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Preprint series, Vol. 41 (2003), 904

QUARTIC HALF-ARC-TRANSITIVE GRAPHS WITH LARGE VERTEX STABILIZERS

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ISSN 1318-4865

December 5, 2003

Ljubljana, December 5, 2003

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Abstract

A $\frac{1}{2}$-*arc-transitive graph* is a vertex- and edge- but not arc-transitive graph. In all known constructions of quartic $\frac{1}{2}$-arc-transitive graphs, vertex stabilizers are isomorphic to $\mathbb{Z}_2, \mathbb{Z}_2^2$ or to D_8. In this article, for each positive integer $m \geq 1$, an infinite family of quartic $\frac{1}{2}$-arc-transitive graphs having vertex stabilizers isomorphic to $\mathbb{Z}_2^m$, is constructed.

1 Introductory remarks

Throughout this paper graphs are assumed to be finite, simple and, unless stated otherwise, connected and undirected (but with an implicit orientation of the edges when appropriate). For group-theoretic concepts not defined here we refer the reader to [3, 7, 22], and for graph-theoretic terms not defined here we refer the reader to [4].

Given a graph X we let $V(X)$, $E(X)$, $A(X)$ and $\operatorname{Aut} X$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. A graph X is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* if its automorphism group $\operatorname{Aut} X$ acts transitively on $V(X)$, $E(X)$ and $A(X)$ respectively. We say that X is $\frac{1}{2}$-*arc-transitive* provided it is vertex- and edge- but not arc-transitive. More generally, by a $\frac{1}{2}$-*arc-transitive* action of a subgroup $G \leq \operatorname{Aut} X$ on X we mean a vertex- and edge- but not arc-transitive action of G on X. In this case we say that the graph X is $(G, \frac{1}{2})$-*arc-transitive*, and we say that the graph X is $(G, \frac{1}{2}, H)$-*arc-transitive* when it needs to be stressed that the vertex stabilizers G_v (for $v \in V(X)$) are isomorphic to a particular subgroup $H \leq G$. By a classical result of

[1]Supported in part by "Ministrstvo za šolstvo, znanost in šport Slovenije", reserach program no. 101-506.

Tutte [20, 7.35, p.59], a graph admitting a $\frac{1}{2}$-arc-transitive group action is neceessarily of even valency. A few years later Tutte's question as to the existence of $\frac{1}{2}$-arc-transitive graphs of a given even valency was answered by Bouwer [5] with a construction of a $2k$-valent $\frac{1}{2}$-arc-transitive graph for every $k \geq 2$. The smallest graph in Bouwer's family has 54 vertices and valency 4. In 1981, Holt [9] found one with 27 vertices, a graph that is now known to be the smallest $\frac{1}{2}$-arc-transitive graph [1]. There has recently been an increased interest in the study of this class of graphs (see [1, 2, 6, 8, 11, 13, 15, 16, 17, 18, 19, 21, 23]). Among them, *quartic* graphs, that is, graphs of valency 4 (the smallest admissible valency), deserve special attention.

With the exception of an infinite family of quartic $\frac{1}{2}$-arc-transitive graphs with vertex stabilizers isomorphic to $\mathbb{Z}_2^2$, constructed in [12] and the first known example of a quartic $\frac{1}{2}$-arc-transitive graph with a nonabelian vertex stabilizer (more precisely, the dihedral group D_8 of order 8) given in [6], in all other constructions vertex stabilizers are isomorphic to $\mathbb{Z}_2$. It is the main aim of this article to construct, for each positive integer m, an infinite family of quartic $\frac{1}{2}$-arc-transitive graphs having vertex stabilizers isomorphic to $\mathbb{Z}_2^m$.

To describe this construction, two complementary points of view may be taken. We start with the one based on the connection between graphs admitting $\frac{1}{2}$-arc-transitive group actions and transitive permutation groups having non-self-paired suborbits.

Let G be a transitive permutation group acting on a set V and let $v \in V$. There is a 1-1 correspondence between the set of *suborbits* of G, that is, the set of orbits of the stabilizer G_v on V, and the set of *orbitals* of G, that is, the set of orbits in the natural action of G on $V \times V$, with the trivial suborbit $\{v\}$ corresponding to the diagonal $\{(v, v) : v \in V\}$. For a suborbit W of G, let $\Delta = \Delta_W$ be the corresponding orbital of G. Then the *orbital graph* $X(G, V; W)$ *of* (G, V) *relative to* W is the graph with vertex set V and arc set Δ.

The paired orbital of an orbital Δ is $\Delta^t = \{(v, w) : (w, v) \in \Delta\}$. The orbital Δ is said to be *self-paired* if $\Delta^t = \Delta$, and *non-self-paired* otherwise; in the latter case $\Delta \cap \Delta^t = \emptyset$. This notion of (non)-self-pairedness also carries over to suborbits in a natural way, and it is important to note that for a non-self-paired suborbit W of G, the orbital graph $X(G, V; W)$ is an oriented graph, whereas the underlying undirected graph $X^*(G, V; W)$ admits a $\frac{1}{2}$-arc-transitive action of G.

In the specific instance (of the situation described above) where $V = \mathcal{H}$ is the set of left cosets of a subgroup H of G and W is a non-self-paired suborbit of length 2 in the action of G on $\mathcal{H}$ (by right multiplication), it follows that $X^*(G, \mathcal{H}, W)$ is a quartic $(G, \frac{1}{2}, H)$-arc-transitive graph. In view of these remarks, our construction of a quartic $\frac{1}{2}$-arc-transitive graph with the desired properties will be based on a group G for which there exists a positive integer m such that the action of G on the set of left cosets of a subgroup $H \leq G$ isomorphic to $\mathbb{Z}_2^m$ gives rise to a non-self-paired suborbit of length 2. More precisely, the following theorem captures the main result of this paper. (For convenience we take A_n to be the group of all even permutations, respectively, on the set $\wp_n = \{0, 1, \dots, n-1\}$ of n letters).

