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Short Cycles Connectivity

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Abstract

Short cycles connectivity is a generalization of ordinary connectivity. Instead by a path (sequence of edges), two vertices have to be connected by a sequence of short cycles, in which two adjacent cycles have at least one common vertex. If all adjacent cycles in the sequence share at least one edge, we talk about edge short cycles connectivity.

It is shown that the short cycles connectivity is an equivalence relation on the set of vertices, while the edge short cycles connectivity components determine an equivalence relation on the set of edges. Efficient algorithms for determining equivalence classes are presented.

Short cycles connectivity can be extended to directed graphs (cyclic and transitive connectivity). For further generalization we can also consider connectivity by small cliques or other families of graphs.

Key words: connectivity, short cycles, large networks, algorithm.

Math. Subj. Class. (2000): 05 A 18, 05 C 70, 05 C 85, 05 C 90, 68 R 10, 68 W 40, 93 A 15.

1 Introduction

The idea of connectivity by short cycles emerges in different contexts. In hierarchical decompositions of networks [3] the long cycles can be violations of the assumed hierarchical structure – and related to general structure these nonhierarhical (cyclic) links can be identified. The symmetric connectivity from paper [3] is essentially the connectivity by 2-cycles.

In [1] we were looking at subgraphs formed by complete triads – triangles. Triangular connectivity also appears to be important in different applications [8, 10, 4, 11].

The next stimulus was a reference in Scott [9] to the early work of M. Everett on this subject [5, 6, 7]. It seems that his ideas can be elaborated to provide a powerful and efficient tool for analysis of large networks.

In this paper we first present connectivity by cycles of length 3 – triangular connectivity. Afterward we generalize the results to connectivity by cycles of length at most k, and at the end we propose further generalizations.

The theorems 1, 2, 3, 5, 6, 7 are generalized by theorems 9, 10, 11, 12, 13 and 14. Therefore they are stated without proofs.

2 Triangular connectivity

2.1 Undirected graphs

Let $\mathbf{K}$ denotes the *connectivity* relation and $\mathbf{B}$ denotes the *biconnectivity* relation in a given undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $n = |\mathcal{V}|$ denotes the number of vertices and let $m = |\mathcal{E}|$ denotes the number of edges.

Vertex $u \in \mathcal{V}$ is in relation $\mathbf{K}$ with vertex $v \in \mathcal{V}$, $u\mathbf{K}v$, iff $u = v$ or there exists a path in $\mathcal{G}$ from u to v.

Vertex $u \in \mathcal{V}$ is in relation $\mathbf{B}$ with vertex $v \in \mathcal{V}$, $u\mathbf{B}v$, iff $u = v$ or there exists a cycle in $\mathcal{G}$ containing u and v.

We call a *triangle* a subgraph isomorphic to a 3-cycle C_3. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is *triangular*, if each its vertex and each its edge belong to at least one triangle in $\mathcal{H}$.

Definition 1 *A sequence* $(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s)$ *of triangles of* $\mathcal{G}$ *(*vertex*)* triangularly connects *vertex* $u \in \mathcal{V}$ *with vertex* $v \in \mathcal{V}$*, iff*

1. $u \in \mathcal{V}(\mathcal{T}_1)$,
2. $v \in \mathcal{V}(\mathcal{T}_s)$, *and*
3. $\mathcal{V}(\mathcal{T}_{i-1}) \cap \mathcal{V}(\mathcal{T}_i) \neq \emptyset \quad$ *for* $\quad i = 2, \ldots, s$.

*Such a sequence is called a (*vertex*)* triangular chain*, see Figure 1.*

Definition 2 *Vertex* $u \in \mathcal{V}$ *is (*vertex*)* triangularly connected *with vertex* $v \in \mathcal{V}$, $u\mathbf{K}_3v$*, iff* $u = v$ *or there exists a (vertex) triangular chain that (vertex) triangularly connects vertex* u *with vertex* v.

Theorem 1 *The relation* $\mathbf{K}_3$ *is an equivalence relation on the set of vertices* $\mathcal{V}$.

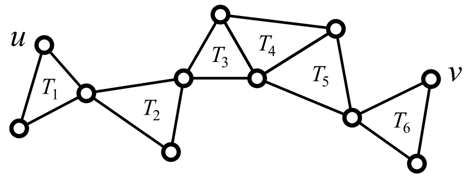

Figure 1: *Triangular chain*

A triangular connectivity component is *trivial* iff it consists of a single vertex.

Theorem 2 *The sets of vertices of maximal connected triangular subgraphs are exactly nontrivial (vertex) triangular connectivity components.*

But subgraphs, induced by nontrivial (vertex) triangular connectivity components are not necessary triangular subgraphs and therefore they are not maximal connected triangular subgraphs. We can see this from example in Figure 2, where all vertices are in the same triangular connectivity component, but the graph is not triangular because of edge e, which does not belong to a triangle.

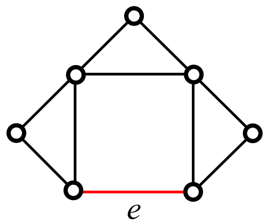

Figure 2: *This graph is not triangular*

An algorithm for determining the relation $\mathbf{K}_3$ is simple, see Algorithm 1. It partitions the set of vertices into k sets (equivalence classes) labeled $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$.

First we choose any vertex $u \in \mathcal{V}$ and put it into a new set, which at the end will become one of the equivalence classes. Then we add to it all vertices, which can be reached from vertex u by triangles. We repeat this procedure until all vertices are assigned to equivalence classes.

