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Abstract We present a short proof based on the Pontryagin-Thom construction of the following Pontryagin theorem, whose original proof was complicated and has never been published in details: *Let M be a connected orientable closed smooth 3-manifold. Let $L_1(M)$ be the set of framed links in M up to a framed cobordism. Let* $\deg : L_1(M) \to H_1(M;\mathbb{Z})$ *be the map taking each $L \in L_1(M)$ to its homology class in $H_1(M;\mathbb{Z})$ (we fix the orientation of L agreeing with the framing). Then for each $\alpha \in H_1(M;\mathbb{Z})$ we have $|\deg^{-1}\alpha| = \infty$ if $\alpha \cdot H_2(M;\mathbb{Z}) = 0$ and $|\deg^{-1}\alpha| = 2\min\{|\alpha \cdot c| : c \in H_2(M;\mathbb{Z}), \alpha \cdot c \neq 0\}$ otherwise.*

Keywords Framed link, Stiefel-Whitney class, manifold, normal bundle, degree map, cohomotopy group, Pontryagin-Thom construction

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RUNNING HEAD: Framed links in 3-manifolds

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1 Introduction

Throughout this paper let M be a connected orientable closed smooth 3-dimensional manifold. Denote by $L_1(M)$ the set of 1-dimensional framed links in M up to framed cobordism. Applying the Pontryagin-Thom construction one can show that the set $L_1(M)$ is in 1–1 correspondence with the set $\pi^2(M) = [M; S^2]$ of continuous maps $M \to S^2$ up to homotopy. The main purpose of this paper is to describe the set $L_1(M) = \pi^2(M)$.

This classification is based on the notions of *natural orientation* on a framed link and *degree* of a framed link, defined as follows. Take a point x on a framed link L and let f_1, f_2 be the frame at this point. The orientation of L is *positive*, if the vector of this orientation together with f_1, f_2 gives a positive basis in M. Thus the natural orientation of L is defined. The *degree* $\deg L$ of L is the homology class (with integral coefficients) of naturally oriented L. So we have a map

$$\deg : L_1(M) \to H_1(M; \mathbb{Z}).$$

The classical Hopf-Whitney theorem (1932-35) asserts that this map is always surjective.

Theorem 1.1 (L. S. Pontryagin) Let M be a connected orientable closed smooth 3-manifold. Then for each $\alpha \in H_1(M; \mathbb{Z})$ we have

$$|\deg^{-1}\alpha| = \begin{cases} \infty, & \text{if } \alpha \cdot H_2(M;\mathbb{Z}) = 0 \\ 2\min\{|\alpha \cdot c| \ : \ c \in H_2(M;\mathbb{Z}),\ \alpha \cdot c \neq 0\}, & \text{otherwise.} \end{cases}$$

In this paper we prove Theorem 1.1 which was stated without proof in [1]. In fact, Theorem 1.1 was not stated in [1] (written in English), but only in the abstract (written in Russian) without any indication of the proof. The statement in the abstract is not clear, so we have borrowed it from [2]. The statement from [2] asserts that there is a 1-1 correspondence between

$$\deg^{-1}\alpha \qquad \text{and} \qquad \frac{\mathbb{Z}}{2\alpha \cap H_2(M;\mathbb{Z})},$$

which is equivalent to our statement of Theorem 1.1. There are reasons to believe that our proof is the original Pontryagin's proof, which he never published, going instead straight ahead to the general case when M is an arbitrary polyhedron.

2 Preliminaries

We are going to use the following *geometric definition of the normal Euler class*, equivalent to other definitions. Let M^4 be a closed orientable connected 4-manifold. Let L^2 be a connected orientable manifold immersed in M^4. Let $\nu(L)$ be the normal bundle of L. Identify L with the zero section of $\nu(L)$. Fix an orientation of L and M, and a natural orientation of $\nu(L)$. Take a general position section L' of $\nu(L)$. *The normal Euler class* $\bar{e}(L) = \bar{e}(\nu(L)) \in \mathbb{Z}$ is the difference between the numbers of positive and negative intersection points of L and L'. Further denote by $X \cap Y$ the difference between the numbers of positive and negative intersection points of X and Y. The sign of the integer $\bar{e}(L)$ is defined in the usual way and depends on the choice of the orientations of L and M.