Theorem 1.1 *For each positive integer m there exist infinitely many quartic $\frac{1}{2}$-arc-transitive graphs with vertex stabilizers isomorphic to $\mathbb{Z}_2^m$.*

More precisely, let $r \geq 2$ be a positive integer, let $n = 2k+1 \geq 4r+3$, let $a = a_n = (0, 1, \dots, 2k) \in A_n$ and $b = b_{n,r} = a^t = t^{-1}at$, where $t = t_{n,r} = (0, r)(2r, 3r+1) \in A_n$, and for each j let $\sigma_j = a^j b^{-j}$. Further let $G = G_{n,r} = \langle a, b \rangle$, let $H_1 = 1$ and for each $2 \leq m \leq r$ let $H_m = \langle \sigma_1, \dots, \sigma_{m-1} \rangle$. Then

- *(i) $G \cong A_n$;*
- *(ii) for each m the set $\{aH_m, bH_m\}$ is a non-self-paired suborbit of length 2 in the action of G on the set $\mathcal{H}$ of left cosets of H_m on G; and*
- *(iii) the underlying undirected graph $X_{n,r,m}$ of the corresponding orbital graph $X(G, H_m; \{aH_m, bH_m\})$ is a quartic $\frac{1}{2}$-arc-transitive with vertex stabilizers isomorphic to $\mathbb{Z}_2^m$ and its automorphism group, isomorphic to $A_n \times \mathbb{Z}_2$, is generated by G and an additional automorphism α_t (induced by the permutation t) which, for each $x \in G_{n,r}$, maps the coset xH_m to the coset $txtH_m$.*

Following theory developed in [16], these permutations a and b were chosen in such a way that the permutations σ_i, $i \in \{1, 2, \dots, r-1\}$ are all (mutually commuting) involutions, forcing each H_m, $m \leq r$, to be an elementary abelian group of order 2^{m-1} (and not normal in $G_{n,r}$). By [16, Theorem 4.1], the graph $X_{n,r,m}$ admits a $\frac{1}{2}$-arc-transitive action of $G_{n,r}$ with vertex stabilizers isomorphic to $\mathbb{Z}_2^{m-1}$. Moreover, as we shall see in the last section, the full automorphism group of $X_{n,r,m}$ coincides with the group $\langle G_{n,r}, \alpha_t \rangle$.

As remarked above there is an alternative description, more geometric in nature, for the graphs given in Theorem 1.1.

An even length cycle C in a given oriented graph is an *alternating cycle* if every other vertex of C is the tail and every other vertex of C is the head of its two incident arcs. By an *edge* of an oriented graph we mean an edge of the underlying undirected graph.

Let X be a graph of valency 4 admitting a $\frac{1}{2}$-arc-transitive action of some subgroup $G \leq \mathrm{Aut}\, X$. Let us assign an orientation to a given edge of X. Then, via the $\frac{1}{2}$-arc-transitive action of G, this orientation extends uniquely to a balanced orientation of the edge set of X, thus giving rise to an oriented graph whose underlying undirected graph is X. (An oriented graph is *balanced* if all vertices of its vertices have equal in- and out-degrees). The above defined concept of alternating cycles may be extended to graphs of valency 4 admitting $\frac{1}{2}$-arc-transitive group action via the above orientation of the edge set induced by this group. In particular, let us mention that the G-alternating cycles in X are all of equal length and they decompose the edge set of X (see [13]). (When $G = \mathrm{Aut}\, X$, these cycles are referred to as the *alternating cycles* of X.) We define the graph $Al(X) = Al_G(X)$ as the intersection graph of X with respect to the G-alternating cycles in X. If the G-alternating cycles of X have length 4, then $Al(X)$ has valency 4 and, as it can easily be seen (see also [15]) it admits a $\frac{1}{2}$-arc-transitive action of G with vertex stabilizer having twice as many elements as that of G in X. Providing the alternating cycles of $Al(X)$ again have length 4, we may repeat the operation and construct $Al^2 = Al(Al(X)$. In this context the meaning of $Al^j(X)$, $j \geq 1$ an integer, is self-explanatory. (A more formal definiton of thes e concepts is given in Section 2.)

Now let n and r have the meaning described in the statement of Theorem 1.1. Clearly, the graph $X_{n,r,1}$ is isomorphic to the Cayley graph $Cay(G_{n,r}; \{a, b, a^{-1}, b^{-1}\})$. (A *Cayley graph* of a group G relative to a subset Q of G such that $id \notin Q$ and $Q = Q^{-1}$ has vertex set G and edges of the form $[g, gq]$, for all $g \in G$ and $q \in Q$.) The choice of permutations a and b forces the $G_{n,r}$-alternating cycles in $X_{n,r,1}$ to be of length 4, as we shall see in Section 3. We may thus apply the operator Al. The graph $Al(X_{n,r,1})$ turns out to be isomorphic to the graph $X_{n,r,2}$. More generally, all of the graphs $X_{n,r,m}$, $2 \leq m \leq r$, have $G_{n,r}$-alternating cycles of length 4, thus allowing a recursive application of the operator Al. It may be seen that the graph $Al^{m-1}(X_{n,r})$ is isomorphic to $X_{n,r,m}$. We will give a formal argument for these facts in Section 4.

This article is organized as follows.

First, in Section 2 we introduce some combinatorial concepts, mostly dealing with walks in oriented graphs, and prove a general lemma on the automorphism group of balanced oriented graphs. In Section 3 a proof of the fact that the group $G_{n,r}$ is isomorphic to the alternating group A_n is given, followed by an analysis of certain relations in this group. All of these results are then used in Section 4 where the proof of Theorem 1.1 is given. The argument consists of two crucial steps. First, based on the results from Section 3, we determine the full automorphism group of $X_{n,r,1}$, thus proving the theorem for $m = 1$. Second, using the above mentioned isomorphism $Al^{m-1}(X_{n,r}) \cong X_{n,r,m}$ together with an analysis of 4-cycles in $X_{n,r,m}$ for $m \leq r-1$, we complete the proof following a recursive argument.

