) such that $\chi_{\rm ac}(G) = 5$.

For surfaces other than the plane, Borodin (see [7]) conjectured that the maximum acyclic chromatic number equals the maximum chromatic number of graphs on that surface. Alon, Mohar, and Sanders [2] proved that the acyclic chromatic number of an arbitrary surface with Euler genus g is at most $O(g^{4/7})$. They also proved that this is nearly tight; for every g there are graphs with Euler genus g whose acyclic chromatic number is at least $\Omega(g^{4/7}/(\log g)^{1/7})$. Therefore, the conjecture of Borodin is false for all surfaces with large Euler genus (and may very well be false for all surfaces).

The nonseparating edge-width, $\texttt{new}(G)$, is the length of a shortest surface nonseparating cycle in an embedded graph G. This notion is introduced and discussed in more details in Section 2. The main result of this paper is:

Theorem 1.2 *For every surface S there is a constant w such that every graph G embedded in S with $\texttt{new}(G) \geq w$ is acyclically 8-colorable.*

The proof is given in Section 4. The constant w, depending on the Euler genus g of the surface, is shown to be of order $O(g^3 2^g)$. This is not best possible but our main priority is to keep the proofs as short and elementary as possible. On the other hand, there are graphs with arbitrarily large girth and arbitrarily large chromatic number. Since the nonseparating edge-width (in any embedding) cannot be smaller than the girth, the number w in Theorem 1.2 must depend on the genus of S.

It is likely that the 8-coloring bound of Theorem 1.2 is not best possible. We believe that the best bound may be 6 or even 5, possibly with a different answer for nonorientable and orientable surfaces. We have no clue which one of these possibilities would be the right one to conjecture.

2 Locally planar graphs

There are two widely recognized notions of "local planarity" – that of large edge-width and that of large face-width, see [11]. For coloring problems, the natural one is the large edge-width condition, meaning that all non-contractible cycles of an embedded graph are large. It is known (see, for example, [11] and [4]) that graphs embedded in a fixed surface with sufficiently large edge-width have similar chromatic properties as planar graphs. For example, Thomassen [12] proved that graphs on a fixed surface, embedded with sufficiently large edge-width are 5-colorable.

In this paper we show that locally planar graphs have bounded acyclic chromatic number. We introduce a weaker local planarity condition which is based on homology instead of homotopy. It turns out that this notion is the right one in relation to graph coloring problems. It has been used previously only in the paper by Fisk and Mohar [5].

Let G be a graph that is Π-embedded in some surface S. An Eulerian subgraph C of G (possibly disconnected) is *surface separating* (or *null-homologous*) if there is a set $\mathcal{F}$ of Π-faces such that the edges of C are precisely those edges of the graph which occur precisely once on the boundaries of faces in the set $\mathcal{F}$. The set of vertices, edges and faces in $\mathcal{F}$ form a graph embedded in a bordered surface (where boundary components are allowed to touch) whose boundary is C. We denote this surface by $\mathrm{Int}(C)$ and call it the *interior* of C. The submap consisting of faces (and their incident vertices and edges) that are not in $\mathcal{F}$ is also a bordered surface whose boundary is C. It is denoted by $\mathrm{Ext}(C)$ and called the *exterior* of C. When speaking of interiors and exteriors, we will usually have a vertex $x \notin V(C)$, and then the interior will always be selected so that $x \in \mathrm{Int}(C)$.

If C and D are subgraphs of G, then their *sum* $C + D$ is the subgraph whose edge-set is the symmetric difference of $E(C)$ and $E(D)$. If C and D are surface separating Eulerian subgraphs, then also their sum is a surface separating Eulerian subgraph. We will refer to this fact as the *3P-property*, the name coming from its relation to the 3-path-property (see [11]). The 3P-property implies that every surface separating Eulerian subgraph of G with the minimum number of edges is an induced cycle of G.

The *nonseparating edge-width* of a Π-embedded graph G, denoted by $\mathtt{new}(G, \Pi)$ (or just $\mathtt{new}(G)$ if the embedding Π is clear from the context), is the length of a shortest surface nonseparating cycle in G. If there are no surface nonseparating cycles, then we set $\mathtt{new}(G, \Pi) = \infty$; this happens if and only if the embedding has genus 0. Clearly, $\mathtt{new}(G, \Pi)$ is always greater or equal to the edge-width of the embedding.

Next, we prove that graphs with large nonseparating edge-width have some properties that are known for graphs with large edge-width.

Lemma 2.1 *Suppose that G is a triangulation of a surface of Euler genus $g \geq 1$, x is a vertex of G, and $q \leq \frac{1}{2}\mathtt{new}(G) - 1$ is a positive integer.*

(a) *If C is an Eulerian graph in G whose vertices are all at distance at most q form x, then C is surface separating.*

(b) *There is a uniquely determined collection of edge-disjoint surface separating cycles $B_1, \ldots, B_p$ $(1 \leq p \leq g)$ whose vertices are all at distance q from x such that $x \in \mathrm{Int}(B_i)$, and such that $\mathrm{Ext}(B_i)$ is not a disk, for $i = 1, \ldots, p$. Any two of these cycles have at most one vertex in common, and $\mathrm{Ext}(B_i) \cap \mathrm{Ext}(B_j) = B_i \cap B_j$ for $1 \leq i < j \leq p$. Moreover, the induced embedding of $\cap_{i=1}^{p} \mathrm{Int}(B_i)$ has genus 0. Under this embedding all cycles $B_1, \ldots, B_p$ are facial.*

Proof. (a) For every vertex $v \in V(C)$, let P_v be a shortest path in G from v to x. If $e = uv \in E(C)$, let W_e be the closed walk in G consisting of e, P_u, and P_v. Clearly, the length of W_e is at most $2q + 1 < \mathtt{new}(G)$, hence the Eulerian graph C_e corresponding to W_e is surface separating. By consecutively applying the 3P-property, we conclude that the sum of all C_e $(e \in E(C))$ is also surface separating. Since C is Eulerian, this sum is easily seen to be equal to C. This completes the proof.