We also are going to use the following *geometric definition of the relative normal Euler class*. Fix an orientation of $M^3 \times [0;1]$. Let $L_1 \subset M \times 1$ and $L_2 \subset M \times 0$ be a pair of framed links, let $L \subset M \times [0;1]$ be a (unframed) cobordism between them. Fix a natural orientation of L, i. e. an orientation that induces natural orientations of L_1 and L_2. Fix a natural orientation of $\nu(L)$. The first vector field of the framings of L_1 and L_2 can be considered as a section of $\partial\nu(L)$. Let L' be a general position extension of this section to a section of $\nu(L)$. *The relative normal Euler class* $\bar{e}(L) \in \mathbb{Z}$ is the difference between the numbers of positive and negative intersection points of L and L'. If we reverse the orientation of $M^3 \times [0;1]$ (and, consequently, of L, because L is naturally oriented), then the sign of the integer $\bar{e}(L)$ changes.

It can be shown that for orientable L and M the class $\bar{e}(L)$ is the complete obstruction to existence of a framing of L, if $\partial L = \emptyset$, and to extension of the framing of ∂L to a framing of L, if $\partial L \neq \emptyset$.

Lemma 2.1 Let L^2 and M^4 be a pair of connected oriented manifolds (M may have boundary). Suppose that L is immersed into M. Denote by $[L] \in H_2(M;\mathbb{Z})$ the class of L. Denote by σ the difference between the numbers of positive and negative self-intersection points of L. Then

$$\bar{e}(L) = [L] \cap [L] - 2\sigma,$$

where we identify the group $H_0(M;\mathbb{Z})$ with $\mathbb{Z}$. In particular, if $M = N^3 \times I$ for some 3-manifold N^3, then $\bar{e}(L) = -2\sigma$.

Remark. This well-known lemma implies Theorem 1.2b in [3].

Proof of Lemma. Let π be the natural projection of a neighbourhood of L in $\nu(L)$ to a small neighbourhood of L in M. Take a general position section L' of $\nu(L)$ close to zero. The lemma now follows from

$$\bar{e}(L) = L \cap L' = \pi L \cap \pi L' - 2\sigma = [L] \cdot [L] - 2\sigma. \quad \square$$

3 Proof of Theorem 1.1

Denote by $a = 2\min\{|\alpha \cdot c| \, : \, c \in H_2(M;\mathbb{Z}), \alpha \cdot c \neq 0\}$. We put $a = \infty$ if $\alpha \cdot H_2(M;\mathbb{Z}) = 0$. It suffices to construct a bijection $I : \deg^{-1}\alpha \to \mathbb{Z}_a$.

In order to construct I, fix a framed circle L_1 such that $\deg L_1 = \alpha$ (clearly, such a circle exists). Take an arbitrary framed link L_2 such that $\deg L_2 = \alpha$. Since L_1 and L_2 are homologous, it follows that there is a (not framed) cobordism L between them. By definition, put $I(L_2) = \bar{e}(L)$ mod a.

It will follow from (1) and (2) below that I is well-defined:

(1) $I(L_2)$ does not depend on the choice of L; and

(2) if L_2 and L_2' are framed cobordant then $I(L_2) = I(L_2')$.

Let us first prove (2). Assume that $L_1 \subset M \times 1$, $L_2 \subset M \times 0$, $L_2' \subset M \times (-1)$, $L \subset M \times [0,1]$. Let $L' \subset M \times [-1,0]$ be a framed cobordism between L_2 and L_2'. By the geometric definition of the relative normal Euler class it follows that $\bar{e}(L \cup L') = \bar{e}(L) + \bar{e}(L')$. Since the cobordism L' is framed, it follows that $\bar{e}(L') = 0$. Thus $\bar{e}(L \cup L') = \bar{e}(L)$, and we obtain the required equality $I(L_2) = I(L_2')$.

