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JADRANSKA 19, 1111 LJUBLJANA, SLOVENIA

Preprint series, Vol. 41 (2003), 908

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ZARICHNYI MAP

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ISSN 1318-4865

December 8, 2003

Ljubljana, December 8, 2003

ON LINEAR REALIZATIONS AND LOCAL SELF-SIMILARITY OF THE UNIVERSAL ZARICHNYI MAP

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ABSTRACT. Answering a question M.Zarichnyi we show that the universal Zarichnyi map $\mu : \mathbb{R}^\infty \rightarrow Q^\infty$ is not locally self-similar. Also we characterize linear operators homeomorphic to μ and on this base give a simple construction of a universal Zarichnyi map μ .

In this paper we investigate the properties of the universal map $\mu : \mathbb{R}^\infty \rightarrow Q^\infty$ constructed by M.Zarichnyi in [Za₃] and subsequently studied in [Za₄] and [Za₅]. Answering a question posed in [Za₅] we prove that the map μ is not locally self-similar. Also we characterize linear operators homeomorphic to μ and on this base give a simple construction of a universal Zarichnyi map μ .

STRONGLY UNIVERSAL MAPS

All topological spaces considered in this paper are Tychonov, all compact spaces are metrizable, and all maps are continuous; $\omega = \{0, 1, 2, \dots\}$ stands for the set of all finite ordinals.

Given a class \mathcal{C} of compacta, by \mathcal{C}^∞ we denote the class of topological spaces X admitting a countable cover \mathcal{U} by subsets of the class \mathcal{C} , generating the topology of X in the sense that a subset $F \subset X$ is closed in X if and only if $F \cap K$ is closed in K for every $K \in \mathcal{U}$. In our subsequent considerations $\mathcal{C} = \mathcal{K}$ or \mathcal{K}_{fd} , where \mathcal{K} (\mathcal{K}_{fd}) is the class of all (finite-dimensional) metrizable compacta.

Given a space X with a fixed point $*$ let X^∞ denote the set

$$X_f^\omega = \{(x_i)_{i \in \omega} \in X^\omega : x_i = * \text{ for almost all } i\}$$

endowed with the strongest topology inducing the product topology on each space $X^n = \{(x_i)_{i \in \omega} \in X : x_i = * \text{ for all } i \geq n\}$, $n \in \omega$. If the space X is topologically homogeneous (like the real line \mathbb{R} or the Hilbert cube $Q = [0, 1]^\omega$), then the topology of the space X^∞ does not depend on the particular choice of a fixed point $* \in X$.

Among the spaces X^∞ the spaces \mathbb{R}^∞ and Q^∞ occupy the special place: they are universal for the classes \mathcal{K}_{fd}^∞ and \mathcal{K}^∞ in the sense that each space from the class \mathcal{K}^∞ (resp. \mathcal{K}_{fd}^∞) is homeomorphic to a closed subspace of Q^∞ (resp. \mathbb{R}^∞). Topological copies of the spaces \mathbb{R}^∞ and Q^∞ very often appear in topological algebra and

2001 *Mathematics Subject Classification.* 57N20, 57N17.

Key words and phrases. Universal Zarichnyi map, local self-similarity.

The authors were supported in part by the Slovenian-Ukrainian research grant SLO-UKR 02-03/04.

functional analysis, see [Ba₂]—[BS₂], [Sa₂]—[Za₂]. In particular, every infinite-dimensional linear topological space $X \in \mathcal{K}_{fd}^\infty$ is homeomorphic to \mathbb{R}^∞ [Ba₂] while each locally convex space $Y \in \mathcal{K}^\infty$ with uncountable Hamel basis is homeomorphic to Q^∞ [Ba₃]. A topological characterization of the spaces \mathbb{R}^∞ and Q^∞ was given by K.Sakai [Sa₁]: *Up to a homeomorphism \mathbb{R}^∞ (resp. Q^∞) is a unique strongly \mathcal{K}_{fd} -universal (resp. strongly \mathcal{K} -universal) space in the class \mathcal{K}_{fd}^∞ (resp. \mathcal{K}^∞).*

A topological space X is defined to be *strongly \mathcal{C} -universal* if every embedding $f : B \rightarrow X$ of a closed subset B of a space $A \in \mathcal{C}$ can be extended to an embedding $\bar{f} : A \rightarrow X$. Replacing the words “embedding” by “map” we obtain the definition of an *absolute extensor* for the class \mathcal{C} (briefly, $\text{AE}(\mathcal{C})$).