(b) Let $\mathcal{F}$ be the collection of all faces of G that contain at least one vertex whose distance from x is less than q. Since G is a triangulation, the boundary B of $\mathcal{F}$ consists only of edges whose endvertices are at distance q from x. Clearly, B is an Eulerian graph. Let C be a cycle in B. By (a), C is surface separating. We define its interior so that $x \in \mathrm{Int}(C)$. If $\mathrm{Ext}(C)$ is a disk, add all faces in $\mathrm{Ext}(C)$ into $\mathcal{F}$ and repeat the argument with the new set $\mathcal{F}$. Let us observe that after such a change, precisely the edges of C disappear from the boundary of $\mathcal{F}$. By the 3P-property, every cycle in the extended set $\mathcal{F}$ is still surface separating. Therefore, the boundary of $\mathcal{F}$ cannot vanish (otherwise, all cycles in G would be surface separating and this would contradict the assumption that $g \geq 1$).

We end up with a set $\mathcal{F}$ whose nonempty boundary B has the property that every one of its cycles has a nondisk exterior. It is easy to see that if C is a cycle in B, then its exterior cannot contain a face that is in $\mathcal{F}$. This implies that B has a unique decomposition into edge-disjoint cycles $B_1, \ldots, B_p$ whose exteriors are pairwise disjoint, except possibly for one

vertex that a pair of cycles may have in common. By the additivity of the Euler genus, we conclude that $p \leq g$. □

We further transform the cycles $B_1, \ldots, B_p$ in Lemma 2.1(b). Suppose that B_i $(1 \leq i \leq p)$ has a chord e. Let D_1, D_2 be the two cycles in $B_i + e$ distinct from B_i. By Lemma 2.1(a), these cycles are surface separating. If one of their exteriors, say $\mathrm{Ext}(D_2)$, is a disk, then we replace B_i with the other cycle D_1 and continue the reduction with the new cycle until every remaining chord gives rise only to nondisk exteriors. It is easy to see that the resulting collection of cycles, $Q_1, \ldots, Q_p$ is uniquely determined (the cycles do not depend on the order used when processing the chords). They are called the *q-canonical cycles* for x in G, and it is clear that they have the same properties as stated in Lemma 2.1(b) for the cycles $B_1, \ldots, B_p$. Let us remark that this generalizes the definition of the q-canonical cycle in [12].

Every $(q+1)$-canonical cycle Q' is in the exterior of some q-canonical cycle Q. We say that Q' is a *successor* of Q. A q-canonical cycle is *clean* if it has precisely one successor. In particular, a clean canonical cycle is an induced cycle.

Suppose that Q is a clean q-canonical cycle for x and let Q' be its successor. Since Q' is $(q+1)$-canonical, every vertex v' of Q' has at least one incident edge whose other endvertex is in Q. Let $K = \mathrm{Ext}(Q) \cap \mathrm{Int}(Q')$ be the cylinder bounded by Q and Q'. If K has no separating triangles, then the above mentioned property of vertices of Q' implies that $K - V(Q')$ is 2-connected. The boundary of the face obtained after deleting Q' is therefore bounded by a cycle Q'', which we call the *repaired q-canonical cycle* corresponding to Q. We will make use of such cycles mainly because of the following property:

Lemma 2.2 *Suppose that Q is a clean q-canonical cycle for x in a triangulation G. Let Q' be the successor of Q. Suppose that there are no separating triangles in the cylinder $K = \mathrm{Ext}(Q) \cap \mathrm{Int}(Q')$. Then there exists the repaired q-canonical cycle $Q'' \subseteq K - Q'$, and in the cylinder between Q'' and Q' there are only edges joining Q'' and Q'.*

A *k-cylinder* (or a *generalized cylinder*) is a connected graph D embedded in the plane with k distinguished facial walks $D_1, \ldots, D_k$, called the *boundary parts* of D, such that all faces of D distinct from the boundary parts are triangles. If every two distinct boundary parts are at distance at least w in D, then D is said to be *w-wide*.

Lemma 2.1 shows that after cutting the surface along the q-canonical cycles $Q_1, \ldots, Q_p$ (and removing their exteriors), a generalized p-cylinder is obtained. We need slightly more.

