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# UNCOUNTABLY MANY INEQUIVALENT LIPSCHITZ HOMOGENEOUS CANTOR SETS IN $\mathbb{R}^3$

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# UNCOUNTABLY MANY INEQUIVALENT LIPSCHITZ HOMOGENEOUS CANTOR SETS IN $\mathbb{R}^3$

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Abstract. General techniques are developed for constructing Lipschitz homogeneous Cantor sets in $\mathbb{R}^3$. These techniques along with link invariants and previous results on Antoine Cantor sets are used to construct uncountably many topologically inequivalent wild Cantor sets in $\mathbb{R}^3$. These Cantor sets have the same number of components in the interior of each stage of the defining sequence and are Lipschitz homogenous.

## 1. Introduction

In [MR99] Malešič and Repovš construct one specific example of a wild Cantor set in $\mathbb{R}^3$ that is Lipschitz homogeneously embedded. This answered negatively a question in [RSŠ96] as to whether Lipschitz homogeneity characterized Lipschitz submanifolds of manifolds. In this paper, we introduce more general techniques for detecting the Lipschitz homogeneity of Cantor sets in $\mathbb{R}^n$. These techniques allow us to construct uncountably many topologically distinct wild Cantor sets in $\mathbb{R}^3$. These Cantor sets are all simple Antoine Cantor sets with the same Antoine graph as defined in [Wri86]. The fact that the constructed Cantor sets are all topologically distinct is a consequence of a result of Sher [She68] and a computation of a link invariant for the center lines of certain tori used in the construction. It is hoped that the techniques in this paper may also prove to be applicable to showing that certain Blankinship type Cantor sets [Bla51, Eat73] in $\mathbb{R}^n$ for $n \geq 4$ can be constructed so as to be Lipschitz homogeneously embedded.

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## 2. Notation and Background

**Lipschitz maps and similitudes.**

A map $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be a *Lipschitz map* if there exists a constant $\lambda$ such that

$$|S(x) - S(y)| \le \lambda |x - y| \text{ for every } x, y \in \mathbb{R}^n$$

and the smallest such $\lambda$ is called the *Lipschitz constant of* $S$. In the special case when

$$|S(x) - S(y)| = \lambda |x - y| \text{ for every } x, y \in \mathbb{R}^n$$

the map $S$ is called a *similarity* and the number $\lambda$ is called the *coefficient of similitude.* Finally, when $\lambda = 1$ the map $S$ is called an *isometry.*

A Cantor set $C$ in $\mathbb{R}^3$ is *Lipschitz homogeneously embedded* if for each pair of points $x$ and $y$ in $C$ there is a Lipschitz homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ with $h^{-1}$ also Lipschitz such that $h(C) = C$ and $h(x) = y$.

**Coordinates of points in Cantor sets.**

Let $G_i$, $1 \le i \le M$, be finite index sets and let $\mathcal{S}_i = \{S_g \colon \mathbb{R}^n \to \mathbb{R}^n \,|\, g \in G_i\}$ be a set of similarities having the same coefficient $\lambda_i$ of similitude. Let $\mathcal{S} = \cup \mathcal{S}_i$. Additionally, suppose that there exists a compact set $X \subset \mathbb{R}^n$ such that

(1) $S_g(X) \subset Int(X)$ for each $g \in G_i$ and
(2) the sets $S_g(X)$ are pairwise disjoint, $g \in G_i$.

Let $\mathcal{T} = (n_1, n_2, \cdots)$ be a fixed sequence where each $n_i$ is in $\{1, \ldots, M\}$. Let $G^k = G_{n_1} \times G_{n_2} \times \cdots \times G_{n_k}$, $G_k^\infty = \prod_{i=k}^{\infty} G_{n_i}$ and $G^\infty = \prod_{i=1}^{\infty} G_{n_i}$. For each multiindex $\gamma = (g_1, g_2, \ldots g_k) \in G^k$ denote:

$$S_\gamma = S_{g_1} \circ S_{g_2} \circ \ldots \circ S_{g_k}$$

and

$$X_\gamma = S_\gamma(X).$$

In particular,

$$X_g = S_g(X) \ \text{ for } \ g \in G.$$

The number of components of a multiindex $\gamma$ is called the *dimension of* $\gamma$:

$$\dim(\gamma) = k \quad \text{if } \gamma \in G^k.$$

Denote

$$X_k = \bigcup_{\dim(\gamma)=k} X_\gamma.$$

It is well-known [Hut81] that the intersection of the sequence of sets $X \supset X_1 \supset X_2 \supset \ldots$ is a Cantor set that is self-similar if $\mathcal{T}$ is repeating. Denote this set by $|(\mathcal{S}, \mathcal{T})|$.

Note that the set $|(\mathcal{S}, \mathcal{T})|$ does not depend on the choice of $X$.

For an infinite multiindex $\gamma = (g_1, g_2, g_3, \ldots) \in G^\infty$ denote

$$\gamma^k = (g_1, g_2, \ldots g_k)$$

and

$$X_\gamma = \bigcap_{k=1}^{\infty} X_{\gamma^k}.$$

Obviously, each $X_\gamma$ is a singleton, consisting of a point from the Cantor set $|(\mathcal{S}, \mathcal{T})|$ and for each point from $|(\mathcal{S}, \mathcal{T})|$ there exists exactly one such multiindex $\gamma$. The components of $\gamma$ are called *coordinates* of the corresponding point from the Cantor set $|(\mathcal{S}, \mathcal{T})|$.

Finally define a juxtaposition of multiindices as follows. If $\delta = (d_1, d_2, \ldots d_k)$ is a finite multiindex and $\gamma = (g_1, g_2, \ldots)$ is finite or infinite then let

$$\delta\gamma = (d_1, d_2, \ldots d_k, g_1, g_2, \ldots).$$

In the special case when $\dim(\gamma) = 1$, hence $\gamma = g_1$ and

$$\delta g_1 = (d_1, d_2, \ldots d_k, g_1).$$

**Antoine Cantor sets.**

We give a brief summary of the necessary results in Sher and Wright [She68, Wri86]. An Antoine Cantor $C$ set in $\mathbb{R}^3$ is a Cantor set satisfying the following conditions.

(1) $C$ has a defining sequence $M_1, M_2, \ldots$,with each $M_i$ consisting of the union of a finite number of pairwise disjoint standard unknotted solid tori in $\mathbb{R}^3$ and with $M_1$ consisting of a single solid torus.
(2) The tori in $M_i$ $i \geq 2$, can be listed in a sequence $T_{i,1}$, $T_{i,2}$, $\ldots$, $T_{i,n(i)}$ so that $T_j$ and $T_k$ are of simple linking type if $j - k = \pm 1 \bmod n$ and do not link if $j - k \neq \pm 1 \bmod n$.
(3) The linked chain of tori $T_{i,1}, T_{i,2}, \ldots, T_{i,n(i)}$ have winding number $> 0$ in the torus at the previous stage that contains them.

If in condition 3, the winding number is required to be 1, and if each $n(i) \geq 4$, we call the resulting Cantor set a *simple Antoine Cantor set.* Most Antoine Cantor sets in the literature, including the original one constructed by Antoine [Ant20], are simple.

Sher [She68] shows that if two Antoine Cantor sets $C_1$ and $C_2$ with defining sequences $M_1, M_2, \ldots$ and $N_1, N_2, \ldots$ are equivalently embedded in $\mathbb{R}^3$, then there is a homeomorphism $h$ of $\mathbb{R}^3$ to itself such that for each $i$, $h$ takes the tori in $M_i$ homeomorphically onto the tori in $N_i$. As a consequence, if it can be shown that for some $i$, no such homeomorphism exists, the two Cantor sets are inequivalently embedded. This is the result we will need to construct the uncountably many inequivalently embedded Cantor sets.