In [Za₃] the notion of the strong universality was generalized to maps. A map $\pi : X \rightarrow Y$ between topological spaces is defined to be *strongly \mathcal{C} -universal* if for every embedding $f : B \rightarrow X$ of a closed subset B of a space $A \in \mathcal{C}$ and a map $g : A \rightarrow Y$ with $\pi \circ f = g|_B$ there exists an embedding $\bar{f} : A \rightarrow X$ such that $\bar{f}|_B = f$ and $\pi \circ \bar{f} = g$. Replacing the words “embedding” by “map” we obtain the definition of a *\mathcal{C} -soft* map. Observe that a space X is strongly \mathcal{C} -universal (resp. is an $\text{AE}(\mathcal{C})$) if and only if the constant map $X \rightarrow \{*\}$ is strongly \mathcal{C} -universal (resp. \mathcal{C} -soft).

The following theorem was proven in [Ba₁].

Uniqueness Theorem. *If $\pi : X \rightarrow Y$, $\pi' : X' \rightarrow Y$ are strongly \mathcal{C} -universal maps from spaces $X, X' \in \mathcal{C}^\infty$, then there is a homeomorphism $h : X \rightarrow X'$ such that $\pi' \circ h = \pi$.*

In case of a one-point space Y we obtain Uniqueness Theorem of strongly \mathcal{C} -universal spaces (see [Sa₁]): *Any two strongly \mathcal{C} -universal spaces $X, X' \in \mathcal{C}^\infty$ are homeomorphic.*

Thus (up to a homeomorphism) there is at most one strongly \mathcal{K}_{fd} -universal map from a space $X \in \mathcal{K}_{fd}^\infty$ onto a given space Y . For which spaces Y does such a map exist? If $Y \in \mathcal{K}_{fd}^\infty$, then the answer is easy: just consider the projection $\pi : Y \times \mathbb{R}^\infty \rightarrow Y$. If $Y \notin \mathcal{K}_{fd}^\infty$ (for example, if $Y = Q^\infty$) the situation is not so obvious. Nonetheless, applying certain non-trivial results of A.Dranishnikov [Dr], M.Zarichnyi has constructed in [Za₃] a strongly \mathcal{K}_{fd} -universal map $\mu : \mathbb{R}^\infty \rightarrow Q^\infty$. Afterwards, he proved that this map μ is homeomorphic to a group homomorphism [Za₅] and to an affine map between suitable spaces of probability measures [Za₄], thus giving an alternative and simpler constructions of the map μ . The Zarichnyi map $\mu : \mathbb{R}^\infty \rightarrow Q^\infty$ contains any map $f : A \rightarrow B$ from a finite-dimensional metrizable compactum A into a metrizable compactum B in the sense that there are two embeddings $e_A : A \rightarrow \mathbb{R}^\infty$ and $e_B : B \rightarrow Q^\infty$ such that $\mu \circ e_A = e_B \circ f$.

CHARACTERIZING LINEAR OPERATORS HOMEOMORPHIC TO THE MAP μ

We define two maps $\pi : X \rightarrow Y$ and $\pi' : X' \rightarrow Y'$ to be *homeomorphic* if $\pi' \circ h = H \circ \pi$ for some homeomorphisms $h : X \rightarrow X'$ and $H : Y \rightarrow Y'$. In this section we characterize linear operators homeomorphic to the universal Zarichnyi map and classify \mathcal{K}_{fd} -invertible linear operators from a linear topological space $X \in \mathcal{K}_{fd}^\infty$ onto a locally convex space $Y \in \mathcal{K}^\infty$. Under a “linear operator” we understand a linear continuous operator between linear topological spaces.