Lemma 2.3 *Let G be a triangulation of Euler genus $g \geq 1$ and let $x \in V(G)$. Let $w \geq 1$ and $a \geq 0$ be integers, and let $k = \lceil \frac{1}{2}w \rceil + 1$. Suppose that $\mathtt{new}(G) \geq 2kg + 2a + 4$. Then there is an integer q, $k + a \leq q \leq kg + a$, such that the q-canonical cycles for x are all clean and at distance at least w from each other. In particular, the intersection of the interiors of the q-canonical cycles is a w-wide generalized cylinder containing x.*

Proof. By Lemma 2.1, the q-canonical cycles exist for every $q \leq kg + a + 1$, and let us denote their number by $p(q)$. By Lemma 2.1(b), $p(q) \leq g$. Clearly, every $(q-1)$-canonical cycle has at least one successor, hence $p(1) \leq p(2) \leq \cdots \leq p(kg + a + 1)$. If two of the q-canonical cycles are at distance less than w, then they are successors of a common $(q - k + 1)$-canonical cycle (if $q \geq k$; and are successors of a common 1-canonical cycle if $q < k$), hence $p(q) \geq p(q - k + 1) + 1$. Similarly, if some q-canonical cycle is not clean, then $p(q + 1) \geq p(q) + 1 \geq p(q - k + 1) + 1$. If the conclusion of the lemma would not hold, then this would imply that

$$2 \leq p(k + a + 1) \leq p(2k + a + 1) - 1 \leq \cdots \leq p(gk + a + 1) - (g - 1).$$

However, this would show that $p(kg + a + 1) > g$, which is a contradiction. □

If $C_1, \ldots, C_k$ is a collection of pairwise disjoint cycles in an embedded graph G, then we say that these cycles form a *planarizing collection of cycles* if cutting along all of $C_1, \ldots, C_k$ results in a connected graph G' embedded in the sphere. Since cutting along a surface nonseparating cycle C_i reduces the Euler genus by 1 (if C_i is 1-sided) or 2 (if C_i is 2-sided), it is clear that $\frac{1}{2}g \leq k \leq g$, where g is the Euler genus of the embedding of G.

Let G' be the embedded graph obtained after cutting the surface along disjoint cycles $C_1, \ldots, C_k$. Every twosided cycle C_i gives rise to two facial cycles C_i', C_i'' in G', called *copies* of C_i. Similarly, a onesided cycle C_i gives rise to a single facial cycle C_i' in G' which doubly covers C_i and is also referred to as the *copy* of C_i. We say that $C_1, \ldots, C_k$ are *d-apart* if the distance in G' between copies of cycles $C_1, \ldots, C_k$ is at least d. Thomassen [12] proved that every triangulation with large edge-width contains a planarizing collection of cycles that are far apart. We prove an extension of this result to graphs with large nonseparating edge-width.

Theorem 2.4 *If $d \geq 1$ and $g \geq 1$ are integers and G is a triangulation of a surface with Euler genus g and with $\mathtt{new}(G) \geq (2d+3)(2^{g+3}+g)$, then G contains a planarizing collection of induced surface nonseparating cycles $C_1, \ldots, C_k$ that are d-apart.*

Proof. The proof is by induction on g. Let C be a shortest noncontractible cycle in G. If $|C| \geq (d+1)2^{g+3}$, then the edge-width of G is at least $(d+1)2^{g+3}$, and the required planarizing cycles exist as shown by Thomassen [12]. (Thomassen's proof, see also [11, Theorem 5.11.1], is given only for orientable surfaces, but it can be extended to nonorientable ones.) Hence, we may assume that $|C| < (d+1)2^{g+3}$, so that C is surface separating and hence $g \geq 2$. For $i = 1, 2$, let S_i be the two surfaces obtained after cutting G along C, let G_i be the corresponding graphs and D_i be the copy of C in G_i. Since C is noncontractible, the Euler genus g_i of S_i is smaller than g. We add a new vertex x_i to G_i and join it to all vertices on D_i, so that we get a new triangulation G_i' of S_i.

Let R be a shortest surface nonseparating cycle in G_i'. If $x_i \notin V(R)$, let $R' = R$; otherwise, let R' be the closed walk obtained from R by replacing the two edges incident with x_i by the shorter of the two segments on D_i joining the ends of those edges on D_i. By the 3P-property, R is an induced cycle. Therefore, R' is also a cycle in G. By the 3P-property, R' is surface nonseparating in G_i and hence also in G. Therefore,

$$\mathtt{new}(G_i') = |R| \geq |R'| - \tfrac{1}{2}|C| + 2 \geq \mathtt{new}(G) - \tfrac{1}{2}(d+1)2^{g+3} + 2. \qquad (1)$$

Next, we apply Lemma 2.3 (with $w = d$ and $a = 0$). Since $\mathtt{new}(G_i') \geq (d+3)g_i + 4$ (which follows from (1)), there is a q, $k \leq q \leq kg$, where $k = \lceil \frac{1}{2}d \rceil + 1$, such that the q-canonical cycles $B_1, \ldots, B_p$ for x_i are clean and at least d-apart. For $j = 1, \ldots, p$, let $M_j = \mathrm{Ext}(B_j)$, and let M_j' be the triangulation obtained from M_j by adding a new vertex y_j joined to all vertices of B_j.

Let R be a shortest surface nonseparating cycle in M_j'. If $y_j \notin V(R)$, let $R' = R$; otherwise, consider a closed walk that is obtained from R by replacing the two edges incident with y_j by paths in G_i' of length q to x_i. By the 3P-property, this walk contains a surface nonseparating cycle R'. Using (1), we estimate:

$$\begin{aligned} \mathtt{new}(M_j') &= |R| \geq |R'| - 2q + 2 \geq \mathtt{new}(G_i') - 2kg + 2 \\ &\geq (2d+3)(2^{g+3}+g) - (d+1)2^{g+2} - (d+3)g + 4 \qquad (2) \\ &\geq (2(d+1)+3)(2^{g+2}+g-1). \end{aligned}$$

By the induction hypothesis, M'_j has a set of induced planarizing cycles that are $(d+1)$-apart. If one of these cycles contains y_j, then we replace its edges incident with y_j by a segment of B_j and obtain another collection of planarizing cycles that are d-apart. If the new cycle C has a chord e, it can be replaced by one of the other two cycles of $C+e$. This follows from the 3P-property of surface nonseparating cycles in the surface obtained after cutting along all cycles in the planarizing collection distinct from C. Thus, we can achieve that the planarizing cycles are induced.