Let us now prove (1), which is the main step in our proof. Take another general position cobordism L' between L_1 and L_2. Assume that $L_2 \subset M \times 0$, two copies of L_1 are contained in $M \times (\pm 1)$ and $L, L' \subset M \times [0,1]$. Let $-L' \subset M \times [-1,0]$ be the cobordism symmetric to L' (we consider the symmetry $x \times t \to x \times (-t)$ on $M \times \mathbb{R}$). Take a general position framed circle $-L_1' \subset M$ such that $L_1 \cup L_1'$ is framed cobordant to zero, i. e. to an empty submanifold. Denote by Δ the corresponding framed cobordism. Assume that two copies of L_1' are contained in $M \times (\pm 1)$, and $\Delta \subset [1; +\infty)$. Let $-\Delta \subset (-\infty, -1]$ be the cobordism, symmetric to Δ. Denote by

$$K = (-L') \cup L \cup \Delta \cup (L_1' \times [-1,1]) \cup (-\Delta).$$

By the geometric definition of the relative normal Euler class we obtain

$$\bar{e}(K) = \bar{e}(-L') + \bar{e}(L) + \bar{e}(\Delta) + \bar{e}(L_1' \times [-1,1]) + \bar{e}(-\Delta).$$

Here Δ, $-\Delta$ and $L_1' \times [-1,1]$ can be framed, so $\bar{e}(\Delta) = \bar{e}(-\Delta) = \bar{e}(L_1 \times [-1,1]) = 0$. Since the symmetry $x \times t \to x \times (-t)$ reverses the orientation of $M \times [-1;1]$, it follows by the geometric definition of the relative normal Euler class that $\bar{e}(-L') = -\bar{e}(L')$. Thus $\bar{e}(K) = \bar{e}(L) - \bar{e}(L')$. Now (1) follows from

$$\bar{e}(K) = -2\sigma = 2(-L' \cup L) \cap (L_1' \times [-1,1]) = 2K \cap (L_1'' \times \mathbb{R}) = 2[pK] \cdot \alpha = 0 \mod a.$$

Here σ is the difference between the numbers of positive and negative self-intersections of K, and the first equality follows from Lemma 2.1. The second equality follows from the construction of K. Then, $L_1'' \subset M$ is a general position circle close to L_1' and homologous to it. By general position L_1' and L_1'' are disjoint, so $(-L' \cup L) \cap (L_1' \times [-1,1]) = K \cap (L_1'' \times [-1,1])$. Since $-\Delta$ is obtained from Δ by the symmetry $x \times t \to x \times (-t)$, it follows that $K \cap (L_1'' \times [1, +\infty)) = -K \cap (L_1'' \times (-\infty, -1])$, and the third equality follows. Denote by $p : M \times I \to M$ the projection. Then by general position we obtain the fourth equality, because the homological class of L_1'' is α. Let us prove the last equality. If $\alpha \cdot H^2(M;\mathbb{Z}) = 0$, then it is obvious. Otherwise let $\beta \in H^2(M;\mathbb{Z})$ be the class such that $2\alpha \cap \beta = a$. Let $2[pK] \cdot \alpha = xa + y$, $x \in \mathbb{Z}$, $0 \leq y < a$. Then $y = 2([pK] - x\beta) \cdot \alpha$, hence $y = 0$ by the definition of a. So the proof of (1) is completed.

Injectivity of I. Let L_2 and L_2' be a pair of framed 1-submanifolds such that $I(L_2) = I(L_2')$. Let us prove that L_2 and L_2' are frame cobordant. Assume that $L_2 \subset M \times 1$, $L_1 \subset M \times 0$ and $L_2' \subset M \times (-1)$. Let $L \subset M \times [0,1]$ and $-L' \subset M \times [-1,0]$ be the cobordisms between L_1 and L_2, L_1 and L_2' respectively. Since $I(L_2) = I(L_2')$, it follows that $\bar{e}(L) = -\bar{e}(-L') \mod a$. Then $\bar{e}(-L' \cup L) = xa$ for some $x \in \mathbb{Z}$.