$N(u) = \{v \in \mathcal{V} : (u : v) \in \mathcal{E}\}$ denotes the set of all neighbors of vertex u. If the sets of neighbours are ordered we can use merging to compute $N(u) \cap N(v)$ in $\mathcal{O}(\Delta)$, Δ is the maximum degree of $\mathcal{G}$. In this case the time complexity of this algorithm is $\mathcal{O}(\Delta m)$. We have to assign each vertex to corresponding equivalence class. To assign vertex u, we have to visit all its neighbors and for each neighbor v we have to find intersection of $N(u)$ and $N(v)$.

Algorithm 1: Equivalence classes of the relation $\mathbf{K}_3$

$k := 0$
while $\mathcal{V} \neq \emptyset$ **do begin**
  choose $u \in \mathcal{V}$
  $k := k + 1$
  $\mathcal{C}_k := \emptyset$
  $\mathcal{L} := \{u\}$
  while $\mathcal{L} \neq \emptyset$ **do begin**
    choose $u \in \mathcal{L}$
    $\mathcal{C}_k := \mathcal{C}_k \cup \{u\}$
    for each $v \in N(u)$ **do begin**
      $\mathcal{N} := N(u) \cap N(v)$
      if $\mathcal{N} \neq \emptyset$ **then** $\mathcal{L} := \mathcal{L} \cup \mathcal{N} \cup \{v\}$
    end
    $\mathcal{V} := \mathcal{V} \setminus \{u\}$
    $\mathcal{L} := \mathcal{L} \setminus \{u\}$
  end
end

Definition 3 *The* triangular network $\mathcal{N}_3(\mathcal{G}) = (\mathcal{V}, \mathcal{E}_3, w_3)$ *determined by graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is a subgraph* $\mathcal{G}_3 = (\mathcal{V}, \mathcal{E}_3)$ *of* $\mathcal{G}$ *which edges are defined by:* $e \in \mathcal{E}_3$*, iff* $e \in \mathcal{E}$ *and* e *belongs to a triangle. For edge* $e \in \mathcal{E}_3$ *its weight* $w_3(e)$ *equals to the number of different triangles in* $\mathcal{G}$ *to which* e *belongs.*

Theorem 3

$$\mathbf{K}_3(\mathcal{G}) = \mathbf{K}(\mathcal{G}_3)$$

An algorithm for determining $\mathcal{E}_3$ and w_3 is simple, see Algorithm 2 and Figure 3. If the sets of neighbors are ordered the time complexity of computing $w_3(e)$ is $\mathcal{O}(\Delta)$ and the total time complexity of the algorithm is $\mathcal{O}(\Delta m)$.

With $t(v)$ we denote the number of different triangles of $\mathcal{G}$ that contain vertex v. It is easy to verify the following relation between t and w.

Theorem 4 $$2t(v) = \sum_{e:e(v:u)} w_3(e)$$

Definition 4 *A sequence* $(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s)$ *of triangles of* $\mathcal{G}$ edge triangularly connects *vertex* $u \in \mathcal{V}$ *with vertex* $v \in \mathcal{V}$*, iff*

Algorithm 2: Triangular network construction

$\mathcal{E}_3 := \emptyset$
for each $e(u:v) \in \mathcal{E}$ **do begin**
$w_3(e) := |N(u) \cap N(v)|$
if $w_3(e) > 0$ **then** $\mathcal{E}_3 := \mathcal{E}_3 \cup \{e\}$
end

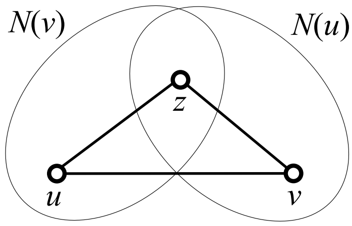

Figure 3: $w_3(e) := |N(u) \cap N(v)|$

1. $u \in \mathcal{V}(\mathcal{T}_1)$,
2. $v \in \mathcal{V}(\mathcal{T}_s)$, *and*
3. $\mathcal{E}(\mathcal{T}_{i-1}) \cap \mathcal{E}(\mathcal{T}_i) \neq \emptyset$ *for* $i = 2, \ldots, s$.

Such a sequence is called an edge triangular chain, *see Figure 4.*

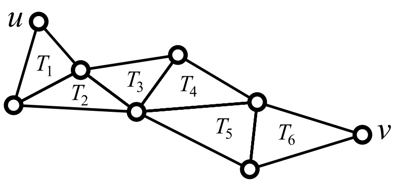

Figure 4: *Edge triangular chain*

Definition 5 *Vertex* $u \in \mathcal{V}$ *is* edge triangularly connected *with vertex* $v \in \mathcal{V}$, $u\mathbf{L}_3v$, *iff* $u = v$ *or there exists an edge triangular chain that edge triangularly connects vertex* u *with vertex* v.

In the biconnected graph in Figure 5 the vertices u in v are edge triangularly connected, while the vertices x and z are not. The relation $\mathbf{L}_3$ is not transitive: $x\mathbf{L}_3v$, $v\mathbf{L}_3z$, but not $x\mathbf{L}_3z$.

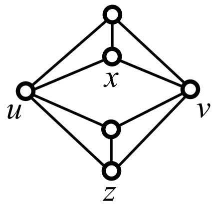

Figure 5: *Biconnected triangular graph*

Theorem 5 *The relation $\mathbf{L}_3$ determines an equivalence relation on the set of edges $\mathcal{E}$.*

An algorithm for determining the relation $\mathbf{L}_3$ is simple, see Algorithm 3. It partitions the set of edges into k sets (equivalence classes of the relation on $\mathcal{E}$) labeled $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$. Vertex u is in relation $\mathbf{L}_3$ with vertex v, if both vertices are end-points of an edge from the same class.

$$u\mathbf{L}_3 v \Leftrightarrow \exists i\ \exists e, f \in \mathcal{C}_i : u \in \mathcal{V}(e) \wedge v \in \mathcal{V}(f)$$

Here $\mathcal{V}(e)$ denotes the set of end-points of edge e.