It is easy to see that the union of the planarizing collections of cycles for G_1 and for G_2 is a planarizing collection for G and, clearly, its members are d-apart. □

It is worth mentioning that the above proof is essentially self-contained since the application of [12] at its beginning can be replaced by nearly the same arguments, as used when C was surface separating, also in the case when it was not.

A generalized *Möbius band* is a connected graph D embedded in the projective plane with $p \geq 1$ distinguished facial walks $C_1, \ldots, C_p$, called the *boundary parts* of D, such that all faces in D distinct from $C_1, \ldots, C_p$ are triangles. If the distance between any two boundary parts of D is at least w, then D is said to be *w-wide*. If D has a 1-sided cycle whose distance from $C_1, \ldots, C_p$ is at least d, then D is said to be *d-deep*.

Corollary 2.5 *Let w and g be positive integers and let G be a triangulation of Euler genus g such that $\mathtt{new}(G) \geq 4(w+3)g(2^{g+3}+g)$ and such that every contractible 3-cycle of G is a facial triangle. Then the vertex set of G can be partitioned into sets $U_1, \ldots, U_r$ and $V_1, \ldots, V_h$ such that the following holds:*

(a) *For every $j = 1, \ldots, r$, the subgraph D'_j induced on U_j is a w-wide generalized cylinder with at most g boundary parts.*

(b) *For every $i = 1, \ldots, h$, the induced subgraph D_i on vertices V_i is either a w-wide generalized cylinder with at most $2g-2$ boundary parts, or a w-wide and w-deep generalized Möbius band with at most $g-1$ boundary parts.*

(c) *Every edge that is not in one of $D'_1, \ldots, D'_r$ and not in $D_1, \ldots, D_h$ connects a boundary part of some D'_j with a boundary part of some D_i. For every boundary part Q' of D'_j, there is a unique boundary part Q of some D_i that is adjacent to Q', and $Q \cup Q'$ together with the edges joining Q and Q' form a 2-cylinder with boundary parts Q and Q'.*

Proof. The case when $g = 1$ has to be done separately and is left to the reader. So we assume that $g \geq 2$. By Theorem 2.4, G has a collection of planarizing cycles $C_1, \ldots, C_h$ that are d-apart, where $d = 2(w+3)g - 2 = 2(w+3)(g-1) + 2w + 1$. Let us cut the surface along the planarizing cycles to obtain a d-wide g-cylinder. If we triangulate one of its boundary parts by adding a new vertex x and paste a very wide Möbius bands in all other boundary parts, we obtain a triangulation H of Euler genus $g-1$ whose nonseparating edge-width is as large as we want. By Lemma 2.3, there is an integer q, $k+l \leq q \leq k(g-1)+l$, where $k = \lceil \frac{1}{2}w \rceil + 1$ and $l = \lfloor \frac{1}{2}w \rfloor - 1$, such that the q-canonical cycles for x are clean and w-apart. By Lemma 2.2, there exist the repaired q-canonical cycles. Let us consider the generalized cylinder containing x whose boundary parts are the repaired q-canonical cycles for x.

If C_i is 2-sided, let $D(C_i)$ be the union of such generalized cylinders for both copies of C_i. Clearly, D_i is a w-wide generalized cylinder in G. If C_i is 1-sided, the generalized cylinder for its copy determines a w-wide and w-deep generalized Möbius band $D(C_i)$ in G (since $q \geq k+l = w$). Finally, consider the $(q+1)$-canonical cycles for x, and let Q' be the family of all such cycles taken for all copies of the cycles C_i, $i = 1, \ldots, h$. (Of course, different copies of the cycles C_i may have different values of Q.) Since $C_i \subseteq D(C_i)$ and $C_1, \ldots, C_h$ are planarizing, Q is the boundary of one or more generalized cylinders $D'_1, \ldots, D'_r$ that are disjoint from $D(C_1), \ldots, D(C_h)$ and satisfy (c). Since $2(q+1) + w \leq d$, every D'_j is w-wide.

By Lemma 2.1(b), the number of q-canonical cycles for x in H is at most the Euler genus of H which is equal to $g-1$. This implies that $D(C_i)$ has at most $g-1$ boundary parts if C_i is 1-sided, and has at most $2(g-1)$ boundary parts if it is 2-sided. Every D'_j and every generalized cylinder in H corresponding to a copy of C_i have at most one adjacent pair of boundary parts. Since the number of copies of cycles $C_1, \ldots, C_h$ is equal to g, D'_j has at most g boundary parts. This completes the proof. □

3 Permuting colors in a generalized cylinder

Let $E_n = \{1, \ldots, n\}$. Let $V_{k,n}$ be the set of all pairs (A, a) where A is a k-subset of E_n and $a \in E_n \setminus A$. Let $\Pi_{k,n}$ be the graph whose vertex set is $V_{k,n}$ and two vertices (A, a) and (B, b) are adjacent if $a \notin B$ and $b \notin A$.

