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Preprint series, Vol. 41 (2003), 901

IRREDUCIBLE (v_3) CONFIGURATIONS AND GRAPHS

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ISSN 1318-4865

November 17, 2003

Ljubljana, November 17, 2003

Irreducible (v_3) configurations and graphs

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Abstract

Cubic bipartite graphs with girth at least 6 correspond to symmetric combinatorial (v_3) configurations. In 1887 V. Martinetti described a simple construction method which enables one to construct all combinatorial (v_3) configurations from a set of so-called *irreducible* configurations. The result has been cited several times since its publication, both in the sense of configurations and graphs. But after a careful examination, the list of irreducible configurations given by Martinetti has turned out to be incomplete. We will give the description of all irreducible configurations and corresponding graphs, including those which are missing in the Martinetti's list.

1 Introduction

Let us start with basic definitions. A *(combinatorial) configuration* (v_r, b_k) is an incidence structure of points and lines with the following properties.

1. There are v points and b lines.
2. There are r lines through each point and k points on each line.
3. Two different points are connected by at most one line and two lines intersect in at most one point.

Note that configurations considered here are purely combinatorial objects and that there is no geometric significance associated with the terms point and line. For this reason we will omit the adjective combinatorial and speak only of configurations. However, we will briefly discuss the geometric representation of configurations at the end of the last section.

A (v_r, b_k) configuration is called *symmetric* if $v = b$ (which is equivalent to saying that $r = k$) and is denoted by (v_r).

Incidence structures and hence configurations are closely related to graphs. Let $G(\mathcal{C})$ be a bipartite graph with v *black* vertices representing points of the incidence structure $\mathcal{C}$, b *white* vertices representing lines of $\mathcal{C}$, and with an edge joining two vertices if and only if the corresponding point and line are incident

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in $\mathcal{C}$. We call $G(\mathcal{C})$ *incidence graph* or *Levi graph* or just *graph* of the incidence structure $\mathcal{C}$. The following proposition characterizes symmetric configurations in terms of their graphs.

Proposition 1. *An incidence structure is a (v_r) configuration if and only if its graph is r-regular with girth at least* 6.

For the proof and more about correlations between configurations and graphs see [6, 9, 10]. For enumeration results about (v_3) configurations the reader is referred to [2].

With each (v_r, b_k) configuration $\mathcal{C}$ the *dual* (b_k, v_r) configuration $\mathcal{C}^*$ may be associated by reversing the roles of points and lines in $\mathcal{C}$. Both $\mathcal{C}$ and $\mathcal{C}^*$ share the same incidence graph, only the black-white coloring of its vertices is reversed. If $\mathcal{C}$ is isomorphic to its dual we say that $\mathcal{C}$ is *self-dual* and a corresponding isomorphism is called a *duality*. A duality of order 2 is called a *polarity*. Configurations which admit a polarity are called *self-polar*.

If $P = \mathbb{Z}_v = \{0, 2, \ldots, v-1\}$ represents a set of points and

$$\mathcal{B} = \{\{0, b, c\}, \{1, b+1, c+1\}, \ldots, \{v-1, b+v-1, c+v-1\}\}, \quad b, c \in P,$$

represents a set of lines of some (v_3) configuration $\mathcal{C}$ then $\mathcal{C}$ is called a *cyclic* (v_3) *configuration with base line* $\{0, b, c\}$. Of course, the idea can be generalized to cyclic (v_r) configurations for general values of r.

The Fano plane or projective plane of order 2, the smallest (v_3) configuration, is a cyclic (7_3) configuration with base line $\{0, 1, 3\}$. Its incidence graph is the well-known Heawood graph. The second one in this family, cyclic (8_3) configuration with base line $\{0, 1, 3\}$, is the only (8_3) configuration and is called Möbius-Kantor configuration [6]. Let us mention also that incidence graphs of cyclic configurations correspond precisely to so-called cyclic Haar graphs of girth at least 6, see [15].

In 1887 V. Martinetti suggested the following construction method for symmetric (v_3) configurations [16]. Suppose that in the given (v_3) configuration exist two parallel (non-intersecting) lines $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ such that points a_1 and b_1 are not on a common line. By removing these two lines, adding one new point c and three new lines $\{c, a_2, a_3\}$, $\{c, b_2, b_3\}$, $\{c, a_1, b_1\}$ we obtain a $((v+1)_3)$ configuration. It is not possible to obtain every (v_3) configuration from some $((v-1)_3)$ configuration by using this method. We will call (v_3) configurations which can not be constructed in this way from a smaller one *irreducible configurations* and the others *reducible configurations*.

2 Irreducible graphs and configurations

In [16] V. Martinetti gave a list of irreducible configurations. He claimed that, in addition to some special cases for $v \leq 10$, there are two infinite families of irreducible (v_3) configurations. The result has been cited several times since its publication, both in the sense of configurations and graphs, for example in [1, 5, 8, 14].

But in [14] the author expressed a certain amount of doubt about the result when saying: "The proof [of Martinetti's theorem] is, not surprisingly, involved and long; I have not checked the details, and I do not know it as a fact that anybody has. The statement has been accepted as true for these 112 years, and it may well be true. On the other hand, Daublebski's enumeration of the (12_3) configurations was also considered true for a comparable length of time..."

And indeed, after a careful examination, the list of irreducible configurations given by Martinetti has proved to be incomplete. The aim of this paper is to give the complete list of irreducible configurations and corresponding graphs, including those which are missing in the Martinetti's list.

To do this we observe the Martinetti method on graphs of (v_3) configurations. For the sake of simplicity we will use the notion (v_3) *graph* instead of graph of (v_3) configuration, i.e. (v_3) graph is a bipartite cubic graph with girth ≥ 6. We define reducible and irreducible (v_3) graphs corresponding to reducible and irreducible configurations respectively as follows. A (v_3) graph G is *reducible* if there exists an edge $uv \in EG$ such that $(G - \{u, v\}) + x_1y_1 + x_2y_2$ and $(G - \{u, v\}) + x_1y_2 + x_2y_1$ are also (v_3) graphs, where x_1, x_2, y_1, y_2 are neighbors of u and v as it is shown in Figure 1. Otherwise a (v_3) graph is *irreducible*.