Algorithm 3: Equivalence classes of the relation on $\mathcal{E}$

$k := 0$
while $\mathcal{E} \neq \emptyset$ **do begin**
  choose $e \in \mathcal{E}$
  $k := k + 1$
  $\mathcal{C}_k := \emptyset$
  $\mathcal{L} := \{e\}$
  while $\mathcal{L} \neq \emptyset$ **do begin**
    choose $e(u : v) \in \mathcal{L}$
    $\mathcal{C}_k := \mathcal{C}_k \cup \{e\}$
    $\mathcal{E} := \mathcal{E} \setminus \{e\}$
    $\mathcal{N} := N(u) \cap N(v)$
    $\mathcal{L} := \mathcal{L} \cup \{(u : w), w \in \mathcal{N}\} \cup \{(v : w), w \in \mathcal{N}\}$
    $\mathcal{L} := \mathcal{L} \setminus \{e\}$
  end
end

In each iteration of the inner loop we move one edge from $\mathcal{E}$ into $\mathcal{C}_k$. So the inner loop repeats m-times. Each assignment or comparison takes constant time, except the statement where the intersection of two neibourhoods is determined. If the sets of neighbours are ordered, this statement has time complexity of $\mathcal{O}(\Delta)$, so the total time complexity of the algorithm is $\mathcal{O}(\Delta m)$.

Note, that in the inner loop the edge e is actually removed from $\mathcal{E}$, so the neighborhoods of vertices are dynamical – they depend on the current set of edges $\mathcal{E}$. This means, that after the edge is removed from $\mathcal{E}$ (and from $\mathcal{L}$), it can not appear in $\mathcal{L}$ again.

Definition 6 *Let* $\mathbf{B}_3 = \mathbf{B} \cap \mathbf{K}_3$.

Theorem 6 *In graph $\mathcal{G}$ hold:*

a. $\mathbf{B} \subseteq \mathbf{K}$
b. $\mathbf{K}_3 \subseteq \mathbf{K}$
c. $\mathbf{L}_3 \subseteq \mathbf{B}_3$
d. $\mathbf{B}_3 \subseteq \mathbf{B}$
e. $\mathbf{B}_3 \subseteq \mathbf{K}_3$

2.2 Directed graphs

If the graph $\mathcal{G}$ is mixed we replace edges with pairs of opposite arcs. In the following let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be a simple directed graph without loops.

For a selected arc $a(u, v) \in \mathcal{A}$ there are four different types of directed triangles: **cyc**lic, **tra**nsitive, **in**put and **out**put.

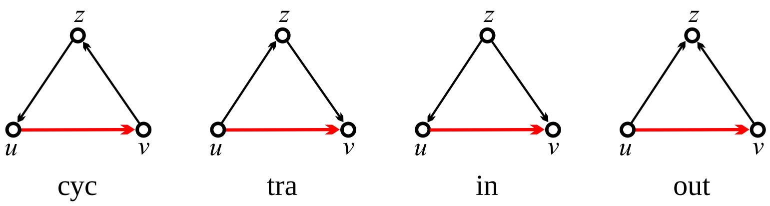

Figure 6: *Types of directed triangles*

For cyclic triangles we define (similarly as for undirected graphs):

$\mathbf{C}_3$ - cyclic triangular connectivity,

$\mathbf{D}_3$ - arc cyclic triangular connectivity,

and the corresponding networks $\mathcal{N}_{cyc}$, $\mathcal{N}_{tra}$, $\mathcal{N}_{in}$ and $\mathcal{N}_{out}$. The algorithms for determining relations $\mathbf{C}_3$ and $\mathbf{D}_3$ and networks $\mathcal{N}_{cyc}$, $\mathcal{N}_{tra}$, $\mathcal{N}_{in}$, $\mathcal{N}_{out}$ are similar to the algorithms for undirected graphs and have the same complexities.

Theorem 7 *A weakly connected cyclic triangular graph is also strongly connected.*

For $\mathbf{C}_3$ and $\mathbf{D}_3$ similar theorems hold as for $\mathbf{K}_3$ and $\mathbf{L}_3$. Besides these two connectivities there is another possibility. Both graphs in Figure 7 are (weakly) triangular. The left side graph is also cyclic triangularly connected, but the right side graph is not. This leads to the following definition.

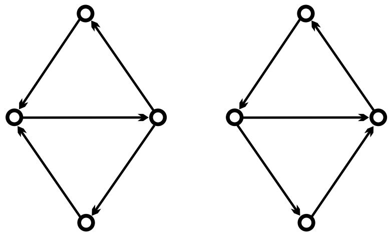

Figure 7: *Strongly triangularly connected graphs*

The vertices $u, v \in \mathcal{V}$ are (*vertex*) *strongly triangularly connected*, $u\mathbf{S}_3v$, iff $u = v$ or there exists strongly connected triangular subgraph that contains u and v.

2.3 Transitivity

Let $\mathbf{R}$ denotes the *reachability* relation in a given directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$. Vertex v is *reachable* from vertex u, $u\mathbf{R}v$, iff $u = v$ or there exists a walk from u to v.

Theorem 8 *If we remove from a graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ all (or some) arcs belonging to a triangularly transitive path π (all arcs of π are transitive) the reachability relation does not change:* $\mathbf{R}(\mathcal{G}) = \mathbf{R}(\mathcal{G} \setminus \mathcal{A}(\pi))$.