Wright [Wri86] associates an Antoine graph $\Gamma(C)$ with a Antoine Cantor $C$ with defining sequence $M_1, M_2, \ldots$ . $\Gamma$ is a countable union of nested subgraphs $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots$.The subgraph $\Gamma_0$ is a single vertex. For each vertex $v$ of $\Gamma_i - \Gamma_{i-1}$ , there is a polygonal simple closed curve with at least 4 vertices $P(v)$ contained in $\Gamma_{i+1} - \Gamma_i$ so that if $v$ and $w$ are distinct vertices of $\Gamma_i - \Gamma_{i-1}$, then $P(v)$ and $P(w)$ are disjoint. $\Gamma_{i+1}$ consists of $\Gamma_i$, the union of the $P(v)$ for $v$ in $\Gamma_i$, and the union of edges running from $v$ to the vertices of $P(v)$. The vertices of $\Gamma_i - \Gamma_{i-1}$ correspond to the components of $M_i$. See page 252 in [Wri86] for a diagram. Wright shows that if $C_1$ and $C_2$ are simple Antoine Cantor sets with different Antoine graphs $\Gamma(C_1)$ and $\Gamma(C_2)$, then the Cantor sets are inequivalently embedded.

In our construction, all of the Cantor sets constructed have the same Antoine graph, but are inequivalently embedded.

## 3. Constructing Lipschitz homogenous Cantor sets

Let $G_i$, $1 \leq i \leq M$, $\mathcal{S}_i = \{S_g \colon \mathbb{R}^n \to \mathbb{R}^n \mid g \in G_i\}$, $X$, $X_g$ and $\mathcal{T} = (n_1, n_2, \cdots)$ be as above. The setting to keep in mind when reading Theorem 1 below is that of a Simple Antoine Cantor set defined by tori where each stage $m$ torus has $|G_{n_m}|$ stage $m+1$ tori in its interior. For Theorem 1, also assume that each $G_i$ is a finite cyclic group, with the group operation written additively.

**Theorem 1.** *For each $i$, $1 \leq i \leq M$, suppose that $f_i \colon \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz homeomorphism,*
*(i)* $f_i|_{\mathbb{R}^n - Int(X)} = id$

*(ii)* $f_i(X_g) = X_{g+1}$ *for* $g \in G_i$ *and the following diagram commutes*

$$\begin{array}{ccccc} & & X & & \\ S_g & \swarrow & & \searrow & S_{g+1} \\ X_g & & \xrightarrow{f_i} & & X_{g+1} \end{array}$$

Then $|(\mathcal{S}, \mathcal{T})|$ is Lipschitz homogeneous in $\mathbb{R}^n$.

*Proof.* The approach to the proof is similar to that used in Lemma 1 of [MR99]. The main modification needed is to take into account the presence of more than one finite index set. Fix an arbitrary pair of points $a$ and $b$ in $|(\mathcal{S}, \mathcal{T})|$. We will construct a homeomorphism

$$h\colon (\mathbb{R}^n, |(\mathcal{S}, \mathcal{T})|, a) \to (\mathbb{R}^n, |(\mathcal{S}, \mathcal{T})|, b)$$

and prove that both $h$ and $h^{-1}$ are Lipschitz. Let $\alpha = (a_1, \cdots, a_k, \cdots)$ and $\beta = (b_1, \cdots, b_k, \cdots) \in G^\infty$ be the coordinates of $a$ and $b$. Let:

$$\begin{aligned} &r_1 = f_{n_1}^{b_1 - a_1}, r_2 = f_{b_1}^{b_2 - a_2}, r_3 = f_{(b_1, b_2)}^{b_3 - a_3}, \dots, r_{k+1} = f_{\beta^k}^{b_{k+1} - a_{k+1}}, \dots \\ &g_i = r_i^{-1} \\ &h_k = r_k \circ r_{k-1} \circ \cdots \circ r_2 \circ r_1 \end{aligned}$$

It follows by Lemma 2 (iv) below that the sequences of homeomorphisms $h_1, h_2, \dots$ and $h_1^{-1}, h_2^{-1}, \dots$ converge pointwise at all points different from the point $a$ and $b$, respectively. The convergence of the sequences at the point $a$ and at the point $b$ follows from Lemma 2, (ii). Denote the limits of the sequences by $h\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\tilde{h}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, respectively. It also follows by Lemma 2 that $h(a) = b$, that $h(|\mathcal{S}|) = |\mathcal{S}|$, and that $h \circ \tilde{h} = \tilde{h} \circ h = \mathrm{id}_{\mathbb{R}^n}$. It follows from Lemma 3 that $h$ and $\tilde{h}$ are Lipschitz. Thus Theorem 1 is proved. □