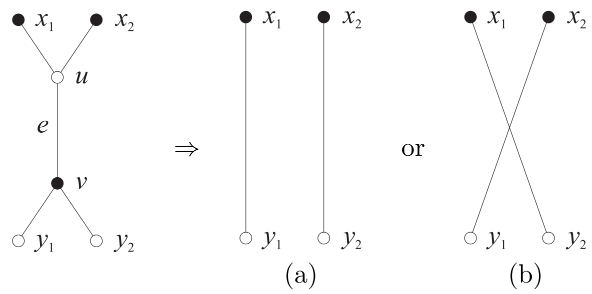


Figure 1: Reduction of the edge uv in a (v_3) graph.

In the proof of the Martinetti theorem, the following characterization of irreducible configurations will be useful.

Lemma 2. *A (v_3) graph G is irreducible if and only if for each edge e of G one of the following is true:*

- *edge e and one of its neighboring edges are the intersection of two 6-cycles, or*
- *there exists a path efg which is the intersection of two 6-cycles.*

Proof. Let G be an irreducible (v_3) graph and let e be an arbitrary edge of G. Then the graph obtained from G by reducing e contains a cycle of length 4. We obtain this 4-cycle by adding edge x_1y_1 or x_2y_2 in the case of type (a) reduction (see Figure 1) or by adding edge x_1y_2 or x_2y_1 in the case of type (b) reduction. It follows that in G there is a 6-cycle containing the edges x_1u, e, vy_1 or x_2u, e, vy_2 (before type (a) reduction) and a 6-cycle containing the edges x_1u, e, vy_2 or x_2u, e, vy_1 (before type (b) reduction). Altogether, there are four ways in which these two 6-cycles can intersect each other. Let us assume that the two 6-cycles are C_1 containing the edges x_1u, e, vy_2 and C_2 containing the edges x_1u, e, vy_1. The cycles C_1 and C_2 can intersect in two or three edges. In the first case the intersection consists of two adjacent edges x_1u and e. In the other case the intersection consists of the edges $e = uv$, x_1u, zx_1, where z is the third vertex in $C_1 \cap C_2$. These three edges construct a path of length three (counting the number of edges) where e is not the middle edge.

The converse is also true. Let $e = uv$ and let e and vw form an intersection of two 6-cycles $C_1 = uvwx_1x_2x_3u$ and $C_2 = uvwy_1y_2y_3u$. After the reduction of e we obtain by type (a) reduction 4-cycle $wx_1x_2x_3w$ and by type (b) reduction 4-cycle $wy_1y_2y_3w$. It follows that e can not be reduced. Now let $e = uv$ and let e and the path $uvwz$ of length three form an intersection of two 6-cycles $C_1 = uvwzx_1x_2u$ and $C_2 = uvwzy_1y_2u$. Again, after the reduction of e we obtain either the cycle wzx_1x_2w or the cycle wzy_1y_2w. Since both cycles are of length four e can not be reduced. Therefore, the two conditions of the theorem imposed on each edge e of the graph G ensure that G is irreducible. □

Next, we define several families of (v_3) graphs.

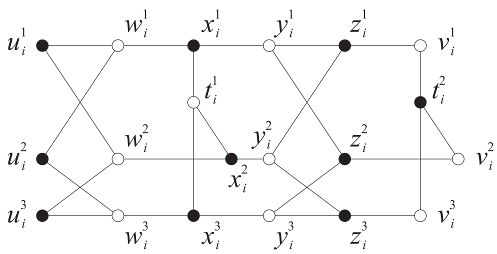
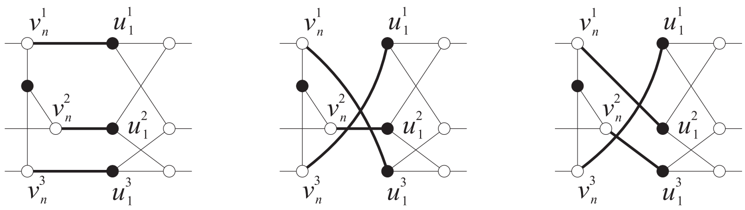


Figure 2: Segment G_T defining the families $T_1(n)$, $T_2(n)$ and $T_3(n)$.

Let $T(n)$, $n \geq 1$, denote a graph on $20n$ vertices which is a union of n segments G_T shown in Figure 2 where the i-th segment $(i \geq 2)$ and the $(i-1)$-th segment are joined together by the edges $v_{i-1}^1u_i^1$, $v_{i-1}^2u_i^2$, $v_{i-1}^3u_i^3$. We will use $T(n)$ in the following definitions. Let $T_1(n)$ be the graph which is obtained from $T(n)$ by adding the edges $u_1^1v_n^1$, $u_1^2v_n^2$, and $u_1^3v_n^3$. Let $T_2(n)$ be the graph obtained from $T(n)$ by adding the edges $u_1^3v_n^1$, $u_1^2v_n^2$, and $u_1^1v_n^3$. And finally, let $T_3(n)$ be the graph obtained from $T(n)$ by adding the edges $u_1^1v_n^3$, $u_1^2v_n^1$, and $u_1^3v_n^2$. See Figure 3.



(a) (b) (c)

Figure 3: The construction of the graphs $T_1(n)$ (a), $T_2(n)$ (b), and $T_3(n)$ (c) from $T(n)$ by adding three edges (shown thick) joining the last and the first segment.

Note that due to the symmetries of the graph G_T (see the list (1) in the proof of the next proposition) it is not important how the vertices v_{i-1}^j are connected to the vertices u_i^k. We always obtain the graph $T(n)$.

Proposition 3. *For each fixed $n \geq 1$, no two of the graphs $T_1(n)$, $T_2(n)$, $T_3(n)$ are isomorphic, and every other irreducible graph on $20n$ vertices, that can be obtained from $T(n)$ by adding three edges, is isomorphic to one of them.*

Proof. Obviously, $T_1(n)$, $T_2(n)$, $T_3(n)$ are cubic and bipartite graphs with girth 6, thus, they are (v_3) graphs. It is easy to check, that each edge satisfies the conditions of Lemma 2 which ensures that the graphs are also irreducible. (It is sufficient to check the conditions for all edges in the segment and the edges joining the segments together.)

At each fixed n, the claim of the theorem can be verified by using some computer program which checks the existence of an isomorphism between two graphs. To prove the statement that the three graphs are non-isomophic in general, we show that the numbers of orbits for the action of the automorphism group on the set of edges are 3, 6, 4 respectively for $T_1(n)$, $T_2(n)$, $T_3(n)$.