If $\alpha \cdot H_2(M;\mathbb{Z}) = 0$, then $\bar{e}(-L' \cup L) = 0$, hence the cobordism $-L' \cup L$ between L_2 and L_2' can be framed (see the remark after the geometric definition of the Euler class), so L_2 and L_2' are frame cobordant.

If $\alpha \cdot H_2(M;\mathbb{Z}) \neq 0$, then take $\beta \in H_2(M;\mathbb{Z})$ such that $2\alpha \cdot \beta = a$. Let $K \subset M \times 0$ be a general position connected submanifold realizing the class $x\beta$. Notice that $\bar{e}(K) = 0$ by Lemma 2.1. Denote by $K' = (-L' \cup L)\sharp K$. By the geometric definition of the relative normal Euler class it follows that $\bar{e}(K') = \bar{e}(-L' \cup L) + \bar{e}(K) = xa$. The cobordism K' has $|x\beta \cap \alpha| = xa/2$ self-intersection points. Let K'' be a new cobordism between L_2 and L_2' obtained from K' by removing of the self-intersection points. Here we use a move in a neighbourhood of each self-intersection point analogous to the move taking the pair of the planes $x = 0$, $y = 0$ and $z = 0$, $t = 0$ to the surface

$$\begin{cases} x(\tau, \varphi) = \tau \cos \varphi, \\ y(\tau, \varphi) = \tau \sin \varphi, \\ z(\tau, \varphi) = (1 - \tau) \cos \varphi, \\ t(\tau, \varphi) = (1 - \tau) \sin \varphi; \end{cases}$$

in $\mathbb{R}^4$ with coordinats (x, y, z, t). By the geometric definition of the normal Euler class it can be proved easily that removing of each self-intersection point decreases $\bar{e}(K')$ by ± 2, depending on the sign of the point (since our move is local, it suffices to prove it for a closed submanifold K', and this latter case follows from Lemma 2.1). So $\bar{e}(K'') = \bar{e}(K') - xa = 0$. Thus K'' can be framed and L_2 and L_2' are framed cobordant.

Surjectivity of I. Let us construct a sequence $L_1, L_2, \dots, L_a$ of framed 1-submanifolds such that for $j = 1, \dots, a$ $I(L_j) = j - 1$. Fix a homeomorphism $L_1 \cong S^1$. Denote by $f_1(x)$ the basis vector of the fixed framing of L_1 at the point $x \in S^1$. Take a map $\varphi : S^1 \to SO(2)$ realizing the generator $\pi_1(SO(2)) \cong \mathbb{Z}$. For $j = 2, \dots, a$ define the framing f_j of L_1 by the formula $f_j(x) = \varphi^{j-1}(x) f_1(x)$. Let L_j be the submanifold L_1 with framing f_j. Without loss of generality we may assume that $I(L_2) \geq 0$. Let us prove that $I(L_2) = 1$. It can be shown analogously that $I(L_j) = j - 1$. Take $L = L_1 \times I$. It suffices to construct a general position normal vector field on L extending the first field of the framing of L_1 and L_2 with one singular point. The normal bundle to L in $M \times \mathbb{R}$ is trivial. Identify this bundle with $\mathbb{R} \times \mathbb{R} \times L$ and denote by $p_1, p_2 : \mathbb{R} \times \mathbb{R} \times L \to \mathbb{R}$ the projections to the first and the second multiples respectively. Further denote by f_2 the first vector field of the framing f_2. Clearly, $p_1 f_2(x)$, where $x \in L_2$, has exactly two zeros. Join them by an arc $A \subset L$. Analogously join by an arc B the pair of zeros of $p_2 f_2(x)$. Clearly, we can choose the arcs A and B intersecting transversally at a single point. Take a general position normal vector field F_1 on L extending the fields $p_1 f_2$, $p_1 f_1$ and such that $p_2 F_1 = 0$, $p_1 F_1 |_A = 0$. Analogously, extend $p_2 f_2$ and $p_2 f_1$ to a normal vector field F_2 such that $p_1 F_2 = 0$, $p_2 F_2 |_B = 0$. The sum $F_1 + F_2$ with a single zero at the point $A \cap B$ is the required vector field. □

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