We define a map $\pi : X \rightarrow Y$ to be *\mathcal{C} -invertible* if for every map $g : A \rightarrow Y$ of a space $A \in \mathcal{C}$ there is a map $f : A \rightarrow X$ such that $\pi \circ f = g$. It is clear that each

strongly \mathcal{C} -universal or \mathcal{C} -soft map is \mathcal{C} -invertible. For linear operators the converse statement is also true. In [Za₅, 2.1] M.Zarichnyi proved that a continuous group homomorphism $h : G \rightarrow H$ is \mathcal{K}_{fd} -soft if and only if h is \mathcal{K}_{fd} -invertible and its kernel $\text{Ker } h$ is an $\text{AE}(\mathcal{K}_{fd})$. Since each linear topological space is an $\text{AE}(\mathcal{K}_{fd})$ (see, e.g., [Ba₂]), we get

Theorem 1. *A linear operator between linear topological spaces is \mathcal{K}_{fd} -soft if and only if it is \mathcal{K}_{fd} -invertible.*

Next, we find conditions under which a given linear operator is strongly \mathcal{K}_{fd} -universal.

Theorem 2. *A linear operator $T : X \rightarrow Y$ from a linear topological space $X \in \mathcal{K}_{fd}^\infty$ to a linear topological space Y is strongly \mathcal{K}_{fd} -universal if and only if the operator T is \mathcal{K}_{fd} -invertible and has infinite-dimensional kernel.*

In the proof we will exploit two lemmas.

Lemma 1. *If $X \in \mathcal{K}_{fd}^\infty$ is an infinite-dimensional linear topological space, then for every compactum $K \subset X$ there is a non-zero point $x \in X$ such that $([-1, 1] \cdot x) \cap K \subset \{0\}$.*

Proof. Replacing K by $[-1, 1] \cdot K$, if necessary, we may assume that $K = [-1, 1] \cdot K$. Since K is a compact subset of the space $X \in \mathcal{K}_{fd}^\infty$, $\dim K < n$ for some $n \in \mathbb{N}$. The linear space X , being infinite-dimensional, contains an n -dimensional linear space \mathbb{R}^n . We claim that there is a point x on the unit sphere S of \mathbb{R}^n such that $([-1, 1] \cdot x) \cap K = \{0\}$. Assuming the converse, for every $x \in S$ we would find a number $n(x) \in \mathbb{N}$ such that $([-1, 1] \cdot x) \cap K \supset [0, \frac{1}{n(x)}] \cdot x$. It can be shown that for every $n \in \mathbb{N}$ the set $S_n = \{x \in S : n(x) \leq n\}$ is closed in S . Since $S = \bigcup_{n \in \mathbb{N}} S_n$, the Baire Theorem guarantees that S_n has non-empty interior in S for some n . Then $\dim S_n = n - 1$. Since $K \supset [0, 1/n] \cdot S_n$ and $n > \dim K > \dim([0, 1/n] \cdot S_n) = n$, we get a contradiction. \square

Lemma 2. *If $T : X \rightarrow Y$ is a linear operator with infinite-dimensional kernel from a linear topological space $X \in \mathcal{K}_{fd}^\infty$, then for every compact subset $C \subset X$ there exists an embedding $e : C \times [0, 1] \rightarrow X$ such that $e(c, 0) = c$ and $T \circ e(c, t) = T(c)$ for all $c \in C$ and $t \in [0, 1]$.*

Proof. Let $L = \text{Ker } T$ and $K = L \cap (C - C)$. By Lemma 1, there exists a non-zero point $x_0 \in L$ such that $([-1, 1] \cdot x_0) \cap K \subset \{0\}$. Define the map $e : C \times [0, 1] \rightarrow X$ letting $e(c, t) = c + tx_0$ for $(c, t) \in C \times [0, 1]$. It is clear that $e(c, 0) = c$ and $T \circ e(c, t) = T(c)$ for every $c \in C$ and $t \in [0, 1]$. To show that the map e is injective, fix two points $(c, t), (c', t') \in C \times [0, 1]$ with $e(c, t) = e(c', t')$. Then $c - c' = (t' - t)x_0$ and $T(c - c') = 0$ which implies $c - c' \in K$. Since $K \ni c - c' = (t' - t)x_0 \in [-1, 1]x_0$ and $K \cap ([-1, 1]x_0) \subset \{0\}$, we get $c - c' = 0$ and $t' - t = 0$, i.e., $(c, t) = (c', t')$. \square