The fact about the number of orbits will be easier to see if we first list the automorphisms of G_T. These are: identity,

$$\begin{aligned}
\alpha_2 &= (u_i^3\ u_i^2)(w_i^2\ w_i^1)(x_i^2\ x_i^1)(y_i^2\ y_i^1)(z_i^3\ z_i^2)(v_i^3\ v_i^2),\\
\alpha_3 &= (u_i^2\ u_i^1)(w_i^3\ w_i^2)(x_i^3\ x_i^2)(y_i^3\ y_i^2)(z_i^2\ z_i^1)(v_i^2\ v_i^1),\\
\alpha_4 &= (u_i^2\ u_i^3\ u_i^1)(w_i^3\ w_i^2\ w_i^1)(x_i^3\ x_i^2\ x_i^1)(y_i^3\ y_i^2\ y_i^1)(z_i^2\ z_i^3\ z_i^1)(v_i^2\ v_i^3\ v_i^1),\\
\alpha_5 &= (u_i^3\ u_i^2\ u_i^1)(w_i^2\ w_i^3\ w_i^1)(x_i^2\ x_i^3\ x_i^1)(y_i^2\ y_i^3\ y_i^1)(z_i^3\ z_i^2\ z_i^1)(v_i^3\ v_i^2\ v_i^1),\\
\alpha_6 &= (u_i^3\ u_i^1)(w_i^3\ w_i^1)(x_i^3\ x_i^1)(y_i^3\ y_i^1)(z_i^3\ z_i^1)(v_i^3\ v_i^1).
\end{aligned} \tag{1}$$

The three edge orbits of the graph $T_1(n)$, $n \geq 2$, are the sets

$$\begin{aligned}
O_1 &= \{u_i^1w_i^1, u_i^1w_i^2, u_i^2w_i^1, u_i^2w_i^3, u_i^3w_i^2, u_i^3w_i^3,\\
&\qquad y_i^1z_i^1, y_i^1z_i^2, y_i^2z_i^1, y_i^2z_i^3, y_i^3z_i^2, y_i^3z_i^3 : i = 1, 2, \dots, n\},\\
O_2 &= \{w_i^1x_i^1, w_i^2x_i^2, w_i^3x_i^3, x_i^1y_i^1, x_i^2y_i^2, x_i^3y_i^3,\\
&\qquad z_i^1v_i^1, z_i^2v_i^2, z_i^3v_i^3, v_i^1u_{i+1}^1, v_i^2u_{i+1}^2, v_i^3u_{i+1}^3 : i = 1, 2, \dots, n\},\\
O_3 &= \{t_i^1x_i^1, t_i^1x_i^2, t_i^1x_i^3, t_i^2v_i^1, t_i^2v_i^2, t_i^2v_i^3 : i = 1, 2, \dots, n\},
\end{aligned}$$

see Figure 4(a). The existence of an automorphism which maps an edge from O_i to another edge from the same set is evident from the definition of $T_1(n)$ and the fact about automorphisms of the graph G_T, (1). Since the edges from O_1 are contained in three 6-cycles, edges from O_2 in two 6-cycles, and edges from O_3 in four 6-cycles, the sets O_1, O_2, and O_3 are indeed three different orbits.

The graph $T_2(n)$, $n \geq 2$, has 6 edge orbits:

$$\begin{aligned}
P_1 &= \{u_i^1w_i^1, u_i^3w_i^3, y_i^1z_i^1, y_i^3z_i^3 : i = 1, 2, \dots, n\},\\
P_2 &= \{u_i^1w_i^2, u_i^2w_i^1, u_i^2w_i^3, u_i^3w_i^2, y_i^1z_i^2, y_i^2z_i^1, y_i^2z_i^3, y_i^3z_i^2 : i = 1, 2, \dots, n\},\\
P_3 &= \{w_i^1x_i^1, w_i^3x_i^3, x_i^1y_i^1, x_i^3y_i^3, z_i^1v_i^1, z_i^3v_i^3, v_i^1u_{i+1}^1, v_i^3u_{i+1}^3 : i = 1, 2, \dots, n\},\\
P_4 &= \{w_i^2x_i^2, x_i^2y_i^2, z_i^2v_i^2, v_i^2u_{i+1}^2 : i = 1, 2, \dots, n\},\\
P_5 &= \{t_i^1x_i^1, t_i^1x_i^3, t_i^2v_i^1, t_i^2v_i^3 : i = 1, 2, \dots, n\},\\
P_6 &= \{t_i^1x_i^2, t_i^2v_i^2 : i = 1, 2, \dots, n\},
\end{aligned}$$

see Figure 4(b). (In the set P_3 the last two edges, at $i = n$, should be replaced by $v_n^1u_1^3$ and $v_n^3u_1^1$.) Now we must show that $P_1 \neq P_2$, $P_3 \neq P_4$, and $P_4 \neq P_5$.

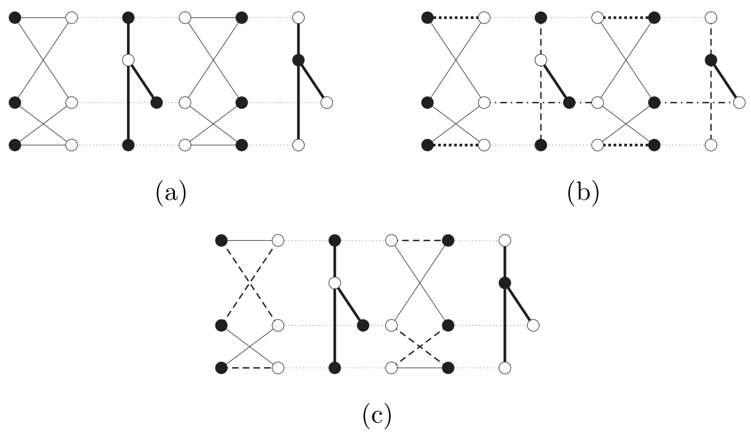


Figure 4: The edge orbits of the graphs $T_1(n)$ (a), $T_2(n)$ (b), $T_3(n)$ (c) shown on a segment. Edges depicted with the same line style belong to the same orbit.