**Lemmas needed for proof of Theorem.** For an arbitrary $\gamma = (g_1, \cdots, g_k) \in G^k$ define the homeomorphism

$$f_\gamma = S_\gamma \circ f_{n_{k+1}} \circ S_\gamma^{-1} : R^n \to \mathbb{R}^n.$$

**Lemma 1.** *The homeomorphism* $f_\gamma$ *is Lipschitz with Lipschitz constant equal to the Lipschitz constant of* $f_{n_{k+1}}$ *and the following holds*
*(i)* $f_\gamma|_{\mathbb{R}^n - Int(X_\gamma)} = id$
*(ii) For arbitrary* $g \in G_{n_{k+1}}$, $f_\gamma(X_{\gamma g}) = X_{\gamma(g+1)}$ *and the following diagram commutes*

$$\begin{array}{ccccc} & & X & & \\ S_{\gamma g} & \swarrow & & \searrow & S_{\gamma(g+1)} \\ X_{\gamma g} & & \xrightarrow{f_\gamma} & & X_{\gamma(g+1)} \end{array}$$

*and* $f_\gamma|_{X_{\gamma g}}$ *is an isometry*
*(iii) For arbitrary* $(g_{k+1}, g_{k+2}, \ldots) \in G^{\infty}_{k+1}$,

$$f_\gamma(X_{(\gamma, g_{k+1}, g_{k+2}, \ldots)}) = f_\gamma(X_{(\gamma, 1+g_{k+1}, g_{k+2}, \ldots)})$$

*Proof.* This follows in a similar manner to Lemma 2 of [MR99]. Part (i) follows directly from the condition (i) of Theorem 1. Proposition (ii) follows from condition (ii). Finally, (ii) implies (iii). □

**Lemma 2.** *The homeomorphisms* $h_k$ *exhibit the following properties:*
*(i)* $h_k^{-1} = g_1 \circ g_2 \circ \cdots \circ g_{k-1} \circ g_k$,
*(ii)* $h_k(X_{\alpha^k}) = X_{\beta^k}$ *and* $h_k(X_{\alpha^k \gamma}) = X_{\beta^k \gamma}$ *for arbitrary multiindex* $\gamma$
*(iii) the restriction* $h_k|_{X_{\alpha^k a_{k+1}}} : X_{\alpha^k a_{k+1}} \to X_{\beta^k a_{k+1}}$ *is an isometry*
*(iv)* $h_k|_{\mathbb{R}^n - Int(X_{\alpha^k})} = h_{k+1}|_{\mathbb{R}^n - Int(X_{\alpha^k})} = h_{k+2}|_{\mathbb{R}^n - Int(X_{\alpha^k})} = \cdots$

*Proof.* This follows in a similar manner to Lemma 3 of [MR99]. Property (i) can be proved directly by examining the construction of $h_k$. Property (ii) follows from Lemma 1, (ii) and (iii). Property (iii) holds since $f_\gamma|_{X_{\gamma g_{k+1}}}$ is an isometry. Property (iv) holds because of Lemma 1 (i). □

**Lemma 3.** $h_k$ *and* $h_{k^{-1}}$ *are Lipschitz with equal Lipschitz constants for all values of* $k$.

*Proof.* This requires the most modification of [MR99] as multiple similarities with different constants of similarity are involved.
We fix the sequence $\alpha = (a_1, a_2, \ldots)$ of coordinates of the point $a \in |\mathcal{S}|$ and introduce the notion of *degree* of a point $x \in \mathbb{R}^n$:

$$\deg x = j \quad \text{if} \quad x \in X_{\alpha^j} - \text{Int}(X_{\alpha^{j+1}}).$$

Additionally, let

$$\deg x = 0 \text{ if } x \in X - \text{Int}(X_{a_1}) \text{ and } \deg x = -1 \text{ if } x \in \mathbb{R}^n - \text{Int}(X).$$

In the case $x \in$Int$(X_{\alpha^j})$ for all $j \in \mathbb{N}$ (i.e. $x = a$) let $\deg x = \infty$.