Proof of Theorem 2. The ‘‘only if’’ part of Theorem 2 is trivial. To prove the ‘‘if’’ part, assume that $T : X \rightarrow Y$ is an \mathcal{K}_{fd} -invertible operator, $X \in \mathcal{K}_{fd}^\infty$, and $\dim \text{Ker } T = \infty$. To prove the strong \mathcal{K}_{fd} -universality of the map T , fix an embedding $f : B \rightarrow X$ of a closed subset B of a space $A \in \mathcal{K}_{fd}$ and a map $g : A \rightarrow Y$ such that $T \circ f = g|_B$. By Theorem 1, the operator T is \mathcal{K}_{fd} -soft and hence there is a map $\tilde{f} : A \rightarrow X$ such that $\tilde{f}|_B = f$ and $T \circ \tilde{f} = g$.

Denote by A/B the quotient space and let $q : A \rightarrow A/B$ be the quotient map. It is clear that the space A/B is finite-dimensional and thus admits an embedding

$i : A/B \rightarrow [0, 1]^n$ for some $n \in \mathbb{N}$ such that $e(\{B\}) = 0^n \in [0, 1]^n$. Applying Lemma 2 several times, construct an embedding $e : \tilde{f}(A) \times [0, 1]^n \rightarrow X$ such that $e(x, 0^n) = x$ and $T \circ e(x, t) = T(x)$ for any $x \in \tilde{f}(A)$ and $t \in [0, 1]^n$. It is easy to verify that the map $\bar{f} : A \rightarrow X$ defined by $\bar{f}(a) = e(\tilde{f}(a), i \circ q(a))$ for $a \in A$ is an embedding satisfying the conditions $\bar{f}|_B = f$ and $T \circ \bar{f} = g$. \square

We apply Theorem 2 to prove the following theorem characterizing linear operators homeomorphic to the universal Zarichnyi map μ .

Theorem 3. *A linear operator $T : X \rightarrow Y$ between linear topological spaces is homeomorphic to the strongly \mathcal{K}_{fd} -universal Zarichnyi map μ if and only if $X \in \mathcal{K}_{fd}^\infty$, Y is homeomorphic to Q^∞ , and the operator T is \mathcal{K}_{fd} -invertible.*

Proof. This theorem will follow from Theorem 2 and Uniqueness Theorem for strongly \mathcal{K}_{fd} -universal map as soon as we prove that each \mathcal{K}_{fd} -invertible linear operator $T : X \rightarrow Y$ from a linear topological space $X \in \mathcal{K}_{fd}^\infty$ onto a linear topological space Y containing a Hilbert cube has infinite-dimensional kernel. Assume to the contrary that $\text{Ker } T$ is finite-dimensional. By Theorem 1, the operator T , being \mathcal{K}_{fd} -invertible, is \mathcal{K}_{fd} -soft. Fix a copy $Q \subset Y$ of the Hilbert cube in Y and an open surjective map $g : A \rightarrow Q$ of a finite-dimensional compactum A onto Q (such a map exists according to [Dr]). Since the operator T is \mathcal{K}_{fd} -invertible, there is a map $f : A \rightarrow X$ such that $T \circ f = g$. Let $B \subset \text{Ker } T$ be any compact neighborhood of the origin in the finite-dimensional linear space $\text{Ker } T$. Next, consider the compact set $K = f(A) + B \subset X$. Since $X \in \mathcal{K}_{fd}^\infty$, $\dim K < n$ for some $n \in \mathbb{N}$. Let $e : I^n \rightarrow Q \subset Y$ be any embedding of the n -dimensional cube I^n into Q . By the \mathcal{K}_{fd} -softness of the map T , there is a map $i : I^n \rightarrow X$ such that $T \circ i = e$ and $i(0^n) \in f(A)$. It is clear that i is an embedding.