2 Preliminary observations

For an oriented graph $\mathcal{O}$ an *antiautomorphism* is a permutation of the vertex set $V(\mathcal{O})$ reversing the orientation of all the arcs. We use the symbol $\operatorname{Anti}\mathcal{O}$ to denote the group of all automorphisms and antiautomorphisms of $\mathcal{O}$. Clearly, either $\operatorname{Aut}\mathcal{O} = \operatorname{Anti}\mathcal{O}$ or $[\operatorname{Anti}\mathcal{O} : \operatorname{Aut}\mathcal{O}] = 2$.

Let X be an arbitrary graph with a fixed orientation giving rise to an oriented graph $\mathcal{O}(X)$. Let $u, v \in V(X)$ be two vertices of X. We write $u \to v$ if u is the tail and v is the head of the arc (oriented edge) (u, v). (By an *edge* of an oriented graph we mean an edge of the underlying undirected graph.) An edge of X is *incident* with v if v is either the head or the tail of the associated oriented graph $\mathcal{O}(X)$. A vertex u is *adjacent* to v (a *neighbor* of v) if either (u, v) or (v, u) is an arc of $\mathcal{O}(X)$. A *walk* in X is an alternating sequence of vertices and arcs $v_0a_0v_1a_1, \ldots, v_{l-1}a_{l-1}v_l$ in $\mathcal{O}(X)$ such that for each i we have that a_i is incident with both v_i and v_{i+1}. A *path* in X is a walk all of whose vertices are distinct. A walk is *closed* if $v_0 = v_l$; a closed walk with no repeated internal vertices is a *cycle*. Paths and cycles are also referred to as *simple walks*. Let W be a walk in X. Then $|W|$ denotes the length of W, that is the number of arcs of W. We say that W *traverses* a vertex v of X if there exists a subpath of length 2 in W with v as its internal vertex.

Let W be a simple walk of length s in X. We assign to each internal vertex v of W one of the *codes* A^+, A^- or D depending on whether v is the tail of both, the head of both, or the tail of one and the head of the other of its two incident edges (in $\mathcal{O}(X)$), respectively. In such a way the walk W is assigned a sequence, of length s in the case of cycles and length $s-1$ in the case of paths, with elements from the set $\{A^+, A^-, D\}$. The equivalence class of all the sequences obtained from it by a cyclic rotation or a reflection, in case W is a cycle, and just a reflection, in case W is a path, will be called the *code* $c(W)$ of W. Clearly, by deleting each of the symbols D from the code $c(W)$, a sequence where symbols A^+ and A^- alternate is obtained. The walk W is *directed* if its code is D^s, and, on the other hand,

W is *alternating* if its code contains no symbol D. In particular, we speak of directed paths and directed cycles, of alternating paths and alternating cycles. Further, a cycle of even length $2s$ is said to be *parallel* if its code is of the form $A^+D^{s-1}A^-D^{s-1}$. Finally, a directed path of length s is sometimes called an *s-arc*.

We observe that possible codes for paths of length 3 belong to the set $\{D^2, DA^+, DA^-, A^+A^-\}$. We shall say that two such codes c_1, c_2 are *equivalent*, and write $c_1 \sim c_2$, if there are 3-paths P_1 and P_2 with respective codes c_1 and c_2 and an automorphism of X mapping P_1 to P_2.

Lemma 2.1 *Let X be a regular graph of valency $2d \geq 2$ and let $\mathcal{O}(X)$ be a balanced orientation of X. (Thus every vertex of $\mathcal{O}(X)$ has indegree as well as outdegree equal to d.)*

Then

(i) $Aut X \cong Anti X$, that is, every automorphism of X either preserves the orientation of the edges of $\mathcal{O}(X)$ or reverses the orientation of the edges of $\mathcal{O}(X)$; or

(ii) any two codes of length 2 (partaining to 3-paths of $\mathcal{O}(X)$) are equivalent.

Proof. The result is clearly true for $d = 1$. Let $d \geq 2$. Assume there is an automorphism $\alpha \in \operatorname{Aut} X$ which does not either preserve the orientation of the edges in $\mathcal{O}(X)$ or reverse the orientation of the edges in $\mathcal{O}(X)$. Then there must be a directed 2-path $P = uvw$ (with $u \to v \to w$) such that $\alpha(P)$ is an alternating 2-path. With no loss in generality we may assume that its code is A^+, meaning that α reverses the orientation of the edge uv and preserves the orientation of the edge vw. Now let $x_1, \dots x_d$ be the d predecessors of u and let $y_1, \dots y_{d-1}$ be the remaining $d-1$ successors of u (other than v).

Suppose first that the orientations of all the edges x_iu, $i = 1, \dots, d$, and y_iu $i = 1, \dots, d-1$, are reversed. Then, on the one hand, $D^2 \sim DA^+$, for x_iuvw is a directed 3-path mapped to the path $\alpha(x_iuvw)$ with code DA^+, and on the other hand, $DA^+ \sim A^+A^-$, for y_iuvw is a 3-path with code A^+D mapped to the path $\alpha(y_iuvw)$ with code A^-A^+.

Assume now that not all of the orientations of the edges x_iu, $i = 1, \dots, d$, and y_iu $i = 1, \dots, d-1$, are reversed. In particular, we may assume that the orientation of the edge x_1u is reversed, whereas the orientation of the edge x_2u is preserved. Now the 3-paths x_1uvw and x_2uvw, both with code

D^2 are mapped by α to 3-paths with respective codes A^-A^+ and DA^+. As in the previous case, $D^2 \sim DA^+ \sim A^+A^-$.