For arbitrary distinct points $x, y \in \mathbb{R}^n$ we now estimate the expression $h_k(x) - h_k(y)$. We may assume that $\deg x \leq \deg y$. As $x$ and $y$ are distinct, case $\deg y = \infty$ and $\deg x = \infty$ is not possible.

**Case 1** Let the Lipschitz constant of the homeomorphism $f_i$ be denoted by $\lambda_i$. Let $\lambda = \max\{\lambda_i;\ 1 \leq i \leq M\}$ and $T = \max\{|G_i|;\ 1 \leq i \leq M\}$, where $|G_i|$ denotes the number of elements of $G_i$. Hence the Lipschitz constants of the homeomorphisms $f_1, f_2, \ldots, g_1, g_2, \ldots$ do not exceed the number $\Lambda = \lambda^T$. Let $|\deg x - \deg y| \leq 1$, i.e.

$$\deg x \in \{j, j+1\}, \ \ \deg y = j+1$$

for some $j \in \mathbb{N}$. By Lemma 2, (iii) and (iv), and because of the construction of $h_k$,

$$|h_k(x) - h_k(y)| = |f_{j+1} \circ f_j(x) - f_{j+1} \circ f_j(y)| \leq \Lambda^2 |x - y|.$$

**Case 2** Let now $|\deg x - \deg y| \geq 2$. First let the degrees be nonnegative, i.e.

$$\deg x = j \geq 0 \quad \text{and} \quad \deg y \geq j + 2$$

for some $j \in \mathbb{N}$. (It may be $\deg y = \infty$ as well.) Then

$$x \in X_{\alpha^j} - \operatorname{Int}(X_{\alpha^{j+1}}), \quad y \in X_{\alpha^{j+2}}.$$

For arbitrary disjoint compact sets $C_1, C_2 \subset \mathbb{R}^n$ denote:

$$\mathrm{d}_{\min}(C_1, C_2) = \min\{|x - y|; x \in C_1,\ y \in C_2\}$$

and

$$\mathrm{d}_{\max}(C_1, C_2) = \max\{|x - y|; x \in C_1,\ y \in C_2\}.$$

The sets $X - \operatorname{Int}(X_1)$ and $X_2$ are compact and disjoint, hence the numbers

$$\mathrm{d}_X = \mathrm{d}_{\min}(X - \operatorname{Int}(X_1), X_2)$$

and

$$\mathrm{D}_X = \mathrm{d}_{\max}(X - \operatorname{Int}(X_1), X_2)$$

exist. Since the similarity $S_{\alpha^k}$ maps the triple $(X, X_{a_1}, X_{(a_1,a_2)})$ onto the triple $(X_{\alpha^k}, X_{\alpha^k a_1}, X_{\alpha^k(a_1,a_2)})$, for each $k \in \mathbb{N}$, the following holds:

$$\frac{d_{\max}(X_{\alpha^k} - \operatorname{Int}X_{\alpha^k a_1}, X_{\alpha^k(a_1,a_2)})}{\mathrm{d}_{\min}(X_{\alpha^k} - \operatorname{Int}X_{\alpha^k a_1}, X_{\alpha^k(a_1,a_2)})} \leq \frac{\mathrm{D}_X}{\mathrm{d}_X}.$$

Hence

$$|h_k(x) - h_k(y)| \leq \frac{\mathrm{D}_X}{\mathrm{d}_X} |x - y|.$$

Finally, let $\deg x = -1$ and $\deg y \geq 1$, i.e. $x \in \mathbb{R}^n - \operatorname{Int}X$ and $y \in X_1$. Then $h_k(x) = x$ and

$$\frac{|h_k(x) - h_k(y)|}{|x - y|} \leq \frac{|x - y| + |y - h_k(y)|}{|x - y|} \leq 1 + \frac{\operatorname{diam}X_1}{m}$$

where

$$m = \inf\{|x - y|; x \in \mathbb{R}^n - \operatorname{Int}X, \quad y \in X_1\}$$