PROOF: Because the graph $\mathcal{G} \setminus \mathcal{A}(\pi)$ is a subgraph of $\mathcal{G}$, it is obvious that $\mathbf{R}(\mathcal{G} \setminus \mathcal{A}(\pi)) \subseteq \mathbf{R}(\mathcal{G})$. Let a be any arc on the transitive path π. Because of its transitivity, its terminal vertex is also reachable from its initial vertex by two supporting arcs. We have only to check, that none of them is a part of the path π, so it can not be deleted. Because the arc a and its supporting arc have a common vertex, the only way to be on the same path is to be subsequent arcs. But this is impossible because of their directions. So also the opposite is true: $\mathbf{R}(\mathcal{G}) \subseteq \mathbf{R}(\mathcal{G} \setminus \mathcal{A}(\pi))$. $\square$

But, we cannot remove all transitive arcs. The counter-example is presented in Figure 8, where we have a directed 6-cycle, which vertices are connected by arcs with additional vertex in its center. The central vertex is reachable from anywhere. All the arcs from the cycle to the central vertex are transitive. If we remove them all, the central vertex is not reachable any more.

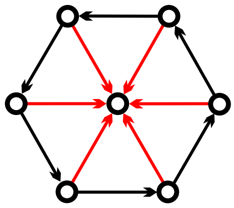

Figure 8: *Graph in which all transitive arcs can not be removed*

3 k-gonal connectivity

3.1 Undirected graphs

We call a k-*gone* a subgraph isomorphic to a k-cycle C_k and a (k)-*gone* a subgraph isomorphic to C_s for some s, $3 \leq s \leq k$. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is k-*gonal*, if each its vertex and each its edge belong to at least one (k)-gone in $\mathcal{H}$.

Definition 7 *A sequence* $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ *of* (k)*-gones of* $\mathcal{G}$ *(*vertex*)* k-gonally connects *vertex* $u \in \mathcal{V}$ *with vertex* $v \in \mathcal{V}$*, iff*

1. $u \in \mathcal{V}(\mathcal{C}_1)$,
2. $v \in \mathcal{V}(\mathcal{C}_s)$, *and*
3. $\mathcal{V}(\mathcal{C}_{i-1}) \cap \mathcal{V}(\mathcal{C}_i) \neq \emptyset \quad$ *for* $\; i = 2, \ldots, s$.

*Such a sequence is called a (*vertex*)* k-gonal chain*, see Figure 9.*

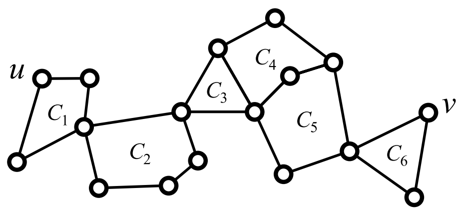

Figure 9: k*-gonal chain*

Definition 8 *Vertex* $u \in \mathcal{V}$ *is (*vertex*)* k-gonally connected *with vertex* $v \in \mathcal{V}$, $u\mathbf{K}_k v$*, iff* $u = v$ *or there exists a (vertex)* k*-gonal chain that (vertex)* k*-gonally connects vertex* u *with vertex* v*.*

Theorem 9 *The relation* $\mathbf{K}_k$ *is an equivalence relation on the set of vertices* $\mathcal{V}$*.*

PROOF: Reflexivity follows directly from the definition of the relation $\mathbf{K}_k$.

Since the reverse of a k-gonal chain from u to v is a k-gonal chain from v to u, the relation $\mathbf{K}_k$ is symmetric.

Transitivity. Let u, v and z be such vertices, that $u\mathbf{K}_k v$ and $v\mathbf{K}_k z$. If these vertices are not pairwise different, the transitivity condition is trivially true. Assume now that they are pairwise different. Because of $u\mathbf{K}_k v$ and $v\mathbf{K}_k z$ there exist (vertex) k-gonal chains from u to v and from v to z. Their concatenation is a (vertex) k-gonal chain from u to z. Therefore also $u\mathbf{K}_k z$. □

Theorem 10 *The sets of vertices of maximal connected k-gonal subgraphs are exactly non-trivial (vertex) k-gonal connectivity components.*

PROOF: Let u and v be any vertices belonging to a connected k-gonal subgraph. If $u = v$, it is obvious that $u\mathbf{K}_k v$. Otherwise there exists a path $\pi = u, e_1, z_1, e_2, z_2, e_3, z_3, \ldots, e_s, v$ from u to v. Because the subgraph is k-gonal, each edge e_i on this path belongs to at least one (k)-gone $\mathcal{C}_i$ in this subgraph. For the obtained k-gonal chain $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ it holds:

- $e_i \in \mathcal{E}(\mathcal{C}_i)$, $i = 1, \ldots, s$
- $u \in \mathcal{V}(\mathcal{C}_1)$, $v \in \mathcal{V}(\mathcal{C}_s)$
- $z_{i-1} \in \mathcal{V}(\mathcal{C}_{i-1}) \cap \mathcal{V}(\mathcal{C}_i)$, $i = 2, \ldots, s$

Therefore $u\mathbf{K}_k v$. So all vertices of any (also maximal) connected k-gonal subgraph belong to the same component of the relation $\mathbf{K}_k$.

Now let u and v be two different vertices of a nontrivial $\mathbf{K}_k$-component $\mathcal{C} \subseteq \mathcal{V}$. Because u is in relation $\mathbf{K}_k$ with v, there exists a k-gonal chain from u to v. It is obvious that all vertices of a k-gonal chain belong to the same maximal connected k-gonal subgraph, so also u and v. But u and v were any two different vertices of $\mathcal{C}$, so all vertices of a nontrivial k-gonal connectivity component belong to the same maximal connected k-gonal subgraph.
□

But, as shown in Figure 2, subgraphs induced by nontrivial (vertex) k-gonal connectivity components are not necessary k-gonal subgraphs and therefore they are not maximal connected k-gonal subgraphs.