Let us, for example, show that there is no automorphism which would map $u_1^1w_1^1$ to $u_1^1w_1^3$ (i.e. that $P_1 \neq P_2$). Now, suppose that this automorphism exists. The first possibility is that it fixes u_1^1 and interchanges w_1^1 and w_1^2. But the transposition $(w_1^1\, w_1^2)$ already induces the rest of the mapping. Hence, the only possibility for this automorphism would be:

$$(u_1^2\, u_1^3)(w_1^1\, w_1^2)(x_1^1\, x_1^2)(y_1^1\, y_1^2)(z_1^2\, z_1^3)(v_1^2\, v_1^3)(u_2^2\, u_2^3)\cdots(v_n^2\, v_n^3).$$

But this mapping is not an automorphism. Since there exist edges $v_n^2u_1^2$ and $v_n^3u_1^1$, the automorphism would have to interchange u_1^1 and u_1^2. But it does not do this. The next possibility is an automorphism which would map u_1^1 to w_1^2 and w_1^1 to u_1^1. This induces the mapping

$$(w_1^2\, u_1^3\, w_1^3\, u_1^2\, w_1^1\, u_1^1)(t_1^1\, t_n^2)(x_1^1\, v_n^3\, x_1^2\, v_n^1\, x_1^3\, v_n^2)(y_1^1\, z_n^3\, y_1^2\, z_n^1\, y_1^3\, z_n^2)$$
$$(z_1^1\, y_n^2\, z_1^3\, y_n^1\, z_1^2\, y_n^3)(t_1^2\, t_n^1)\cdots,$$

which, again, is not an automorphism of $T_2(n)$. A contradiction occurs in the "middle" of the graph. This shows that $P_1 \neq P_2$. The remaining two inequalities, $P_3 \neq P_4$ and $P_4 \neq P_5$, can be justified in a similar way.

The edge orbits of $T_3(n)$ are:

$$\begin{aligned}
Q_1 &= \{u_i^1w_i^1, u_i^2w_i^3, u_i^3w_i^2, y_i^1z_i^2, y_i^2z_i^1, y_i^3z_i^3 : i = 1, 2, \ldots, n\},\\
Q_2 &= \{u_i^1w_i^2, u_i^2w_i^1, u_i^3w_i^3, y_i^1z_i^1, y_i^2z_i^3, y_i^3z_i^2 : i = 1, 2, \ldots, n\},\\
Q_3 &= \{w_i^1x_i^1, w_i^2x_i^2, w_i^3x_i^3, x_i^1y_i^1, x_i^2y_i^2, x_i^3y_i^3, z_i^1v_i^1, z_i^2v_i^2, z_i^3v_i^3,\\
&\qquad v_i^1u_{i+1}^1, v_i^2u_{i+1}^2, v_i^3u_{i+1}^3 : i = 1, 2, \ldots, n\},\\
Q_4 &= \{t_i^1x_i^1, t_i^1x_i^2, t_i^1x_i^3, t_i^2v_i^1, t_i^2v_i^2, t_i^2v_i^3 : i = 1, 2, \ldots, n\},
\end{aligned}$$

see Figure 4(c). (In the set Q_3 the last three edges, at $i = n$, should be replaced by $v_n^1u_1^2$, $v_n^2u_1^3$, $v_n^3u_1^1$.) This time we must show that $Q_1 \neq Q_2$. We can prove this if we try to map $u_1^1w_1^1$ to $u_1^1w_1^3$. It turns out that this is not possible. The reason is the same as in the case when we proved that $P_1 \neq P_2$.

Finally we prove that the remaining three graphs which can be obtained from $T(n)$ are isomorphic to $T_1(n)$ or $T_2(n)$ or $T_3(n)$. Let $T_4(n)$ be the graph obtained from $T(n)$ by adding the edges $v_n^1 u_1^3$, $v_n^2 u_1^1$, $v_n^3 u_1^2$, let $T_5(n)$ be the graph obtained from $T(n)$ by adding the edges $v_n^1 u_1^1$, $v_n^2 u_1^3$, $v_n^3 u_1^2$, and let $T_6(n)$ be the graph obtained from $T(n)$ by adding the edges $v_n^1 u_1^2$, $v_n^2 u_1^1$, $v_n^3 u_1^3$. Obviously $T_4(n) \cong T_3(n)$ and $T_5(n) \cong T_6(n)$ (the constructions are symmetric). Furthermore $T_5 \cong T_6(n) \cong T_2(n)$. The isomorphism between $T_2(n)$ and $T_5(n)$ is

$$\begin{array}{ccccccc} u_i^1 \mapsto u_i^2, & u_i^2 \mapsto u_i^1, & u_i^3 \mapsto u_i^3, & w_i^1 \mapsto w_i^1, & w_i^2 \mapsto u_i^3, & w_i^3 \mapsto w_i^2, & \\ x_i^1 \mapsto x_i^1, & x_i^2 \mapsto x_i^3, & x_i^3 \mapsto x_i^2, & y_i^1 \mapsto y_i^1, & y_i^2 \mapsto y_i^3, & y_i^3 \mapsto y_i^2, & t_i^1 \mapsto t_i^1, \\ z_i^1 \mapsto z_i^2, & z_i^2 \mapsto z_i^1, & z_i^3 \mapsto z_i^3, & v_i^1 \mapsto v_i^2, & v_i^2 \mapsto v_i^1, & v_i^3 \mapsto v_i^3, & t_i^2 \mapsto t_i^2, \end{array}$$

$i = 1, 2, \ldots, n$. □

Proposition 4. *Each of the graphs $T_1(n)$, $T_2(n)$, $T_3(n)$, $n \geq 1$, is an incidence graph of a self-polar irreducible $((10n)_3)$ configuration.*

Remark. $T_1(1)$ is incidence graph of the Desargues (10_3) configuration, while $T_2(1)$ and $T_3(1)$ determine the configurations $(10_3)_2$ and $(10_3)_6$, respectively, according to the classification found in [4].

Proof. We only need to find an automorphism of order two which interchanges vertices of the bipartition for each of the graphs $T_1(n)$, $T_2(n)$, $T_3(n)$. For $T_1(n)$ this automorphism is (given in cycle notation)

$$\begin{gathered} (u_1^1\, w_1^1)(u_1^2\, w_1^2)(u_1^3\, w_1^3)(t_1^1\, t_n^2)(t_1^2\, t_n^1)(x_1^1\, v_n^1)(x_1^2\, v_n^2)(x_1^3\, v_n^3) \\ (y_1^1\, z_n^1)(y_1^2\, z_n^2)(y_1^3\, z_n^3)(z_1^1\, y_n^1)(z_1^2\, y_n^2)(z_1^3\, y_n^3) \\ (v_1^1\, x_n^1)(v_1^2\, x_n^2)(v_1^3\, x_n^3)(u_2^1\, w_n^1)(u_2^2\, w_n^2)(u_2^3\, w_n^3) \\ (w_2^1\, u_{n-1}^1)(w_1^2\, u_{n-1}^2)(w_2^3\, u_{n-1}^3)(t_2^1\, t_{n-1}^2)(t_2^2\, t_{n-1}^1)(x_2^1\, v_{n-1}^1)(x_2^2\, v_{n-1}^2)(x_2^3\, v_{n-1}^3) \\ \cdots \end{gathered}$$