Finally, let z be a predecessor of w different from v. The code of the 3-path $uvwz$ is DA^- and the code of its image $\alpha(uvwz)$ is either A^+A^- or A^+D. In both cases we deduce that all four codes are equivalent. ■

We now formally introduce two operators on balanced oriented quartic graphs. For such a graph X let the *partial line graph* $Y = Pl(X)$ of X be the balanced oriented quartic graph with vertex set $A(X)$ such that there is an arc in Y from $x \in A$ to $y \in A$ in Y if and only if xy is a directed 2-path (a 2-arc) in X. Note that the arc set of Y decomposes into alternating 4-cycles no two of which intersect in more than one vertex. To introduce the inverse operator Al, let the vertex set of $Al(Y)$ be the set of alternating cycles (of length 4) in Y, with two such cycles adjacent in $Al(Y)$ if and only if they have a common vertex in Y. The orientation of the edges of $Al(Y)$ is inherited from that of the edges of Y in a natural way. Letting C_v and C_w be the two alternating 4-cycles in Y, corresponding to two adjacent vertices v and w in $Al(Y)$, we orient the edge $[v, w]$ in $Al(Y)$ from v to w if and only if the two arcs in Y with the tail in $u \in C_v \cap C_w$ have heads on C_w. Observe that $Al(Pl(X) = X$ for every balanced oriented graph X of valency 4. Moreover, $Pl(Al(Y)) = Y$ as long as the graph Y has the above assumed properties.

These two operators are also applied to (undirected) graphs whenever an accompanying oriented graph is (perhaps tacitly) associated with the undirected graph in question. A typical situation is presented by a quartic graph admitting a $\frac{1}{2}$-arc-transitive group action and its two accompanying balanced oriented graphs, or by a Cayley graph arising from a set of non-involutory generators, for each of which one of the two possible orientation is prescribed. Let X be a graph together with an inherited orientation given via an oriented graph $\mathcal{O}(X)$, whose underlying graph it is. Then we let the *partial line graph* $Y = Pl(\mathcal{O}(X))$ of X be the underlying graph of $Pl(\mathcal{O}(X))$. In a similar fashion, also the operator Al may be extended to graphs possessing an implicit orientation of their edge sets. Again, these two operators are inverses of each other for graphs too.

We have the following straightforward generalization of [15, Proposition 2.1].

Proposition 2.2 *If X is a balanced oriented 4-valent graph, then $Aut\,Pl(X) = Aut\,X$ and $Anti\,Pl(X) = Anti\,X$. Conversely, let Y be a balanced oriented graph of valency 4 such that the alternating cycles have length 4, no two intersect in more than one vertex, and they decompose the edge set. Then $Aut\,Al(Y) = Aut\,Y$ and $Anti\,Al(Y) = Anti\,Y$.*

Let a group G act (on the right) on a set V and let Q be a nonempty subset of G. We define the *action digraph* $Act(G, V, Q)$ to be the digraph with vertex set V and arcs of the form $(v, v * q)$, $v \in V$, $q \in Q$. (Note that if $v * q_1 = v * q_2$ for $q_1 \neq q_2$, then the arcs $(v, v * q_1)$ and $(v, v * q_2)$ are considered to be distinct.) Throughout this paper we shall be assuming that the action of G is transitive and that Q is a generating set of G, thus forcing $Act(G, V, Q)$ to be (weakly) connected. In particular, if G acts on itself by right multiplication and if $1 \notin Q = Q^{-1}$ then the graph associated with the digraph $Act(G, G, Q)$ is nothing but the Cayley graph $Cay(G, Q)$.

For a group G and a subset $Q \subseteq G$ we let $\mathrm{Aut}\,(G, Q) = \{\alpha \in \mathrm{Aut}\,(G) : \alpha(Q) = Q\}$. Next, by a *$Q$-sequence* and a *$Q$-relation* we shall mean a word on symbols from $Q \cup Q^{-1}$ which corresponds, respectively, to a simple path and to a simple cycle in $Cay(G, Q \cup Q^{-1})$. (In other words, by a Q-relation we mean a primitive Q-relation and by a Q-sequence a reduced word on symbols from $Q \cup Q^{-1}$ such that no proper subword is a relation.) For a Q-sequence S in G let $l(S)$ denote the *length* of S. We say that two Q-sequences are *equivalent* if one can be obtained from the other by a finite series of transformations of the following three types: a cyclic rotation, taking the Q-sequence in the reverse order with all terms inverted (that is the inverse Q-sequence), or substituting each term in the Q-sequence by its image under an element of $Aut(G, Q \cup Q^{-1})$. Note that the corresponding equivalence relation on Q-sequences distinguishes between relations and nonrelations in G. To each Q-sequence in a group G acting on a set V and a vertex v of $Act(G, V, Q)$, we may associate in a natural way a walk originating in v. Furthermore, if the action of G on V is faithful, then a Q-sequence in G is a relation if and only if it represents a closed walk at every vertex of $Act(G, V, Q)$. In this sense the action digraph is a useful geometric tool for testing whether a given sequence is a group relation or not.

3 Analyzing the group $G_{n,r}$

In this section we prove that the group $G_{n,r}$ is isomorphic to A_n and analyze certain relations in this group.

Recall the definition. Let $r \geq 2$ be an integer and let $n = 2k+1 \geq 4r+3$ be an odd integer. Next set $a = a_{n,r} = (0, 1, \ldots, 2k)$, let $t = t_{n,r} = (0, r)$, let $b = b_{n,r} = a^t$ and let $G_{n,r} = \langle a, b \rangle$.