Definition 9 *The* k-gonal network $\mathcal{N}_k(\mathcal{G}) = (\mathcal{V}, \mathcal{E}_k, w_k)$ *determined by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a subgraph $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k)$ of $\mathcal{G}$ which edges are defined by: $e \in \mathcal{E}_k$, iff $e \in \mathcal{E}$ and e belongs to a (k)-gone. For an edge $e \in \mathcal{E}_k$ the weight $w_k(e)$ equals to the number of different (k)-gones in $\mathcal{G}$ to which e belongs.*

Theorem 11

$$\mathbf{K}_k(\mathcal{G}) = \mathbf{K}(\mathcal{G}_k)$$

PROOF: Let $u\mathbf{K}_k v$ holds in graph $\mathcal{G}$. If $u = v$, it is also true that $u\mathbf{K}v$ in graph $\mathcal{G}_k$. If the vertices are different, there exists (vertex) k-gonal chain in $\mathcal{G}$ from u to v. Each edge in this chain belongs to at least one (k)-gone, so the whole chain is in $\mathcal{G}_k$. So u and v are connected in $\mathcal{G}_k$ or with other words $u\mathbf{K}v$ in $\mathcal{G}_k$.

Let $u\mathbf{K}v$ holds in graph $\mathcal{G}_k$. Then in graph $\mathcal{G}_k$ exists a path from u to v. Because $\mathcal{G}_k$ is k-gonal, each edge on this path belongs to at least one (k)-gone, so we can construct a k-gonal chain from u to v in $\mathcal{G}_k$. Because $\mathcal{G}_k$ is subgraph of $\mathcal{G}$, this chain is also in $\mathcal{G}$, which means that $u\mathbf{K}_k v$ in graph $\mathcal{G}$. □

To determine the equivalence classes of the relation $\mathbf{K}_k$, we can first determine its k-gonal subgraph $\mathcal{G}_k$ and find the connected components of it.

To compute the weight of edge e we have to count to how many (k)-gones it belongs. We are still working on development of an efficient algorithm for this task.

The weights w_k can be used to identify dense parts of a given network. For example, for a selected edge e in r-clique we can count, to how many k-gones it belongs. The end-points of e are the first two vertices of the k-gone. There are $r-2$ ways to choose the third vertex, then $r-3$ ways to choose the fourth vertex, ..., and $r-k+1$ ways to choose the last vertex of the k-gone (which is connected to the first one). So we have $(r-2)(r-3)\cdots(r-k+1)$ different k-gones and $\sum_{i=3}^{k}(r-2)(r-3)\cdots(r-i+1)$ different (k)-gones. It follows that each edge e of r-clique in k-gonal network has weight $w_k(e)$ at least $\sum_{i=3}^{k}(r-2)(r-3)\cdots(r-i+1)$

$$w_k(e) \geq \sum_{i=3}^{k}(r-2)(r-3)\cdots(r-i+1)$$

The *Everett's k-decomposition* of a given undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a partition $\{\mathcal{C}_1, ..., \mathcal{C}_p, \mathcal{B}_1, ..., \mathcal{B}_q\}$ of the set of vertices $\mathcal{V}$, where $\mathcal{C}_i$ are the k-gonally connected components and $\mathcal{B}_j$ are *bridges* – the connected components of the $\mathcal{V} \setminus \cup\mathcal{C}_i$.

A procedure for determining Everett's decomposition is as follows: First determine the k-gonal subgraph $\mathcal{G}_k$. Its connected components $\{\mathcal{C}_i\}$ are by Theorem 11 just the k-gonally connected components. Finally in the graph $\mathcal{G}|\mathcal{V}\setminus\cup\mathcal{C}_i$ determine the connected components – bridges $\{\mathcal{B}_i\}$.

Definition 10 *A sequence $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ of (k)-gones of $\mathcal{G}$* edge k-gonally connects *vertex $u \in \mathcal{V}$ with vertex $v \in \mathcal{V}$, iff*

1. $u \in \mathcal{V}(\mathcal{C}_1)$,
2. $v \in \mathcal{V}(\mathcal{C}_s)$, *and*
3. $\mathcal{E}(\mathcal{C}_{i-1}) \cap \mathcal{E}(\mathcal{C}_i) \neq \emptyset \quad$ *for* $\quad i = 2, \ldots, s.$

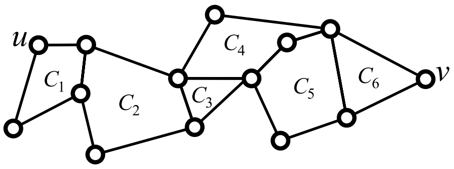

Figure 10: *Edge k-gonal chain*

Such a sequence is called an edge k-gonal chain*, see Figure 10.*

Definition 11 *Vertex $u \in \mathcal{V}$ is* edge k-gonally connected *with vertex $v \in \mathcal{V}$, $u\mathbf{L}_k v$, iff $u = v$ or there exists an edge k-gonal chain that edge k-gonally connects vertex u with vertex v.*

Theorem 12 *The relation $\mathbf{L}_k$ determines an equivalence relation on the set of edges $\mathcal{E}$.*

PROOF: Let the relation $\sim$ on $\mathcal{E}$ be defined as: $e \sim f$, iff $e = f$ or there exists an edge k-gonal chain $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$, where $e \in \mathcal{E}(\mathcal{C}_1)$ and $f \in \mathcal{E}(\mathcal{C}_s)$.

Reflexivity of $\sim$ follows from its definition.

The symmetry is simple too. Let be $e \sim f$. Then there exists an edge k-gonal chain 'from' e 'to' f. Its reverse is an edge k-gonal chain 'from' f 'to' e, so $f \sim e$.