Labeling of the vertices is presented on the segment in Figure 2. Automorphism of $T_2(n)$ is similar and for this reason we can give shorter argumentation

$$\begin{gathered} (u_1^1\, w_1^1)(u_1^2\, w_1^2)(u_1^3\, w_1^3)(t_1^1\, t_n^2)(t_1^2\, t_n^1)(x_1^1\, v_n^3)(x_1^2\, v_n^2)(x_1^3\, v_n^1) \\ (y_1^1\, z_n^3)(y_1^2\, z_n^2)(y_1^3\, z_n^1)(z_1^1\, y_n^3)(z_1^2\, y_n^2)(z_1^3\, y_n^1) \\ \cdots \end{gathered}$$

And, finally, automorphism of $T_3(n)$ is

$$\begin{gathered} (u_1^1\, w_1^1)(u_1^2\, w_1^2)(u_1^3\, w_1^3)(t_1^1\, t_n^2)(t_1^2\, t_n^1)(x_1^1\, v_n^3)(x_1^2\, v_n^2)(x_1^3\, v_n^1) \\ (y_1^1\, z_n^3)(y_1^2\, z_n^1)(y_1^3\, z_n^2)(z_1^1\, y_n^3)(z_1^2\, y_n^1)(z_1^3\, y_n^2) \\ \cdots \end{gathered}$$

□

Let $C(n)$, $n \geq 1$, be the graph on $6n$ vertices, which is a union of n segments (6-cycles) depicted in Figure 5 and the i-th segment is joined with the $(i-1)$-th segment, $i \geq 2$, by the edges $v_{i-1}^1 u_i^1$, $v_{i-1}^2 u_i^4$, and $u_{i-1}^3 u_i^2$. See the Figure 6. Finally, let $D(n)$ be the graph defined in the following way.

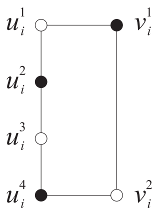


Figure 5: Segment from the definition of the graph $C(n)$.

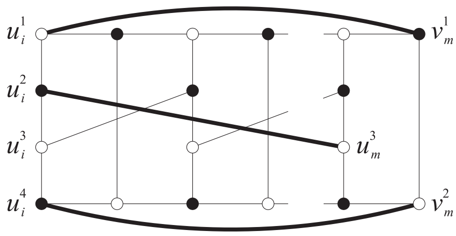


Figure 6: Graph $D(n)$ for $n \equiv 0 \pmod 3$, which is the graph $C(m)$, $m = \frac{n}{3}$, (thin edges) with three edges added (shown thick).

For $n \equiv 0 \pmod 3$ let $D(n)$ be graph $C(m)$, $m = \frac{n}{3}$, with three edges $u_1^1 v_m^1$, $u_1^4 v_m^4$, $u_1^2 u_m^3$ added. See Figure 6.

For $n \equiv 1 \pmod 3$ let $D(n)$ be graph $C(m)$, $m = \frac{n-1}{3}$, with two vertices w_m^1, w_m^2 and six edges $u_1^1 w_m^1$, $u_1^2 v_m^2$, $u_1^4 w_m^2$ $w_m^1 w_m^2$, $w_m^1 u_m^3$, $w_m^2 v_m^1$ added

For $n \equiv 2 \pmod 3$ let $D(n)$ be graph $C(m)$, $m = \frac{n-2}{3}$, with four vertices w_m^1, w_m^2, w_m^3, w_m^4 and nine edges $v_m^1 w_m^1$, $v_m^2 w_m^4$, $u_m^3 w_m^2$, $u_1^1 w_m^4$, $u_1^2 w_m^1$, $u_1^4 w_m^3$, $w_m^1 w_m^2$, $w_m^2 w_m^3$, $w_m^3 w_m^4$ added.

Proposition 5. *For each $n \geq 7$ graph $D(n)$ is an irreducible (v_3) graph on $2n$ vertices. Graph $D(n)$, $n \geq 7$, is an incidence graph of the cyclic (n_3) configuration with base line $\{0, 1, 3\}$. These configurations are self-polar.*

Proof. The construction of graphs $D(n)$ and $C(n)$ assures that $C(n)$ is a cubic bipartite graph. It is also easy to see that $\mathrm{girth}(C(n)) = 6$, so $C(n)$ is a (v_3) graph. It turns out that for every edge e of $C(n)$ there exist edges f and g such that the path efg is the intersection of two 6-cycles. Then, by Lemma 2, it follows that $C(n)$ is also irreducible. Isomorphism between the graph of the cyclic (n_3) configuration with base line $\{0, 1, 3\}$ and graph $D(n)$ is given by the following rules:

$$3i - 3 \mapsto u_i^2, \quad 3i - 2 \mapsto u_i^4, \quad 3i - 1 \mapsto v_j^1, \qquad i = 1, 2, \ldots, \left\lfloor \frac{n}{3} \right\rfloor.$$

If $n \equiv 1 \pmod 3$ then the additional rule is $n - 1 \mapsto w_m^1$, $m = \left\lfloor \frac{n}{3} \right\rfloor$, and if $n \equiv 2 \pmod 3$ then the additional rules are $n - 2 \mapsto w_m^2$ and $n - 1 \mapsto w_m^4$. An automorphism of order 2 which interchanges points and lines of these configurations, i.e. white and black points of their incidence graphs, maps point i to line $\{-i, 1 - i, 3 - i\}$ (arithmetic is modulo n). □

3 The Martinetti theorem

The theory we developed up to this point is already enough to state and prove the main theorem.

Theorem 6. *The only connected irreducible (v_3) graphs are:*

1. *graph of the Pappus configuration, see Figure 7,*
2. *graphs $T_1(n)$, $T_2(n)$, $T_3(n)$ for $n \geq 1$,*

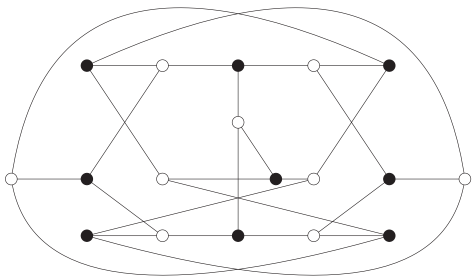


Figure 7: Incidence graph of the Pappus configuration.