Lemma 3.1 *Let (A, a) and (B, b) be vertices of $\Pi_{4,8}$. Then their distance in $\Pi_{4,8}$ is at most 3, and is equal to 3 if and only if $A \cap B = \emptyset$. There is a walk*

from (A,a) *to* (B,b) *of length* 3, $(A,a)=(A_0,a_0),(A_1,a_1),(A_2,a_2),(A_3,a_3)=(B,b)$, *such that for every* $c\in E_8$, *there exists an* $i\in\{0,1,2,3\}$ *for which* $c\notin A_i\cup\{a_i\}$.

Proof. We may assume that $(A,a)=(1234,5)$ (where we use the short notation 1234 for the set $\{1,2,3,4\}$). Let $r=|B\setminus(A\cup\{a\})|$. We will exhibit the color pairs (A_i,a_i) depending on the value of r; in each case, only nonisomorphic cases will be treated. The claim that the distance between (A,a) and (B,b) is 3 if and only if $A\cap B=\emptyset$ is left as an exercise.

$(r=0)$: In this case we may assume that $(B,b)=(123x,y)$, where $xy\in\{45,46,54,56\}$. Let z be the element in $\{4,5,6\}\setminus\{x,y\}$. The required walk is:

$$(1234,5)-(1267,8)-(37xz,8)-(123x,y).$$

$(r=1)$: In this case we may assume that $(B,b)=(126x,y)$, where $xy\in\{34,35,37,53,57\}$. Let z,w be the elements of $\{3,4,5,7\}\setminus\{x,y\}$. The required walk is:

$$(1234,5)-(1267,8)-(6xzw,8)-(126x,y).$$

$(r=2)$: In this case we may assume that $(B,b)=(167x,y)$, where $xy\in\{23,25,28,52,58\}$. Let z be an element of the set $\{2,5,8\}$ that is distinct from x and y. The required walk is:

$$(1234,5)-(3467,8)-(167x,z)-(167x,y).$$

$(r=3)$: In this case we may assume that $(B,b)=(567x,y)$, where $xy\in\{12,15,51\}$. The required walk is:

$$(1234,5)-(3678,5)-(3678,2)-(678x,y).$$

□

Let (A,a) and (B,b) be adjacent color pairs in $\Pi_{4,8}$ and let $t=|A\cap B|$. Let us enumerate the elements of A and B, respectively, as $a_1,\ldots,a_4$ and $b_1,\ldots,b_4$ such that $a_i=b_i$ for $i=1,\ldots,t$. Let π be the permutation of E_8 which interchanges a_j with b_j for $j=t+1,\ldots,4$, interchanges a and b and leaves all other colors fixed. Every such permutation is said to be *compatible* with the (directed) edge $(A,a)(B,b)$ of $\Pi_{4,8}$.

Lemma 3.2 *Suppose that* c *is an acyclic 8-coloring of a graph* D. *Let* (A,a) *and* (B,b) *be adjacent color pairs of* $\Pi_{4,8}$ *and let* π *be a compatible*

permutation. Let $U \subseteq V(D)$ be a set of vertices of D such that their colors and the colors of all vertices at distance at most two from U are all in $A \cup \{a\}$. Let $c' : V(D) \to E_8$ be defined by $c'(v) = c(v)$ if $v \notin U$ and $c'(v) = \pi(c(v))$ if $v \in U$. Then c' is an acyclic 8-coloring of D.

Proof. Let us first verify that c' is a coloring. If not, then $c'(u) = c'(v)$ for some adjacent pair u, v of vertices. Without loss of generality, $u \in U$ and $v \notin U$. Since v is a neighbor of U, $c'(v) = c(v) \in A \cup \{a\}$. Since $c'(u) \neq c(u)$, the changed color $c'(u)$ is not in $A \cup \{a\}$. This contradicts the assumption that $c'(u) = c'(v)$.

To prove that c' is acyclic, suppose that C is a bi-colored cycle under c'. Then C contains a vertex u such that $c'(u) \neq c(u)$, hence $u \in U$ and $c'(u) \notin A \cup \{a\}$. This shows that vertices at distance two from u whose color is equal to $c'(u)$ are also in U. Consequently, every second vertex on C is in U, and hence C is also bi-colored under c, a contradiction. □

Let D be a generalized cylinder (or a generalized Möbius band) with k boundary parts $D_1, \ldots, D_k$. The *extended k-cylinder* (or the *extended Möbius band* $\tilde{D}$ is the triangulation that is obtained from D by adding a new vertex y_i and joining it to all vertices of D_i, for every $i = 1, \ldots, k$. A coloring c of $\tilde{D}$ is *compatible* with color pairs $(A_i, a_i) \in V_{4,8}$ $(i = 1, \ldots, k)$ if $c(y_i) = a_i$ and all colors used on D_i and on vertices at distance at most 3 from D_i are in $A_i \cup \{a_i\}$, $i = 1, \ldots, k$.

Lemma 3.3 *Suppose that $(A_i, a_i) \in V_{4,8}$ are color pairs for $i = 1, \ldots, k$. If D is a 17-wide k-cylinder, then the extended cylinder $\tilde{D}$ admits an acyclic 8-coloring that is compatible with the color pairs (A_i, a_i), $i = 1, \ldots, k$. Under this coloring, no bi-colored path is joining distinct boundary parts of D.*

Proof. By Theorem 1.1, there is an acyclic 5-coloring c_0 of $\tilde{D}$ using colors 1–5. Let U_0 be the set of vertices of D whose distance from the boundary parts is at least 8. For $i = 1, \ldots, k$ and $j = 1, 2, 3$, let $U_{i,j}$ be the set of all vertices of D whose distance from U_0 is at least $2j - 1$ and whose distance from the ith boundary part D_i is at most 7. Since D is 17-wide, the sets U_0 and $U_{i,j}$ partition $V(D)$.