And transitivity. Let e, f and g be such edges, that $e \sim f$ and $f \sim g$. There exist an edge k-gonal chain from e to f and an edge k-gonal chain from f to g. The concatenation of these two chains is an edge k-gonal chain from e to g (the (k)-gones in the contact of the chains both contain the edge f, so their intersection is not empty). Therefore also $e \sim g$. □

Definition 12 *Let $\mathbf{B}_k = \mathbf{B} \cap \mathbf{K}_k$.*

Theorem 13 *In graph $\mathcal{G}$ hold:*

a. $\mathbf{B} \subseteq \mathbf{K}$
b. $\mathbf{K}_k \subseteq \mathbf{K}$
c. $\mathbf{L}_k \subseteq \mathbf{B}_k$
d. $\mathbf{B}_k \subseteq \mathbf{B}$
e. $\mathbf{B}_k \subseteq \mathbf{K}_k$

and for $i < j$ also:

f. $\mathbf{K}_i \subseteq \mathbf{K}_j$
g. $\mathbf{L}_i \subseteq \mathbf{L}_j$
h. $\mathbf{B}_i \subseteq \mathbf{B}_j$

PROOF:

a. Evident from definitions.

b. Let u and v be such vertices, that $u\mathbf{K}_k v$. If $u = v$, it is also $u\mathbf{K}v$ by the definition. Otherwise there exists k-gonal chain from u to v. Therefore there also exists a path from u to v, which means that $u\mathbf{K}v$.

c. Let u and v be such vertices, that $u\mathbf{L}_k v$. If $u = v$, it is also $u\mathbf{B}v$ and $u\mathbf{K}_k v$ by definition, from which it follows that $u\mathbf{B}_k v$. If the vertices are different, there exists an edge k-gonal chain from u to v. But since each edge k-gonal chain is also a vertex k-gonal chain (if two (k)-gones have a common edge, they also have a common vertex), $u\mathbf{K}_k v$ holds. Subgraph in the form of an edge k-gonal chain is biconnected [2], $u\mathbf{B}v$. Both results together give us $u\mathbf{B}_k v$.

d. Follows from the definition of the relation $\mathbf{B}_k$.

e. Follows from the definition of the relation $\mathbf{B}_k$.

f. Let u and v be such vertices, that $u\mathbf{K}_i v$. If $u = v$, it is also $u\mathbf{K}_j v$ by the definition. Otherwise there exists i-gonal chain from u to v, where none of (i)-gones has length greater than i. The same chain is also a j-gonal chain from u to v – therefore $u\mathbf{K}_j v$.

g. Let u and v be such vertices, that $u\mathbf{L}_i v$. If $u = v$, it is also $u\mathbf{L}_j v$ by the definition. Otherwise there exists edge i-gonal chain from u to v, where none of (i)-gones has length greater than i. The same chain is also an edge j-gonal chain from u to v – therefore $u\mathbf{L}_j v$.

h. Follows from the definition of the relation $\mathbf{B}_k$ and item f of this theorem.

□

The relationships from theorem 13 can be presented by a diagram:

$$\begin{array}{ccccc}
 & & \mathbf{B} & \subseteq & \mathbf{K} \\
 & & \cup\!\!\mid & & \cup\!\!\mid \\
\vdots & & \vdots & & \vdots \\
\cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid \\
\mathbf{L}_k & \subseteq & \mathbf{B}_k & \subseteq & \mathbf{K}_k \\
\cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid \\
\mathbf{L}_{k-1} & \subseteq & \mathbf{B}_{k-1} & \subseteq & \mathbf{K}_{k-1} \\
\cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid \\
\vdots & & \vdots & & \vdots
\end{array}$$

3.2 Directed graphs

We shall give a special attention to two special types of Everett's semicycles [5, 6], see Figure 11, related to selected arc $a(u, v) \in A$: **cycles** (arc with a feed-back path) and **transitive semicycles** (arc with a reinforcement path) of length at most k.

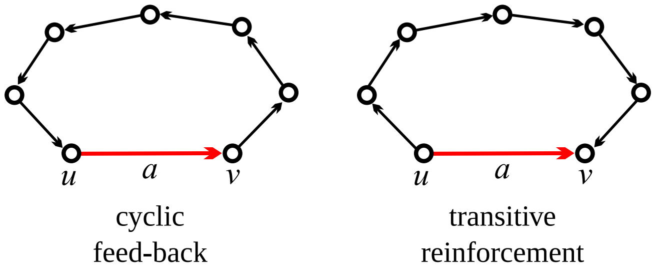

Figure 11: *Cycles on an arc*

For cyclic (k)-gones we define (similarly as for undirected graphs):

Definition 13 *A sequence* $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ *of cycles of length at most* k *and at least 2 of* $\mathcal{G}$ *(*vertex*)* cyclic k-gonally connects *vertex* $u \in \mathcal{V}$ *with vertex* $v \in \mathcal{V}$*, iff*

1. $u \in \mathcal{V}(\mathcal{C}_1)$,
2. $v \in \mathcal{V}(\mathcal{C}_s)$, *and*
3. $\mathcal{V}(\mathcal{C}_{i-1}) \cap \mathcal{V}(\mathcal{C}_i) \neq \emptyset \quad$ *for* $\quad i = 2, \ldots, s$.

*Such a sequence is called a (*vertex*)* cyclic k-gonal chain.

Definition 14 *Vertex* $u \in \mathcal{V}$ *is (*vertex*)* cyclic k-gonally connected *with vertex* $v \in \mathcal{V}$*,* $u\mathbf{C}_k v$*, iff* $u = v$ *or there exists a (vertex) cyclic* k*-gonal chain that (vertex) cyclic* k*-gonally connects vertex* u *with vertex* v*.*

Definition 15 *A sequence* $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ *of cycles of length at most* k *and at least 2 of* $\mathcal{G}$ arc cyclic k-gonally connects *vertex* $u \in \mathcal{V}$ *with vertex* $v \in \mathcal{V}$*, iff*

1. $u \in \mathcal{V}(\mathcal{C}_1)$,
2. $v \in \mathcal{V}(\mathcal{C}_s)$, *and*
3. $\mathcal{A}(\mathcal{C}_{i-1}) \cap \mathcal{A}(\mathcal{C}_i) \neq \emptyset \quad$ *for* $\quad i = 2, \ldots, s$.