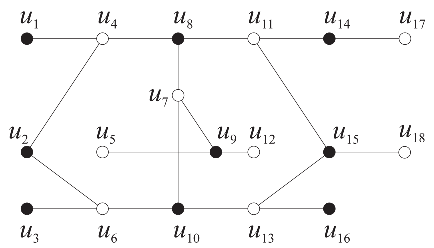


Figure 8: Graph G_0^1 from the proof of Theorem 6.

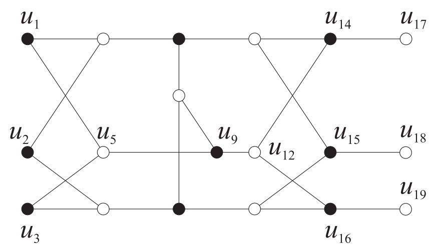


Figure 9: Graph G_3 from the proof of Theorem 6.

3. graphs $D(n)$ for $n \geq 7$.

Proof. We distinguish two cases. First, we assume that in the given irreducible (v_3) graph there exist no two 6-cycles which intersect in three edges (this must be a path of length 3). Then, by Lemma 2, there must exist two 6 cycles intersecting in a path of length two. Locally, the structure in the neighborhood of these two cycles must be such as it is shown in Figure 8. We denote this graph by G_0^1. By Lemma 2, the edge $e = u_7u_9$ and one of its neighbors must lie on two 6-cycles. Since the situation is symmetric we may assume that one of the cycles containing e is $u_7u_9u_{12}u_{14}u_{11}u_8u_7$, i.e. there exist the edge $u_{12}u_{14}$. So, let us denote $G_1^1 = G_0^1 + u_{12}u_{14}$. Vertex u_{12} must have another black neighbor. The only choices in G_1^1 are vertices u_{16} and u_3 (since we assume that in the graph there exist no two 6-cycles intersecting in three edges). Another possibility is that we choose a vertex which is not a vertex in G_1^1. We denote graphs obtained by choosing the neighbor of u_{12} in these three ways by G_2^1, G_2^2, and G_2^3, successively.

First, let us continue with G_2^1. It leads to the graph G_3 shown in Figure 9. This is true since vertex u_{16} must be connected to a new white vertex (denoted by u_{19} in G_3) and vertex u_5 must be connected to u_1 and u_3 because of the edge u_5u_9. Next we focus to the vertices u_{17}, u_{18}, and u_{19}. Each of them should be connected to two black vertices. Some of these black vertices can be chosen from the vertices u_1, u_2, u_3 but some of them must be new. In each case it follows, due to the fact that each of the edges $u_{14}u_{17}$, $u_{15}u_{18}$, and $u_{16}u_{19}$ must be in the intersection of two 6-cycles (Lemma 2), that u_{17}, u_{18}, and u_{19} must have a common new black neighbor, see graph G_4 in Figure 10. If vertex u_{17} (or u_{18}, or u_{19}) has a neighbor in the set $U = \{u_1, u_2, u_3\}$ then vertices u_{18} and

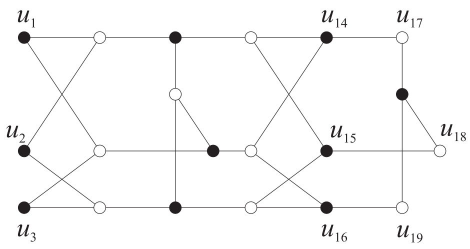


Figure 10: Graph G_4 from the proof of Theorem 6.

u_{19} must also be connected to the remaining two vertices from U. This is true, since it would not be possible for the edges from u_{18} and u_{19} to new vertices to be contained in two 6-cycles. Noticing that the graph G_4 is actually the segment in the Figure 2, we recognize that the described case leads to the three non-isomorphic irreducible (v_3) graphs $T_1(1)$, $T_2(1)$, $T_3(1)$ (by Proposition 3).

We are left with the case where we add three new vertices to the graph G_4 and connect them to vertices u_{17}, u_{18}, u_{19}. We obtain the graph G_5, see Figure 11. Now, the new vertices u_{21}, u_{22}, u_{23} must be connected with new white vertices.

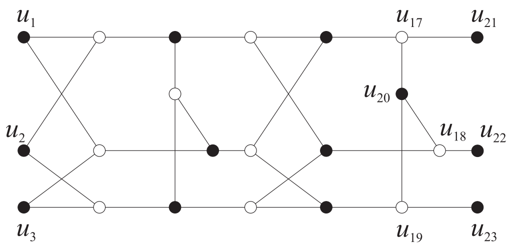


Figure 11: Graph G_5 from the proof of Theorem 6.

The only possibility is the graph G_6 which is shown in Figure 12. This follows easily if we consider the requirements of the Lemma 2 on the edges $u_{17}u_{21}$,

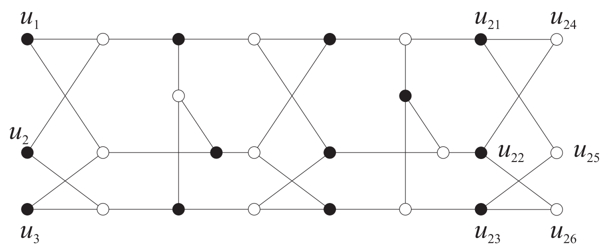


Figure 12: Graph G_6 from the proof of Theorem 6.

$u_{18}u_{22}$, $u_{19}u_{23}$. In the next step, we observe that each of the vertices u_{24}, u_{25}, u_{26} should be connected with one black vertex. We clearly can not use only one from the set U for these black vertices, but we also can not use two or three vertices from U since we would obtain 4-cycles. Hence, the only possibility is to add three new black vertices. Proceeding in this way, we obtain graph G_8 which is shown in Figure 13. Here, as at the time we were considering the graph

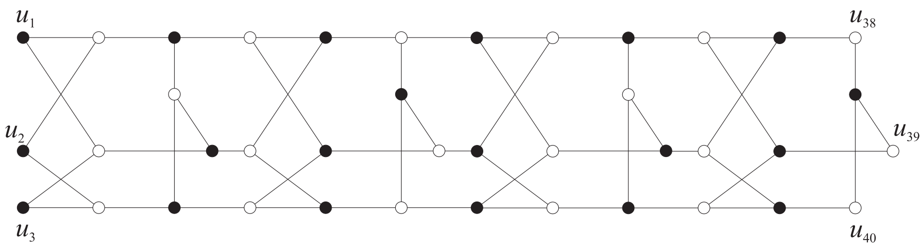


Figure 13: Graph G_8 from the proof of Theorem 6.