Let us fix an $i \in \{1, \ldots, k\}$. By Lemma 3.1, there is a walk of length 3 from $(\{1, 2, 3, 4, 5\} \setminus \{c_0(y_i)\}, c_0(y_i))$ to (A_i, a_i) in $\Pi_{4,8}$. For $j = 1, 2, 3$, let π_j be a permutation of E_8 that is compatible with the jth edge on this walk. Finally, for $j = 1, 2, 3$, let the coloring c_j be obtained from c_{j-1} by setting $c_j(v) = c_{j-1}(v)$ if $v \notin U_{i,j}$, and $c_j(v) = \pi_j(c_{j-1}(v))$ if $v \in U_{i,j}$. By Lemma

3.2, c_1, c_2, and c_3 are acyclic 8-colorings of D. Using the fact that D_i and all vertices at distance at most 3 from D_i are in $U_{i,3}$, it is easy to see that c_3 is compatible with (A_i, a_i) at the boundary part D_i.

After consecutively repeating such a change for every $i = 1, \ldots, k$, we obtain an acyclic 8-coloring of D that is compatible with all color pairs (A_i, a_i), $i = 1, \ldots, k$.

By Lemma 3.1 we may assume that no color occurs in all color pairs on the selected walk from $(\{1,2,3,4,5\} \setminus \{c_0(y_i)\}, c_0(y_i))$ to (A_i, a_i). Therefore, there is no path from D_i to U_0 whose every second vertex would have the same color as its vertex in U_0. Since D is 17-wide, every path joining distinct boundary parts must go through U_0, and hence contains at least three colors. □

A similar result holds for every $n \geq 6$: If D is sufficiently wide and $(A_i, a_i) \in V_{4,n}$ $(i = 1, \ldots, k)$ are color pairs assigned to boundary parts of D, then $\tilde{D}$ has an acyclic n-coloring that is compatible with the given color pairs (A_i, a_i) such that no bi-colored path is joining distinct boundary parts of D.

Further, we shall need a simple lemma.

Lemma 3.4 *Let G be graph. If G can be written as $G = G_1 \cup G_2$, where $G_1 \cap G_2$ is a complete graph, then $\chi_{\mathrm{ac}}(G) = \max\{\chi_{\mathrm{ac}}(G_1), \chi_{\mathrm{ac}}(G_2)\}$.*

Proof. Let $k = \max\{\chi_{\mathrm{ac}}(G_1), \chi_{\mathrm{ac}}(G_2\}$. Since G_1 and G_2 are subgraphs of G, we have $\chi_{\mathrm{ac}}(G) \geq k$. On the other hand, having acyclic k-colorings of G_1 and G_2, we may assume that they coincide on $G_1 \cap G_2$ since any two vertices in the intersection are adjacent and hence have distinct colors under either coloring. The same argument also shows that the combined k-coloring of G is acyclic, so $\chi_{\mathrm{ac}}(G) \leq k$. □

Now, we can prove an extension of Lemma 3.3 for Möbius bands.

Lemma 3.5 *Let D be a generalized Möbius band with k boundary parts and with $\mathtt{new}(D) \geq 6$. Let $(A_i, a_i) \in V_{4,8}$ be color pairs for $i = 1, \ldots, k$. Suppose that D is 17-wide, and $(3k + 11)$-deep. Then the extended Möbius band $\tilde{D}$ admits an acyclic 8-coloring that is compatible with the color pairs (A_i, a_i), $i = 1, \ldots, k$. Under this coloring, every path joining distinct boundary parts of D has at least three distinct colors.*

Proof. By Lemma 3.4 we may assume that every 3-cycle in D whose disk interior contains no boundary part of D is a facial triangle. Let C_0 be a

1-sided cycle in D whose distance from the boundary parts $D_1, \ldots, D_k$ of D is at least $3k + 11$. Let us cut the surface along C_0 and triangulate the resulting face by inserting a new vertex x. Additionally, we add a Möbius band triangulation to every boundary part of D. This can be done in such a way that the nonseparating edge-width of the resulting triangulation D' of Euler genus k is larger than $6k + 6$. By Lemma 2.3 (applied with $g = k$, $w = 3$, and $a = 1$), there is a q, $4 \leq q \leq 3k + 1$, such that the q-canonical cycles $Q_1, \ldots, Q_p$ for x are clean and are 3-apart. Let Q'_i be the $(q + 1)$-canonical cycle that is the successor of Q_i, and let Q''_i be the repaired q-canonical cycle (see Lemma 2.2). The generalized cylinder around x with boundary parts $Q''_1, \ldots, Q''_p$ corresponds to a 3-wide generalized Möbius band $D_0 \subseteq D$ that contains C_0 in its interior, and C_0 is at distance $q - 1 \geq 3$ from $Q''_1, \ldots, Q''_p$. Moreover, the boundary parts of D_0 have distance at least 10 from the boundary parts of D. In the sequel, we shall consider exteriors and interiors of cycles in D and D_0. To be consistent with our choices of interiors made in D', we will always assume that the interior contains C_0 (and this will always be possible).