Such a sequence is called an arc cyclic k-gonal chain.

Definition 16 *Vertex* $u \in \mathcal{V}$ *is* arc cyclic k-gonally connected *with vertex* $v \in \mathcal{V}$*,* $u\mathbf{D}_k v$*, iff* $u = v$ *or there exists an arc cyclic* k*-gonal chain that arc cyclic* k*-gonally connects vertex* u *with vertex* v*.*

For $\mathbf{C}_k$ and $\mathbf{D}_k$ similar theorems hold as for $\mathbf{K}_k$ and $\mathbf{L}_k$.

Theorem 14 *A weakly connected cyclic k-gonal graph is also strongly connected.*

PROOF: Take any pair of vertices u and v. Since $\mathcal{G}$ is weakly connected there exists a semipath connecting u and v. Each arc on this semipath belongs to at least one (k)-cycle. Therefore its end-points are connected by a path in opposite direction – we can construct a walk from u to v and also a walk from v to u. □

Theorem 15 *Cyclic k-gonal connectivity $\mathbf{C}_k$ is an equivalence relation on the set of vertices $\mathcal{V}$.*

PROOF: Reflexivity follows directly from the definition of the relation $\mathbf{C}_k$.

Since the reverse of a cyclic k-gonal chain from u to v is a cyclic k-gonal chain from v to u, the relation $\mathbf{C}_k$ is symmetric.

Transitivity. Let u, v and z be such vertices, that $u\mathbf{C}_k v$ and $v\mathbf{C}_k z$. If the vertices are not pairwise different, the transitivity condition is trivialy true. Assume now that they are pairwise different. Because of $u\mathbf{C}_k v$ and $v\mathbf{C}_k z$ there exist cyclic k-gonal chains from u to v and from v to z. Their concatenation is a cyclic k-gonal chain from u to z. Therefore also $u\mathbf{C}_k z$. □

An arc is *cyclic* iff it belongs to some cycle (of any length) in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$. The cyclic arcs that do not belong to some (k)-cycle are called *k-long* (range) arcs.

Theorem 16 *If the graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ does not contain k-long arcs then its cyclic k-gonal reduction $\mathcal{G}/\mathbf{C}_k = (\mathcal{V}/\mathbf{C}_k, \mathcal{A}^*)$, where for $X, Y \in \mathcal{V}/\mathbf{C}_k : (X, Y) \in \mathcal{A}^* \iff \exists u \in X \exists v \in Y : (u, v) \in \mathcal{A}$, is an acyclic graph.*

PROOF: Suppose that cyclic k-gonal reduction of graph $\mathcal{G}$ is not acyclic. Then it contains a cycle C^*, which can be extended to a cycle C of graph $\mathcal{G}$. Let a^* be any arc of C^* and let a be a corresponding arc of C. Because the end-points of a^* are different, the end-points of a belong to two different components of the relation $\mathbf{C}_k$. So a does not belong to any cyclic (k)-gone. But a is cyclic (it belongs to cycle C), so it is a k-long arc. This is a contradiction. Therefore, the cyclic k-gonal reduction of graph $\mathcal{G}$ must be acyclic. □

From this proof we also see how to identify the k-long arcs. They are exactly the arcs that are reduced to cyclic arcs in $\mathcal{G}/\mathbf{C}_k$.

Theorem 17 *The relation $\mathbf{D}_k$ determines an equivalence relation on the set of arcs $\mathcal{A}$.*

PROOF: Let the relation $\sim$ on $\mathcal{A}$ be defined as: $e \sim f$, iff $e = f$ or there exists an arc cyclic k-gonal chain $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$, where $e \in \mathcal{A}(\mathcal{C}_1)$ and $f \in \mathcal{A}(\mathcal{C}_s)$.

Reflexivity of $\sim$ follows from its definition.

The symmetry is simple too. Let be $e \sim f$. Then there exists an arc cyclic k-gonal chain 'from' e 'to' f. Its reverse is an arc cyclic k-gonal chain 'from' f 'to' e, so $f \sim e$.

And transitivity. Let e, f and g be such arcs, that $e \sim f$ and $f \sim g$. There exist an arc cyclic k-gonal chain from e to f and an arc cyclic k-gonal chain from f to g. The concatenation of these two chains is an arc cyclic k-gonal chain from e to g (the (k)-cycles in the contact of the chains both contain the arc f, so their intersection is not empty). Therefore also $e \sim g$. □

Definition 17 *The vertices $u, v \in \mathcal{V}$ are (*vertex*)* strongly k-gonally connected, *$u\mathbf{S}_k v$, iff $u = v$ or there exists strongly connected k-gonal subgraph that contains u and v.*

It is easy to see that $\mathbf{D}_k \subseteq \mathbf{C}_k \subseteq \mathbf{S}_k$. The relationships between these relations can be presented by a diagram:

$$\begin{array}{ccccc}
 & & & & \mathbf{S} \\
 & & & & \cup\!\! \mid \\
\vdots & & \vdots & & \vdots \\
\cup\!\! \mid & & \cup\!\! \mid & & \cup\!\! \mid \\
\mathbf{D}_k & \subseteq & \mathbf{C}_k & \subseteq & \mathbf{S}_k \\
\cup\!\! \mid & & \cup\!\! \mid & & \cup\!\! \mid \\
\mathbf{D}_{k-1} & \subseteq & \mathbf{C}_{k-1} & \subseteq & \mathbf{S}_{k-1} \\
\cup\!\! \mid & & \cup\!\! \mid & & \cup\!\! \mid \\
\vdots & & \vdots & & \vdots
\end{array}$$

We can define three networks, that can provide us with more detailed picture about the network structure:

- *Feedback network* $\mathcal{N}_F = (\mathcal{V}, \mathcal{A}_F, w_F)$ where $w_F(a)$ is the number of different (k)-cycles containing the arc a.
- *Transitive network* $\mathcal{N}_T = (\mathcal{V}, \mathcal{A}_T, w_T)$ where $w_T(a)$ is the number of different transitive (k)-semicycles containing the arc a as the transitive arc (shortcut).
- *Support network* $\mathcal{N}_S = (\mathcal{V}, \mathcal{A}_S, w_S)$ where $w_S(a)$ is the number of different transitive (k)-semicycles containing the arc a as a nontransitive arc.