Lemma 3.1 *With n and r satisfying the above conditions we have $G_{n,r} \cong A_n$.*

Proof. Observe first that for each j we have

$$a^j b^{-j} = a^j t a^{-j} t = t^{a^{-j}} t = (n-j, r-j)(2r-j, 3r+1-j)(0, r)(2r, 3r+1). \quad (1)$$

In particular, $a^r b^{-r} = (n-r, 0)(r, 2r+1)(0, r)(2r, 3r+1) = (0, n-r, r, 2r+1)(2r, 3r+1)$. Next, observe that possible blocks for $G_{n,r}$ of length at least 2 in the action of $G_{n,r}$ on the set $\wp_n$ must be of the form $\{i, i+d, i+2d, \ldots, i+n-d\}$ for some proper divisor d of n and some $i \in \wp_d$. Also, every such block must clearly be fixed by $(a^r b^{-r})^2 = (0, r)(n-r, 2r+1)$, forcing 0 and r to be in the same block. (Namely, the other possibility is that $(a^r b^{-r})^2$ interchanges $B(0)$ with $B(r)$, and the two blocks have length 2, contradicting the fact that n is odd.) But then applying a^r we have that this block contains $2r$, and applying a^{-r} we have that this block contains $n-r$ and hence also $2r+1$. But then it coincides with the whole of $\wp_n$. Thus $G_{n,r}$ is a primitive group. Next, $(a^{r+1} b^{-r-1})^2 = (2r, r-1, 3r+1)$ and so $G_{n,r}$ contains a 3-cycle and is therefore at least $(n-2)$-transitive in view of the classical result of Jordan [22, Theorem 13.3]. Hence $G_{n,r} \cong A_n$. ■

Next, we study relations in the group $A_n \cong G_{n,r}$. Hereafter, by a sequence in A_n and a relation in A_n we shall always mean a $Q_{a,b}$-sequence and a $Q_{a,b}$-relation in A_n, respectively, where $Q_{a,b} = \{a, a^{-1}, b, b^{-1}\}$. Relations in A_n of length 4 and those of length n are of particular importance.

For a $Q_{a,b}$-sequence S and a vertex v of the action digraph $\Omega_{n,r} = \mathrm{Act}(A_n, \wp, \{a, b\})$ we let $W(S; v)$ denote the walk in $\Omega_{n,r}$ associated with S whose origin is v . Moreover, the vertex of $W(S; v)$ reached from v by the first r steps will be denoted by $v *_r S$. If $r = l(S)$ then the subscript r is omitted.

Lemma 3.2 *A relation of length 4 is equivalent to the relation $(ab^{-1})^2$.*

Proof. The following are all the inequivalent sequences of length 4 in $G_{n,r}$: $(ab^{-1})^2$, $a^2 b^{-2}$, $a^3 b^{-1}$ $(ab)^2$, $a^2 b^2$, $a^3 b^1$ and a^4. Of these it may be easily checked that only the first one is a relation. (For example, by considering the associated closed walks in the corresponding action digraph $\Omega_{n,r}$; see Figure 1). ■

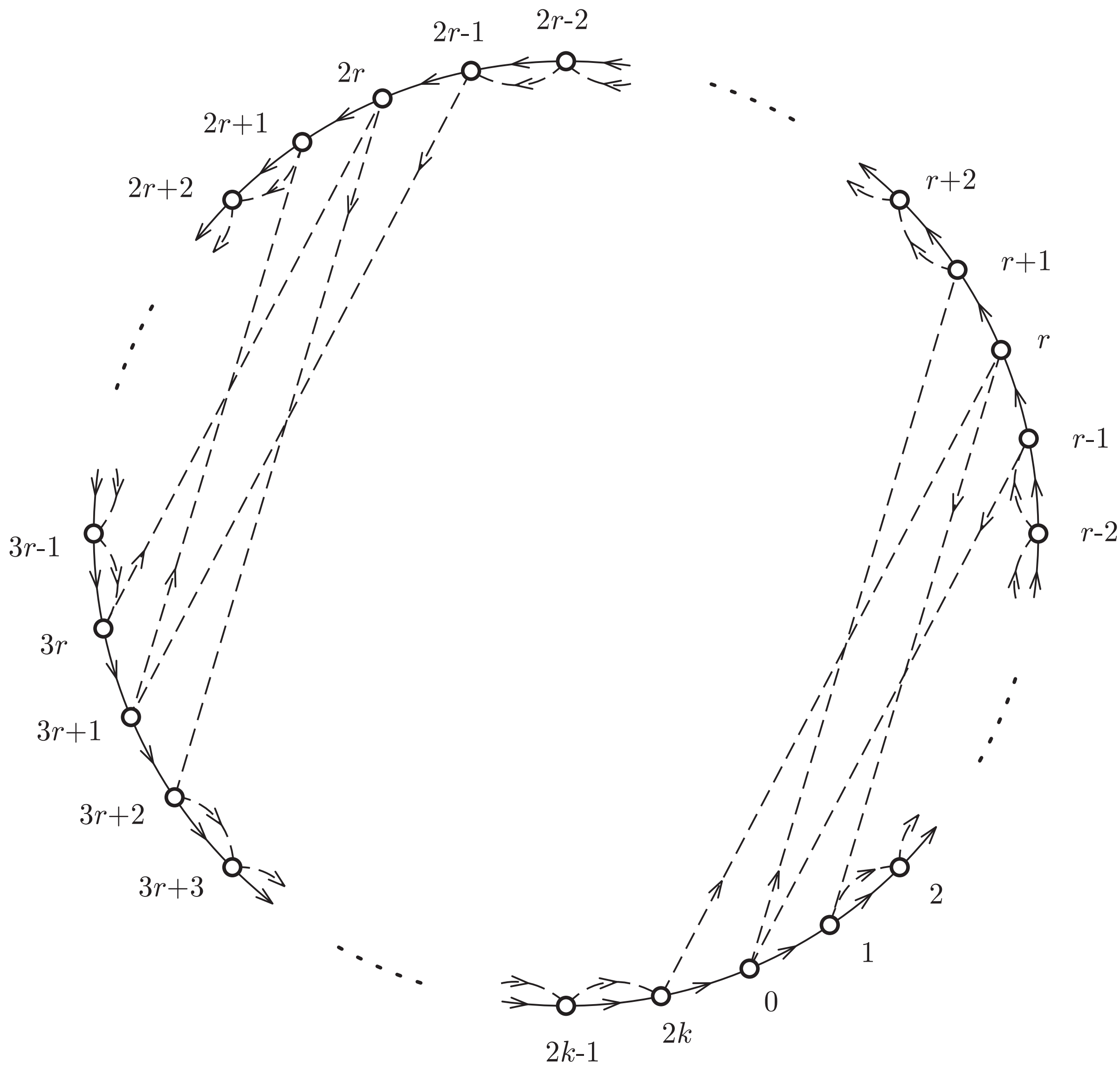

Figure 1: The action digraph $\Omega_{n,r} = \mathrm{Act}(A_n, \wp, \{a, b\})$; unbroken and broken lines correspond to permutations a and b, respectively.