Let C be a shortest 1-sided cycle in D_0. Then C is at distance at least 10 from the boundary parts of D. Now, we consider the graph H which is obtained from D by contracting C to a single vertex x (and removing parallel edges). The graph is planar. By triangulating possible nontriangular faces around x, we obtain a 17-wide k-cylinder H whose boundary parts coincide with those of D. By Lemma 3.3, $\tilde{H}$ has an acyclic 8-coloring c which is compatible with the given color pairs (A_i, a_i), $i = 1, \ldots, k$. Moreover, the proof of Lemma 3.3 shows that x and all vertices at distance at most 2 from x are in the set U_0 used in that proof and hence colored only with colors 1–5. We may assume that $c(x) = 5$. To complete the proof, it suffices to see that c can be extended to an acyclic 8-coloring of D. Note that $D - V(C) \subseteq H$, so we have to show how to color C.

First, we color the vertices of C with colors 5,6,7,8 so that any two vertices that are at distance 2 on C receive different colors. This is possible since $|C| \geq \texttt{new}(D) \geq 6$. Moreover, such a coloring exists in which color 8 is used precisely twice, and the two vertices of color 8 are at distance three on C. On the projective plane, any two noncontractible cycles intersect. So, C and C_0 have a vertex y in common. Then we may assume that the vertices of color 8 are at distance 1 and 2 from y. Since C is induced and since all neighbors of C are adjacent to x in H and therefore use only colors 1–4, this gives rise to an 8-coloring c_1 of D. However, it may happen that this coloring is not acyclic. In the remaining part of the proof we will show that c_1 can be changed (only on vertices adjacent to C) so that an acyclic

8-coloring is obtained.

A 2-*shortcut* is a path uvw of length 2 in D, where u and w are vertices on C and $c_1(u) = c_1(w)$. This implies that the distance from u to w on C is more than 2. Since C is a shortest noncontractible cycle in D_0 and since the noncontractible cycles satisfy the 3P-property, the vertex v is not in D_0. Therefore, u and w are both on the same cycle Q_i'' and by Lemma 2.2, $v \in V(Q_i')$. Moreover, u and w are at distance $q - 1 \geq 3$ from C_0. In particular, $c_1(u) \neq 8$.

Suppose that R is a bi-colored cycle. Then R is composed of 2-shortcuts at the same boundary part Q_i''. (This is obvious when R is (a, b)-colored and $a \neq 5$ and $b \neq 5$. If $a = 5$, this follows from the fact that in $D - V(C) \subseteq H - x$, there is no (a, b)-colored path joining two vertices that are adjacent to C in D since such a path would determine a bi-colored cycle in H.) Since R is composed of 2-shortcuts, Lemma 2.2 implies that the only possibility for R is that it winds once around Q_i''. Thus, if we change the color of one of its vertices v on Q_i' to color 8, R is no longer 2-colored. At distance 2 from v there is at most one vertex colored 8 (the one whose distance from y is 2), so this change does not give rise to new bi-colored cycles. Moreover, no bi-colored cycles at Q_i'' are left. By repeating the same procedure for bi-colored cycles at other boundary parts Q_j'', an acyclic 8-coloring with desired properties is obtained. □

Lemma 3.5 can be proved without the restriction that $\texttt{new}(G) \geq 6$. The only difference occurs when the cycle C in the proof has length 5. In that case the proof is easier, but alternative arguments (similar to those used in [2]) are needed.

4 Acyclic colorings

In this section we give the proof of our main result, Theorem 1.2.

Theorem 4.1 *Let G be a triangulation of a surface of Euler genus g such that $\texttt{new}(G) \geq 6$. Suppose that the vertex set of G can be partitioned into sets $U_1, \ldots, U_r$ and $V_1, \ldots, V_h$ such that the following holds:*

(a) *For every $j = 1, \ldots, r$, the subgraph D_j' induced on U_j is a 3-wide generalized cylinder.*

(b) *For every $i = 1, \ldots, h$, the induced subgraph D_i on vertices V_i is either a 17-wide generalized cylinder or a 17-wide and d-deep generalized*

Möbius band, where $d \geq 3k + 11$ if k is the number of its boundary parts.

(c) *Every edge that is not in one of $D'_1, \ldots, D'_r$ and $D_1, \ldots, D_h$ connects a boundary part of some D'_j with a boundary part of some D_i. For every boundary part Q' of D'_j there is a unique boundary part Q of some D_i that is adjacent to Q', and $Q \cup Q'$ together with the edges joining Q and Q' form a 2-cylinder with boundary parts Q and Q'.*

Then $\chi_{\mathrm{ac}}(G) \leq 8$.

Proof. First, we form the extended generalized cylinders and Möbius bands $\tilde{D}'_1, \ldots, \tilde{D}'_r$ and $\tilde{D}_1, \ldots, \tilde{D}_h$. For a boundary part Q, we will denote by $y(Q)$ the vertex used to triangulate the face bounded by Q. By Theorem 1.1, every $\tilde{D}'_j$ has an acyclic 5-coloring with colors 1–5. Let c_0 be the union of these colorings for $j = 1, \ldots, r$.

For every i, $1 \leq i \leq h$, and every boundary part Q of D_i, we select a color pair $(A, a) \in V_{4,8}$ as follows. Let Q' be the boundary part of D'_j which is adjacent to Q in G. If $c_0(y(Q')) = t$, then we set $A = \{t, 6, 7, 8\}$ and $a = t + 1$ if $t \neq 5$ and $a = 1$ if $t = 5$.

By Lemma 3.3 and Lemma 3.5, every $\tilde{D}_i$ $(1 \leq i \leq h)$ has an acyclic 8-coloring c_i that is compatible with the color pairs that were chosen for its boundary parts. The combination of these colorings together with c_0 is clearly an 8-coloring of G. However, it may not be acyclic.