G_4, we obtain, by connecting vertices u_{38}, u_{39}, u_{40} only to the vertices u_1, u_2, u_3, graphs $T_1(2)$, $T_2(2)$, $T_3(2)$ or continue with three new vertices. In the latter case, we continue in the same manner as with graph G_5, only that this time the graph is for a segment larger. Hence, we conclude that the continuation of the procedure gives precisely families $T_1(n)$, $T_2(n)$, $T_3(n)$.

Now, let us return back to the graph G_2^2. By considering all possibilities and excluding the cases where the situation contradicts Lemma 2 it is possible to see that G_2^2 leads only to the Pappus graph, i.e. to the graph of the Pappus configuration, see Figure 7. Similarly, it is possible to see that G_2^3 leads only to graphs isomorphic to those obtained from G_2^1. Thus we do not get any new irreducible (v_3) graphs.

In the second part of the proof, let us assume that in the given irreducible graph there exist two 6-cycles intersecting in a path of length three (counting the number of edges). Locally, the structure of this graph must correspond to the graph H_0 shown in Figure 14. where the two 6-cycles are $u_1u_2u_3u_4u_6u_5u_1$

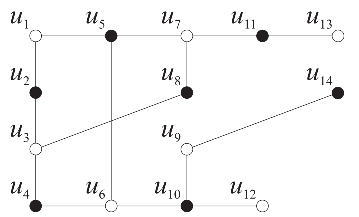


Figure 14: Graph H_0 from the proof of Theorem 6.

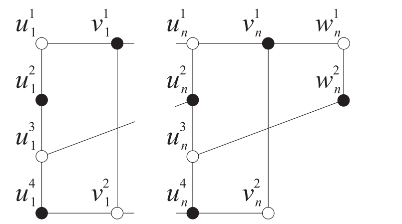


Figure 15: Graph $H_1(n)$ from the proof of Theorem 6.

and $u_3u_4u_6u_5u_7u_8u_3$. Now we imitate considerations we did in the previous case. We systematically add vertices and edges to H_0 and to the subsequent graphs such that they satisfy Lemma 2. First, Lemma 2 used on edges u_9u_{10} and u_6u_{10} implies that there should exist edge u_8u_9. Similarly, it follows that there must exist edges $u_{11}u_{12}$ and $u_{13}u_{14}$. (There are other possibilities but it turns out that they do not give any new graphs.) The current situation is the graph $H_1(2)$ where by $H_1(n)$ we denote graph $C(n)$ with vertices w_n^1, w_n^2 and edges $v_n^1w_n^1$, $u_n^3w_n^2$, $w_n^1w_n^2$ added, see Figure 15. From $H_1(n)$, $n \geq 2$, it is possible to continue in the following ways.

If we choose not to add any new vertex then, to obtain an irreducible (v_3) graph, we must add three more edges. This can be done in only one way; we obtain the graph $D(3n+1)$ which is, by Proposition 5, graph of the cyclic

$((3n+1)_3)$ configuration with base line $\{0,1,3\}$.

Next, we assume that we connect precisely two vertices from the set $U_n = \{v_n^2, w_n^1, w_n^2\}$ to the free vertices from the first segment of $H_1(n)$ and only one to a new vertex. But we can disregard this case since it would be not possible to assure the conditions of Lemma 2 for the edge to a new vertex. Next, we consider the case where we add two new vertices and connect them to two vertices from U_n (and we connect the remaining vertex from U_n to a vertex from the first segment). Using Lemma 2 (on new edges and, in one case, on $w_n^1 w_n^2$) we exclude all pairs but v_n^2 and w_n^2. Finally, it is possible to connect all three vertices from U_n to three new vertices. In the last two cases we also recognize that the new vertices we connect to v_n^2 and w_n^2 must be connected. In general, this leads to the graph $H_2(n)$ which is shown in Figure 16. (The case where w_n^1 is connected to a new vertex will discussed in the next step.) Now, the situation is similar to that at the moment we were considering graph

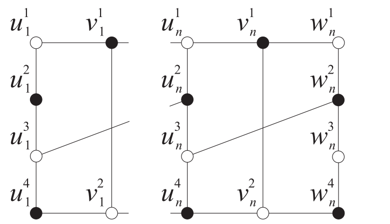


Figure 16: Graph $H_2(n)$ from the proof of Theorem 6.

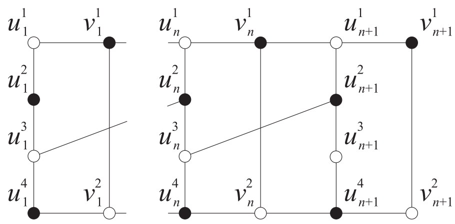


Figure 17: Graph $H_3(n)$ from the proof of Theorem 6.

$H_1(n)$. Vertices from the set $V_n = \{w_n^1, w_n^3, w_n^4\}$ should be connected either to the vertices from the first segment (vertices u_1^1, u_1^2, u_1^4) or to new vertices. If we choose not to add any new vertex we can obtain only graph $D(3n+2)$, $n \geq 2$, while the other cases lead to the graph $H_3(n)$, see Figure 17. The only (v_3) graph we can obtain from $H_3(n)$ by adding edges is graph $D(3n+3)$. The next step, again analogous to those we did above, leads to graph $H_1(n+1)$.

Hence, we got in this part of the proof exactly graphs $D(n)$, $n \geq 3$, which are by Proposition 5 graphs of cyclic (n_3) configurations with base line $\{0,1,3\}$. □

Now we can state the revised form of the Martinetti theorem.

Theorem 7. *All connected irreducible (v_3) configurations are*

1. *cyclic configurations with base line $\{0,1,3\}$,*
2. *configurations with their incidence graphs $T_1(n)$, $T_2(n)$, $T_3(n)$, $n \geq 1$, each of them giving precisely one $((10n)_3)$ configuration, and*
3. *the Pappus configuration.*

Remark. In the theorem stated in the original paper [16] and in its citations [8, 14] configurations arising from graphs $T_2(n)$ and $T_3(n)$ are missing for $n \geq 2$.