Theorem 18 *Let $\mathbf{S}$ be the relation of strong connectivity, $\mathbf{S} = \mathbf{R} \cap \mathbf{R}^{-1}$. Then*

$$\mathbf{C}_k(\mathcal{G}) = \mathbf{S}(\mathcal{G}_F)$$

PROOF: Let $u\mathbf{C}_k v$ holds in graph $\mathcal{G}$. If $u = v$, it is also true, that $u\mathbf{S}v$ in graph $\mathcal{G}_F$. If the vertices are different, there exists cyclic k-gonal chain in $\mathcal{G}$ from u to v. Each arc in this chain belongs to at least one (k)-cycle, so the whole chain is in $\mathcal{G}_F$. Vertices u and v are mutually reachable by arcs of this chain, so $u\mathbf{S}v$ in $\mathcal{G}_F$.

Let $u\mathbf{S}v$ holds in graph $\mathcal{G}_F$. Then in graph $\mathcal{G}_F$ exists a walk from u to v. Because $\mathcal{G}_F$ is cyclic k-gonal, each arc on this walk belongs to at least one (k)-cycle, so we can construct a cyclic k-gonal chain from u to v in $\mathcal{G}_F$. Because $\mathcal{G}_F$ is subgraph of $\mathcal{G}$, this chain is also in $\mathcal{G}$, which means that $u\mathbf{C}_k v$ holds in graph $\mathcal{G}$. □

3.3 Transitivity

Let $\mathbf{T}_k$ denotes the k-transitive reachability relation in a given directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$.

Definition 18 *Vertex v is* k-transitively reachable *from vertex u, $u\mathbf{T}_k v$, iff $u = v$ or there exists a walk from u to v in which each arc is k-transitive – is a base (shortcut arc) of some transitive semicycle of length at most k.*

The vertices u and v are mutually k-transitively reachable, if vertex u is k-transitively reachable from vertex v, and vertex v is k-transitively reachable from vertex u. We denote this relation by $\hat{\mathbf{T}}_k$

$$u\hat{\mathbf{T}}_k v \Leftrightarrow u\mathbf{T}_k v \wedge v\mathbf{T}_k u$$

Theorem 19 *The relation of mutual k-transitive reachability $\hat{\mathbf{T}}_k = \mathbf{T}_k \cap \mathbf{T}_k^{-1}$ is an equivalence relation on the set of vertices V.*

PROOF: It is well known that if $\mathbf{Q}$ is a reflexive and transitive relation then $\hat{\mathbf{Q}} = \mathbf{Q} \cap \mathbf{Q}^{-1}$ is an equivalence relation. The relation $\mathbf{T}_k$ is reflexive by definition, so we have only to prove that it is transitive.

Let u, v and w be such vertices that $u\mathbf{T}_k v$ and $v\mathbf{T}_k w$. If these vertices are not pairwise different, the transitivity condition is trivialy true. Otherwise there exists a walk from u to v and a walk from v to w, in which every arc is k-transitive. Their concatenation is a walk from u to w, in which every arc is k-transitive, so $u\mathbf{T}_k w$. □

4 Further generalizations

Until now we observed the connectivity by triangles and other short cycles. Intersections of two adjacent cycles in the coresponding chains contained at least one vertex (vertex connectivity) or at least one edge/arc (edge/arc connectivity). This can be generalized to other families of grahps.

Definition 19 *Let $\mathbb{H}$ and $\mathbb{H}_0$ be two families of graphs. A sequence $(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_s)$ of subgraphs of $\mathcal{G}$ $(\mathbb{H}, \mathbb{H}_0)$* connects *vertex $u \in \mathcal{V}$ with vertex $v \in \mathcal{V}$, iff*

1. $u \in \mathcal{V}(\mathcal{H}_1)$,
2. $v \in \mathcal{V}(\mathcal{H}_s)$,
3. $\mathcal{H}_i \in \mathbb{H}$ *for* $i = 1, \ldots, s$, *and*
4. $\mathcal{H}_{i-1} \cap \mathcal{H}_i \supseteq \mathcal{H} \in \mathbb{H}_0$ *for* $i = 2, \ldots, s$.

Example: For $r < k$ we can define (k, r)-clique connectivity: $\mathbb{H} = \{K_{r+1}, K_{r+2}, \ldots, K_k\}$, $\mathbb{H}_0 = \{K_r\}$

All the previous types of connectivity are special cases of the generalized connectivity:

$$\begin{aligned} \mathbf{K}_3 &= (3,1)\text{-clique connectivity} \\ \mathbf{L}_3 &= (3,2)\text{-clique connectivity} \\ \mathbf{K}_k &= (\{C_3, \ldots, C_k\}, \{K_1\}) \text{ connectivity} \\ \mathbf{L}_k &= (\{C_3, \ldots, C_k\}, \{K_2\}) \text{ connectivity} \end{aligned}$$

For the generalized connectivity similar theorems hold as for triangular and k-gonal connectivity.

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