Lemma 3.3 *A relation of length n is equivalent to the relation a^n.*

Proof. It suffices to show that a sequence S of the form abT where T is an $Q_{a,b}$-sequence of length $n-2$, is not a relation.

Assume for the contrary, that there exists such a sequence S and consider the closed walk $W = W(S; 2k)$ in the corresponding action digraph $\Omega_{n,r}$ (see Figure 1). The sequence S decomposes into subsequences S_i, $i = 1, 2, \dots, c$, such that $2k * S_i = 2k$ and for every nonempty subsequence S'_i of S_i we have that $2k * S'_i \neq 2k$. In other words, the closed walk W decomposes into c closed walks W_i, $i = 1, 2, \dots, c$, with intial vertex $2k$. Observe that for each i, we have $l(S_i) \geq n - 2r - 1$ for the minimal number of steps occurs when two chords are used (one at $2k$ or 0 and the other one at $2r-1$ or $2r$), thus reducing the number of steps to $n - 2r - 1 = n - (r + r + 1)$. Therefore

$$n = l(S) = \Sigma_{i=1}^{i=c} l(S_i) \geq \Sigma_{i=1}^{i=c}(n - 2r - 1) \geq c(n - 2r - 1).$$

It follows that $n(c-1) \leq c(2r+1)$ and since $n \geq 4r+3$, we have $(c-1)(4r+3) \leq n(c-1) \leq c(2r+1)$. We infer that $(c-1)/c \leq (2r+1)/(4r+3) < 1/2$, forcing $c = 1$. Therefore $S = S_1$. But $2k *_2 S = r+1$ and so $2k * abT = (r+1)*T$. Now in order to get from $r+1$ to $2k$, at most $(n-1)-(r+1) = n-r-2$ additional steps are needed. Hence $l(S) \leq n - r < n$, a contradiction. This completes the proof of Lemma 3.3. ■

We wrap up this section with the following lemma about the the intersection of normalizers of a and b in S_n.

Lemma 3.4 *let $r \geq 2$ be a positive integer, let $n \geq 4r + 3$. Let $a = (0, 1, \dots, n-1) \in A_n$ and $b = a^t = tat$, where $t = (0, r)(2r, 3r+1) \in A_n$. Then $N_{S_n}(a) \cap N_{S_n}(b)$ contains no element x satisfying $\alpha_x(a) = a^{-1}$ and $\alpha_x(b) = b^{-1}$.*

Proof. Suppose on the contrary that such an element x exists. It follows that $\alpha_{txt}(a) = a^{-1}$. A short calculation shows that $x, txt \in \{y_c \ : \ c \in \wp_n\}$, where $y_c \in S_n$ maps according to the rule $y_c(i) = c - i$ for each $i \in \wp_n$. Hence $ty_ct = y_d$ for some $c, d \in \wp_n$. Since $n \geq 4r + 3 \geq 11$ is large enough, there exists $k \in \wp_n \backslash \{0, r, 2r, 3r+1\}$ such that $c - k \in \wp_n \backslash \{0, r, 2r, 3r+1\}$, too. Consequently, $t(k) = k$ and $t(c-k) = c-k$ and so $c - k = ty_ct(k) = y_d(k) = d - k$, implying $c = d$. Hence $ty_ct = y_c$. Applying this relation first to $i = r$ and then to $i = 3r+1$ we obtain, respectively, $t(c) = c - r$ and $t(c - 2r) = c - 3r - 1$. These give us that on the one hand, $c = r$, and on the other, that $c = 5r + 1$. But then $4r + 1 = 0$, impossible as $n \geq 4r + 3$. This contradiction proves Lemma 3.4 ■

4 Proving of Theorem 1.1

Let n, r and m satisfy the conditions in the statement of Theorem 1.1. Recall that $a = a_n = (0, 1, \ldots, 2k) \in A_n$ and $b = b_{n,r} = a^t = t^{-1}at$, where $t = t_{n,r} = (0, r)(2r, 3r+1) \in A_n$, and for each j let $\sigma_j = a^j b^{-j}$. Further let $G_{n,r} = \langle a, b\rangle$, let $H_1 = 1$ and, for each $2 \leq m \leq r$, let $H_m = \langle \sigma_1, \ldots, \sigma_{m-1}\rangle$. We adopt the following shorthand notation. We let $H_{n,r} = \langle G_{n,r}, \alpha_t\rangle$, and $Q = Q_{a,b} = \{a, a^{-1}, b, b^{-1}\}$. Further, for each m, the underlying undirected graph $X_{n,r,m}$ of the orbital graph $X(G_{n,r}, H_m; \{aH_m, bH_m\})$ will be denoted by X_m.