Proof. The theorem follows from Theorem 6, Proposition 4, and Proposition 5. □

An important topic in the study of configurations is the problem of their realization with points and lines in the plane. Configurations which can be realized in the plane will be called *linear*. It is a well known fact that, for example, Fano configuration and Möbius-Kantor configuration are not linear while the Pappus configuration and the other two (9_3) configurations are. The problem of realization has a long history. H. Schröter proved in 1888 the realizability in the plane of the cyclic configurations with base line $\{0, 1, 3\}$. In 1889 he proved that nine of the ten combinatorial (10_3) configurations found earlier by Kantor can be realized geometrically in the real plane, but that the remaining one cannot be realized in such a way. The most important result is due to E. Steinitz (1894) which (roughly) says that *every connected* (v_3) *configuration can be drawn in the plane with at most one curved line.* More about realizations and problems can be found in [11, 7, 14]. The geometric view of the configurations is explicit in the work of B. Grünbaum [12, 13, 14]. Recently, this topic has been investigated in [3] for special types of configurations.

Here, we will only briefly present some known results regarding the irreducible configurations. Geometric representations of the two smallest irreducible configurations, the Fano configuration and the Möbius-Kantor configuration are in Figure 18 and Figure 19, respectively. Cyclic (9_3) and (12_3) configurations

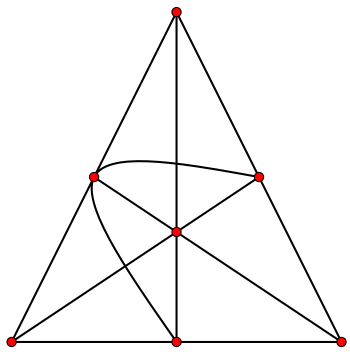


Figure 18: Fano configuration

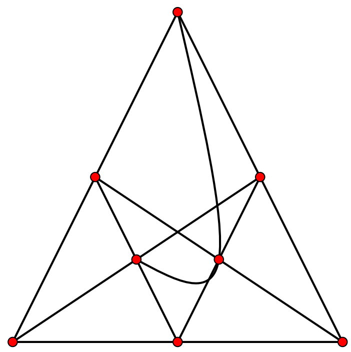


Figure 19: Möbius-Kantor configuration

with base line $\{0, 1, 3\}$ are shown in Figures 20 and 21. Realizations of other cyclic (v_3) configurations with base line $\{0, 1, 3\}$ for $v \equiv 0 \pmod 3$ follow the same principle of mutually inscribed and circumscribed $\frac{v}{3}$-gons which represents the structure of their automorphisms. More about realizations of this kind can be found in [3].

Realizations of the Pappus configuration and the Desargues configuration are in Figures 22 and 23. It is also known that other two irreducible (10_3) configurations are linear. Their realizations can be, for example, found in [4]. With methods presented in [4] and use of a computer it is also not difficult to find realizations for each particular configuration $T_1(n)$, $T_2(n)$, or $T_3(n)$.

But it is more intriguing to give a geometric construction which can be found in [14] for configurations arising from $T_1(n)$. Since the paper does not seem to be widely available we repeat it here. Configurations determined by $T_1(n)$ are built-

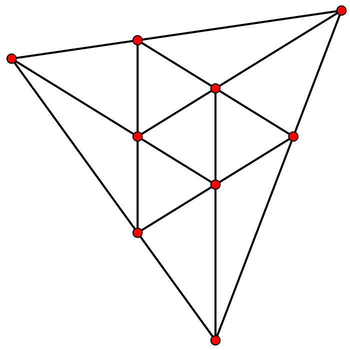


Figure 20: Cyclic (9_3) configuration with base line $\{0, 1, 3\}$.

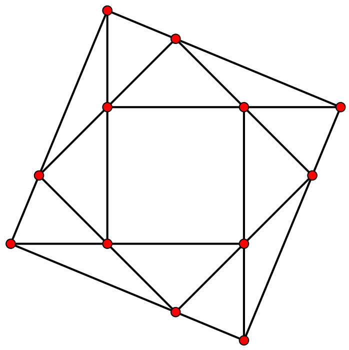


Figure 21: Cyclic (12_3) configuration with base line $\{0, 1, 3\}$.

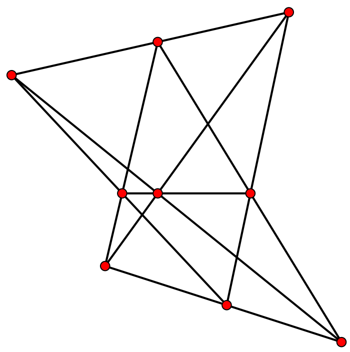


Figure 22: Pappus configuration

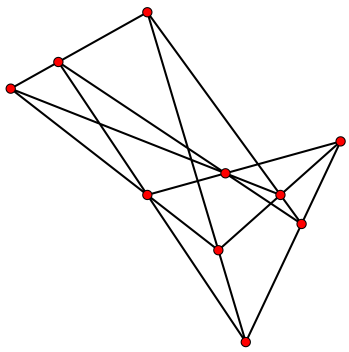


Figure 23: Desargues configuration

up of segments with their graph shown in Figure 2. If we realize these segments geometrically in the way it is done in Figure 24 and appropriately choose the angle between the line $\{v_i^1, v_i^2, v_i^3\}$ and the line $\{v_{i-1}^1, v_{i-1}^2, v_{i-1}^3\}$ (which comes from the previous segment) then we can realize configurations from $T_1(n)$ by attaching n of these segments one next to another. This geometric procedure works for $n \geq 3$. Note that we can redraw the segment in such way that the point t_i^1 which is at infinity in Figure 24 has "Euclidean" coordinates and the construction still works.

The question and an exercise to the reader would now be to find a similar construction for configurations arising from $T_2(n)$ and $T_3(n)$.

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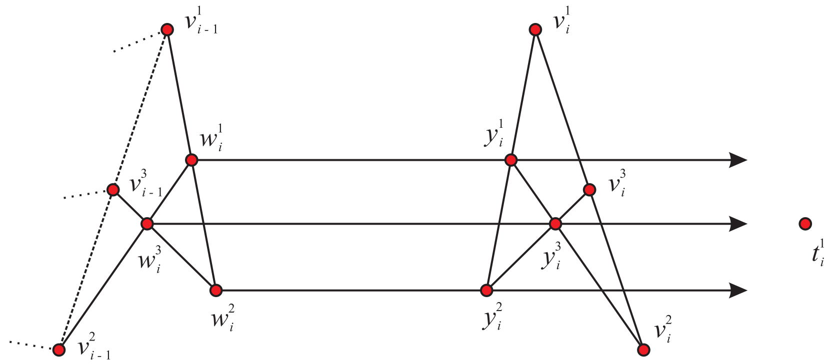

Figure 24: Geometric realization of the configuration segment given by the graph in Figure 2. Horizontal lines meet in point t_i^1 "at infinity".

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