(it is easy to show that $m > 0$). To conclude, denote

$$L = \max\{\Lambda^2, \frac{\mathrm{D}_X}{\mathrm{d}_X}, 1 + \frac{\operatorname{diam}X_1}{m}\}$$

Then

$$|h_k(x) - h_k(y)| \leq L|x - y|$$

for an arbitrary $k \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$.The estimate

$$|h_k^{-1}(x) - h_k^{-1}(y)| \leq L|x - y|$$

can be proved analogously, using Lemma 2 (i). □

## 4. Main result

**Theorem 2.** *There exist uncountably many topologically distinct Lipschitz homogeneous wild Cantor sets in $\mathbb{R}^3$. In fact, these Cantor sets can all be constructed as simple Antoine Cantor sets with the same number of components of stage $n$ inside each component of stage $n-1$ and thus with the same Antoine graphs.*

*Proof.* Use Theorem 1 with $M = 2$, $G_1 = Z_{60}$ and $G_2 = Z_{60}$. Let $\mathcal{T} = (n_1, n_2, \cdots)$ be a fixed sequence of ones and twos. For each such fixed sequence, construct a Lipschitz Homogenous Antoine Cantor set as in Theorem 1. For $G_1$, let the similarities $S_g$, $g \in G_1$ be constructed so as to take the outer torus in figure 1 to the smaller tori in figure 1. Each smaller torus in the chain is obtained from the previous one by rotating the large torus by $2\pi/60$ radians and then by rotating the small tori by $\pi/2$ radians. The homeomorphism $f_1$ needed in Theorem 1 is constructed in a manner similar to that constructed in the example in [MR99].

For $G_2$, let the similarities $S_g$, $g \in G_2$ be constructed so as to take the outer torus in figure 2 to the smaller tori in figure 2. Each smaller torus in the chain is obtained from the previous one by rotating the large torus by $2\pi/60$ radians and then by rotating the small tori by $\pi/4$ radians. The homeomorphism $f_2$ needed in Theorem 1 is constructed in a manner similar to that constructed in the example in [MR99]. The resulting Cantor set is Lipschitz homogeneously embedded by Theorem 1.

Note that the Antoine graphs associated with any two Cantor set constructed in this way are the same.

Let $C_1$ be the Cantor set constructed as in Theorem 1 for one sequence of ones and twos and let $C_2$ be the Cantor set constructed using a different sequence. We need to show that these two Cantor sets are topologically inequivalently embedded. For this, using the above mentioned result from Sher [She68], it suffices to show that there is not a homeomorphism of $\mathbb{R}^3$ to itself taking the large torus in Figure 1 to the large torus in Figure 2, and taking the chain of smaller tori in Figure 1 to the chain of smaller tori in Figure 2. If there was such a homeomorphism, the link formed by the centerlines of the small tori in Figure 1 would be topologically the same as the link formed by the centerlines of the small tori in Figure 2.

A simple computation of the bracket polynomial introduced by Kauffman in [Kau88] shows that these links are topologically distinct, and thus the Cantor sets are topologically distinct

There are uncountably many sequences of 1's and 2's. The above argument shows that each such sequence leads to a topologically distinct Lipschitz homogeneous wild Cantor set. This completes the proof of the theorem. □

Figure 1 below shows a large torus with 60 smaller similar tori linked in a simple chain inside. Each of the smaller tori is rotated by $\pi/2$ radians from the previous one. The figure at the right in Figure 1 show an enlarged view of 6 of the smaller tori.

Figure 2 below shows a large torus with 60 smaller similar tori linked in a simple chain inside. Each of the smaller tori is rotated by $\pi/4$ radians from the previous one. The figure at the right in Figure 1 show an enlarged view of 6 of the smaller tori.