Proposition 4.1 *The graph X_m, $m \in \{1, 2, \ldots, r\}$ admits a $\frac{1}{2}$-arc-transitive action of the group $H_{n,r}$.*

PROOF. In [16, Theorem 4.1] a necessary and sufficient condition is obtained (stated in a group-theoretic language) for a graph of valency 4 to admit a $\frac{1}{2}$-arc-transitive group action with vertex stabilizers isomorphic to an elementary abelian group. Translating this result into the situation we have here, it follows that the graph X_m admits a $\frac{1}{2}$-arc-transitive action of the group $H_{n,r}$ with vertex stabilizers isomorphic to an elementary abelian group of order 2^m. Namely, in view of (1), the subgroup $H_m = \langle \sigma_1, \sigma_2, \ldots, \sigma_{m-1}\rangle$ is elementary abelian of order 2^{m-1}. Moreover, the vertex (coset) H_m is fixed also by the automorphism α_t which interchanges the successors aH_m and bH_m of H_m and fixes both predecessors $a^{-1}H_m$ and $\sigma_{m-1}a^{-1}H_m$. Besides, for any $x \in A_n$ and any $j \in \{1, 2, \ldots m-1\}$, we have that

$$\sigma_j \alpha_t x H_m = \sigma_j txt H_m = a^j b^{-j} txt H_m = tta^j b^{-j} txt H_m =$$
$$tb^j a^{-j} xt H_m = t\sigma_j xt H_m = \alpha_t \sigma_j x H_m.$$

Hence α_t is an involution commuting with each σ_j. It follows that the stabilizer $\langle H_m, \alpha_t\rangle$ of the vertex H_m is isomorphic to $\mathbb{Z}_2^m$, completing the proof of Proposition 4.1. ■

Throughout the rest of this section it will be tacitly assumed that the graph X_m carries the orientation of its associated oriented graph, the orbital graph $X(G_{n,r}, H_m; \{aH_m, bH_m\})$ together with any particular feature arising from this orientation. For example, by an *alternating cycle* in X_m we mean a $\langle G_{n,r,\alpha_t}\rangle$-alternating cycle in $X(G_{n,r}, H_m; \{aH_m, bH_m\})$. Furthermore, we let the symbol Anti X_m denote the group Anti $X(G_{n,r}, H_m; \{aH_m, bH_m\})$ of all automorphisms and antiautomorphisms of the associated oriented graph of X_m.

The next three lemmas are of crucial importance. The first and the third deal with cycles of length 4 in the graph X_m, whereas the second establishes the isomorphism between the graphs X_m and $Al(X_{m-1})$.

Lemma 4.2 *The alternating cycles in X_m, $m \in \{1, 2, \ldots, r\}$, have length 4.*

PROOF. Since $Q_{a,b}$-relations in $G_{n,r}$ correspond to cycles in X_1, Lemma 3.2 establishes the result for $m = 1$.

To prove Lemma 4.2 for $m \geq 2$, recall that the paired suborbit of $\{aH_m, bH_m\}$ is $\{a^{-1}H_m, \sigma_{m-1}a^{-1}H_m\}$. Consider the alternating cycle of X_m containing the vertex (coset) H_m and its two successors $aH_m$and bH_m. By definition, the other predecessor of bH_m is $b\sigma_{m-1}a^{-1}H_m$, which coincides with $\sigma_m H_m$ in view of the fact that $\sigma_{m-1} = a^{m-1}b^{-m+1} = b^{m-1}a^{-m+1}$ is an involution. But $\sigma_m H_m$ is also the remaining predecessor of aH_m. Namely, $a\sigma_{m-1}a^{-1}H_m = \sigma_m\sigma_1 h_m = \sigma_m H_m$. Hence the alternating cycles are indeed of length 4. ■

Lemma 4.3 $X_m = Al^{m-1}(X_1$ *for each* $m \in \{1, 2, \ldots, r-1\}$.

PROOF. The result is trivially true for $m = 1$. By induction it suffices to see that $Al(X_{m-1} \cong X_m$. To this end, consider the cycle

$$C = (H_{m-1}, aH_{m-1}, \sigma_{m-1}H_{m-1}, bH_{m-1}),$$

one of the two alternating cycles in X_{m-1} containing the vertex (coset) H_{m-1}. Clearly, each element of the group H_{m-1} fixes C. Moreover, σ_{m-1} interchanges H_{m-1} and $\sigma_{m-1}H_{m-1}$, as well as aH_{m-1} and bH_{m-1}. Namely, $\sigma_{m-1}bH_{m-1} = a\sigma_{m-2}H_{m-1} = aH_{m-1}$. Similarly,

$$\sigma_{m-1}aH_{m-1} = a^{m-1}b^{-m+1}aH_{m-1} = b^{m-1}a^{-m+1}aH_{m-1} =$$
$$b\sigma_{m-2}H_{m-1} = bH_{m-1} = bH_{m-1}.$$

Hence C is fixed by the group $H_m = \langle H_{m-1}, \sigma_{m-1}\rangle$ and so $Al(X_{m-1})$ is a quartic graph admitting the group $G_{n,r}$ with H_m as one of the vertex stabilizers. It is then a matter of a routine check to see that the two successive alternating cycles of C are represented by cosets aH_m and bH_m. Consequently $Al(X_{m-1}) \cong X_m$, as required. ■

Lemma 4.4 *A cycle of length 4 in X_m, $m \in \{1, 2, \ldots, r\}$, is necessarily alternating.*

PROOF. For $m = 1$ the result follows directly from Lemma 3.2. To prove Lemma 4.4 for $m \geq 2$, recall first that a cycle of length 4 has one of the following four codes: $(A^+A^-)^2$ (alternating cycle), D^4 (directed cycle), DA^+DA^- (parallel cycle) and $D^2A^+A^-$.

Now the existence of a directed cycle in X_m implies the existence of the same length directed cycle also in X_{m-1}. Continuing this way, we see that X_1 would have to have a directed cycle of the same length as X_m. But since X_1 does not have a directed 4-cycle neither does X_m for any $m \leq r$.

Next, suppose that we have a 4-cycle with code $D^2A^+A^-$ in X_m. Then following the proof of [17, Theorem 4.1] we see that either X_m is isomorphic to the circulant $Cay(\mathbb{Z}_{10}; \{1, -1, 3, -3\})$ a graph on 10 vertices (which is clearly impossible for $|V(X_m)| = n!/2^m \neq 10$) or X_m has also a 4-cycle with code DA^+DA^-, a parallel cycle.