Suppose that there is a bi-colored cycle R. Since every coloring c_i is acyclic, R must pass from some boundary part Q' of D'_j to the neighboring boundary part Q of some D_i, where $1 \leq i \leq h$. Let (A, a) be the corresponding color pair. Since every path in D_i joining distinct boundary parts uses at least 3 colors, R must return back to D'_j through Q. Let α and α' be the colors that appear on R, where α appears on Q and α' appears on Q'. If R uses an edge in D_i, then $\alpha = a$. However, in such a case, the bi-colored segment of R in D_i joining two vertices of Q would give rise to an (α, α')-colored cycle in $\tilde{D}_i$. Similarly we see that R cannot have an edge in D_0.

Consequently, R is a cycle that alternates between Q and Q'. We may assume that $t = 5$ and $a = 1$ (as introduced above), so that the color pair at Q is $(5678, 1)$. We may assume that $\alpha \neq 8$ and $\alpha' \neq 2$. Let us consider consecutive vertices of R on Q. There is at least one pair of them which is not connected by an $(\alpha, 8)$-colored path in D_i. Otherwise, the sum of all such paths would contain an $(\alpha, 8)$-colored cycle in D_i. Let u and w be such pair of vertices, and let v be their common neighbor on R. Now we change

the color of v to 8, and change the color 8 to 2 on all vertices on Q and those that are adjacent to Q. Clearly, this defines a new 8-coloring of G, and we claim that there are no bi-colored cycles at Q and Q'.

If there was such a cycle R', it would use one of the vertices whose color has been changed. This vertex cannot be v, since color 8 is not used at distance 2 from v. Hence, this would be a vertex on Q or one of its neighbors that was recolored from 8 to 2. Also, R' cannot be contained in D_i since it would be bi-colored also before. (Note that the new vertices colored 2 cannot be at distance 2 from a vertex in D_i whose color was 2 already before since c_i is compatible with (5678,1) at Q and hence the color 2 does not appear at distance at most 3 from Q.) Hence the colors on R' are either 2 and α or 2 and α'. If the colors are 2 and α, then the color α must appear in D'_j, so $\alpha = 5$. If the colors are 2 and α', then α' must appear in D_i adjacent to Q, so α' is either 1 or 5. It is easy to see that this leads to a contradiction since the segment of R' in D_i (if the colors on R' are 2 and 1) or its segment in D'_j (if the colors are 2 and 5) would give rise to a bi-colored cycle in $\tilde{D}_i$ or in $\tilde{D}'_j$, respectively. This shows that R' does not exist.

If there are other bi-colored cycles, they are located at other boundary parts. We repeat the same procedure with them, and after a finite number of steps we end up with an acyclic 8-coloring of G. □

Finally, we can conclude with the proof of the main result.

Proof. (Of Theorem 1.2). Let g be the Euler genus of S and $w = 4(3g + 14)g(2^{g+3} + g)$. We may assume that G is a triangulation since every graph H embedded in S is contained (as a subgraph) in a triangulation whose nonseparating edge-width is the same as that of H. The acyclic chromatic number of graphs in the projective plane is at most 7 [2], so we may assume that $g \geq 2$. By Lemma 3.4, we may assume that every contractible 3-cycle in G is a facial triangle. By Corollary 2.5, G has a decomposition as used as the assumption in Theorem 4.1, where the width and depth of all generalized cylinders and Möbius bands is at least $3g + 11$. Under this decomposition, every generalized Möbius band has at most $k = g - 1$ boundary parts, so its depth is at least $3k + 11$. The width of the generalized cylinders and Möbius bands is also at least $3g + 11 \geq 17$. Therefore, we can apply Theorem 4.1 to conclude that $\chi_{\mathrm{ac}}(G) \leq 8$. □

References

[1] M. O. Albertson and D. M. Berman, Every planar graph has an acyclic 7-coloring, Israel J. Math. 28 (1977) 169–174.

[2] N. Alon, B. Mohar, D. P. Sanders, On acyclic colorings of graphs on surfaces, Isr. J. Math. 94 (1996) 273–283.

[3] O. V. Borodin, On acyclic colorings of planar graphs, Discrete Math. 25 (1979) 211–236.

[4] M. DeVos, L. Goddyn, B. Mohar, D. Vertigan, X. Zhu, Coloring-flow duality of embedded graphs, Trans. AMS, submitted.

[5] S. Fisk, B. Mohar, Coloring graphs without short non-bounding cycles, J. Combin. Theory, Ser. B 60 (1994) 268–276.

[6] B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (1973) 390–408.

[7] T. Jensen, B. Toft, Graph Coloring Problems, John Wiley, New York, 1995.

[8] A. V. Kostochka, Acyclic 6-coloring of planar graphs, Diskret. Analiz. 28 (1976) 40–54 (in Russian).

[9] A. V. Kostochka and L. S. Melnikov, Note to the paper of B. Grünbaum on acyclic colorings, Discrete Math. 14 (1976) 403–406.

[10] J. Mitchem, Every planar graph has an acyclic 8-coloring, Duke Math. J. 41 (1974) 177–181.

[11] B. Mohar, C. Thomassen, Graphs on Surfaces, The Johns Hopkins University Press, Baltimore and London, 2001.

[12] C. Thomassen, Five-coloring maps on surfaces, J. Combin. Theory Ser. B 59 (1993) 89–105.

[13] G. Wegner, Note on a paper of Grünbaum on acyclic colorings, Israel J. Math. 14 (1973) 409–412.