**Figure 1**

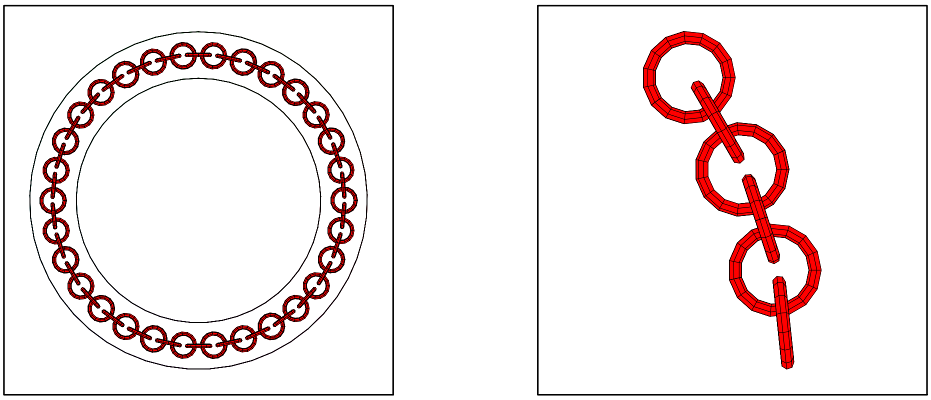
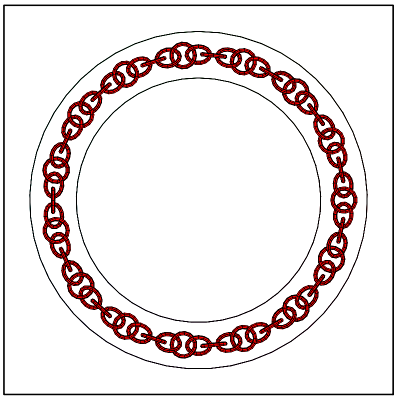
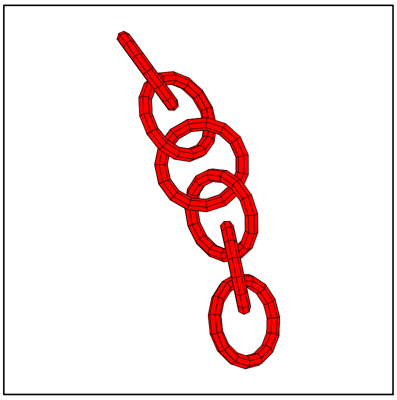


**Figure 2**





## 5. Other results and questions

Using techniques similar to those used in the proof of Theorem 1, we can prove the following result. Note that in this case, we assume that $G$ is of the form $Z_p \times Z_q$ for some positive integers $p$ and $q$.

**Theorem 3.** *For each $i$, $1 \le i \le 2$, suppose that $f_i \colon \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz homeomorphism and*
*(i) $f_i|_{\mathbb{R}^n - Int(X)} = id$*
*(ii) $f_1(X_{(a,b)}) = X_{(a+1,b)}$ for $(a,b) \in G$,*
*(iii) $f_2(X_{(a,b)}) = X_{(a,b+1)}$ for $(a,b) \in G$ and the following diagrams commute*

$$\begin{array}{ccccc} & & X & & \\ S_{(a,b)} & \swarrow & & \searrow & S_{(a+1,b)} \\ X_{(a,b)} & & \stackrel{f_1}{\to} & & X_{(a+1,b)} \end{array} \qquad \begin{array}{ccccc} & & X & & \\ S_{(a,b)} & \swarrow & & \searrow & S_{(a,b+1)} \\ X_{(a,b)} & & \stackrel{f_2}{\to} & & X_{(a,b+1)} \end{array}$$

*Then $|(\mathcal{S}, \mathcal{T})|$ is Lipschitz homogeneous in $\mathbb{R}^n$.*

The construction suggested by the previous theorem is similar to the Blankinship construction for wild Cantor sets in $R^4$.

**Question:** Can the previous theorem be used to show that a Lipschitz homogeneous wild Cantor set in $\mathbb{R}^4$ exists? This would require a more careful Blankinship type construction [Bla51] in which the successive stages in the construction were self similar to the original stage.

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