We may therefore assume that X_m has both parallel and alternating 4-cycles. (Again, using [17, Theorem 4.1] we would have that in this case $X_m \cong C_l[2K_1]$ for a suitable positive integer l.) The existence of parallel 4-cycles in X_m together with vertex- and edge-transitivity of X_m clearly implies that the cosets aH_m and bH_m must have a common successor. But their two successors are, respectively, a^2H_m and abH_m, and b^2H_m and baH_m. We must therefore have $\{a^2H_m, abH_m\} \cap \{b^2H_m, baH_m\} \neq \emptyset$. In other words, H_m has nonempty intersection with the set $\{a^{-2}b^2, a^{-2}ba, b^{-2}ab, b^{-1}a^{-1}ba\}$. However, observe (for example, by adhering to the action digraph $\Omega_{n,r}$) that every element σ_j, $j \leq r - 1$, fixes the element $2 \in \wp_n$ with the exception of σ_{r-2} which interchanges 2 with $n - r + 2$. On the other hand, $a^{-2}b^2$ interchanges 2 with $r + 2$. Therefore $a^{-2}b^2 \notin H_m$. The proof that the remaining three elements are not in H_m is analogous and we omit it. This contradiction shows that X_m does not have parallel 4-cycles, completing the proof of Lemma 4.4. ∎

The next proposition can be easily deduced from [10, Lemma 2.1].

Proposition 4.5 *Let G be a group and Q a generating set of G such that $1 \notin Q = Q^{-1}$. Let $X = Cay(G, Q)$ and $H = Aut\, X$. Then $N_H(G) \cap H_1 \cong Aut(G, Q)$.*

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The fact that a 4-cycle in X_m is necessarily alternating plays an essential role in the proof of Theorem 1.1 as is reflected in our arguments below.

First, since an alternating 2-path is contained on a 4-cycle, as opposed to a directed 2-path, it follows that no alternating 2-path may be mapped to a directed 2-path. Consequently, no alternating 3-path may be mapped to a 3-path which is not alternating. By Lemma 2.1 we have that every automorphism of X_m either preserves the orientation of all the edges or reverses the orientation of all the edges. In other words, $\mathrm{Aut}\, X_m$ coincides with the group $\mathrm{Anti}\, X_m$.

The graph X_m (in fact its oriented counterpart) satisfies the conditions of Proposition 2.2 and so, since Lemma 4.3 implies that $X_{m-1} = Pl(X_m)$, it follows that the two groups coincide, that is, $\mathrm{Aut}\, X_m \cong \mathrm{Aut}\, X_{m-1}$. Continuing this way we see that $\mathrm{Aut}\, X_m \cong \mathrm{Aut}\, X_1$. Consequently, in order to complete the proof of Theorem 1.1, it remains to be seen that $\mathrm{Aut}\, X_1 = H_{n,r}$.

Now, in view of the comments in Section 1, X_1 is defined to be the Cayley graph $Cay(G_{n,r}; Q_{a,b})$, where $Q_{a,b} = \{a, b, a^{-1}, b^{-1}\}$ and its associated oriented graph is the Cayley digraph $Cay(G_{n,r}; \{a, b\})$.

We introduce the following shorthand notation. Let $A = \mathrm{Aut}\, X$, $G = G_{n,r}$ and $H = H_{n,r} = \langle G_{n,r}, \alpha_t \rangle$.

First we show that A_{id}, the stabilizer of vertex id in A, acts faithfully and semiregularly on the neighbors' set $N(id) = \{a, a^{-1}, b, b^{-1}\}$.

The fact that A coincides with $\mathrm{Anti}\, X_1$ may be translated into the following language. In particular, this means that $\{a, b\}$ and $\{a^{-1}, b^{-1}\}$ are blocks of imprimitivity in the action of A_{id} on the neighbors' set $N(id) = \{a, b, a^{-1}, b^{-1}\}$.

Let $\gamma \in A_{id} \cap A_a$. Clearly γ fixes b in view of the uniqueness of alternating 4-cycles in X_1. Moreover, by Lemma 3.3, the directed 2-path (a^{-1}, id, a) is contained on an n-cycle, but the directed 2-path (b^{-1}, id, a) is not. Hence γ fixes also a^{-1} and b^{-1}. Continuing this way, the connectedness of X_1 implies that $\gamma = 1$. It follows that A_{id} is faithful and semiregular on $N(id)$ and so its order is either 2 or4.

In particular, the order of vertex stabilizers is either 2 or 4. In the first case $A = H$ and every automorphism of X_1 preserves the orientation of edges, and in the second case every automorphism of X_1 belonging to H preserves the orientation of edges and every automorphism in $A \setminus H$ reverses the orientation of edges.

We proceed to show that $|A_{id}| = 2$. In other words, $A_{id} = \langle \alpha_t \rangle$ and consequently $A = H$, as required.

Assume the contrary, that is, assume that H is of index 2 in A. First we show that G is normal in A. To do this, it suffices to see that an element $\gamma \in A \setminus H$ normalizes G. By the comments in the previous paragraph, we have that γ must reverse the orientation of edges. Besides, in view of Lemma 3.3, it either preserves or interchanges the two sets of a-colored and b-colored edges of X_1 . But then the conjugate G^γ preserves both the coloring and the orientation of edges of X_1. Therefore G^γ coincides with G and so G is normal in A.

We now use Proposition 4.5 to deduce that A_{id} is isomorphic with $\mathrm{Aut}\,(G, Q)$, that is, it consists of precisely those automorphisms of $G \cong A_n$ which fix the set Q. But the action of γ on Q is such that it interchanges a and its inverse as well as b and its inverse. Besides, being an automorphism of $G \cong A_n$, it coincides with a conjugation α_x by some element $x \in S_n$. Hence $\alpha_x(a) = a^{-1}$ and $\alpha_x(b) = b^{-1}$. But the existence of such an element contradicts Lemma 3.4. This contradiction shows that vertex stabilizers are not of order 4, hence they are of order 2. Therefore $A = H$, which completes the proof of Theorem 1.1. ∎

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Keywords: group, group action, quartic graph, Cayley graph, vertex-transitive graph, edge-transitive graph, 1/2-arc-transitive graph, action digraph.