We claim that K is a neighborhood of the point $x_0 = i(0^n)$ in $i(I^n)$. Assuming that it is not true, we would find a sequence $(x_n)_{n=1}^\infty \in i(I^n) \setminus K$ tending to x_0 . Then the sequence $(T(x_n))_{n=1}^\infty$ converges to $T(x_0)$. Let $a_0 \in A$ be any point with $f(a_0) = x_0$. Since the map $g : A \rightarrow Q$ is open and $g(a_0) = T \circ f(a_0) = T(x_0) = \lim_{n \rightarrow \infty} T(x_n)$, there exists a sequence $(a_n)_{n=1}^\infty \subset A$ tending to a_0 such that $g(a_n) = T(x_n)$ for each n . Then the sequence $(f(a_n))_{n=1}^\infty$ converges to $f(a_0) = x_0$ and has the property: $T \circ f(a_n) = g(a_n) = T(x_n)$ for every n . Hence $x_n - f(a_n) \in \text{Ker } T$ for every n . Since $\lim_{n \rightarrow \infty} x_n = x_0 = \lim_{n \rightarrow \infty} f(a_n)$, we get $\lim_{n \rightarrow \infty} (x_n - f(a_n)) = 0$ and thus $x_m - f(a_m) \in B$ for some m . Then $x_m \in B + f(a_m) \subset B + f(A) = K$, a contradiction with the choice of the sequence $(x_n)_{n=1}^\infty$. Thus K is a neighborhood of the point $x_0 = i(0^n)$ in $i(I^n)$ what is not possible since $\dim K < n = \dim(U)$ for any neighborhood $U \subset i(I^n)$ of $i(0^n)$. This contradiction shows that the kernel of T is infinite-dimensional. \square

We remind that a topological space Y is called a k -space if a subset $F \subset Y$ is closed in Y if and only if for every compact subset $K \subset Y$ the intersection $F \cap K$ is closed in K .

Theorem 4. *A \mathcal{K}_{fd} -invertible linear operator $T : X \rightarrow Y$ from a linear topological space $X \in \mathcal{K}_{fd}^\infty$ onto a locally convex k -space Y is homeomorphic either to the projection $pr : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ for some $n, m \in \omega \cup \{\infty\}$ or to the strongly \mathcal{K}_{fd} -universal Zarichnyi map $\mu : \mathbb{R}^\infty \rightarrow Q^\infty$. The latter case occurs if and only if the space Y has uncountable Hamel basis.*

Proof. First, we show that $Y \in \mathcal{K}^\infty$. Since $X \in \mathcal{K}_{fd}^\infty$, the space X contains a countable collection $\{X_n : n \in \omega\}$ of compact subsets, fundamental in the sense that every compact subset of X lies in some X_n . By [En₁, 3.1.22], for every n the image $f(X_n)$ of the compact set X_n is a metrizable compactum. The operator T , being \mathcal{K}_{fd} -invertible, is surjective. Then $Y = \bigcup_{n \in \omega} f(X_n)$. We claim that the collection $\{f(X_n) : n \in \omega\}$ generates the topology of Y , i.e., $Y \in \mathcal{K}^\infty$.

Assuming the converse, we would find a non-closed subset $F \subset Y$ such that $F \cap f(X_n)$ is closed in $f(X_n)$ for every n . Since Y is a k -space, there is a compactum $K \subset Y$ such that $F \cap K$ is not closed in K . The compactum $K = \bigcup_{n \in \omega} K \cap f(X_n)$, being the countable union of metrizable compacta, is metrizable. Consequently, there is a sequence $(y_n)_{n=1}^\infty \subset F \cap K$ converging to a point $y_0 \in K \setminus F$. Since the map T is \mathcal{K}_{fd} -invertible, there is a sequence $(x_n)_{n=1}^\infty \subset X$ converging to a point $x_0 \in X$ such that $T(x_n) = y_n$ for all $n \in \omega$. The subset $\{x_n : n \in \omega\} \subset X$, being compact, lies in the compactum X_m for some m . Then $\{y_n : n > 0\} \subset f(X_m) \cap F$. Since the intersection $f(X_m) \cap F$ is closed in $f(X_m)$, we get $y_0 = \lim_{n \rightarrow \infty} y_n \in f(X_m) \cap F$, a contradiction with $y_0 \notin F$.

Thus the locally convex space Y belongs to the class \mathcal{K}^∞ . By [Ba₃], Y is homeomorphic either to Q^∞ or to \mathbb{R}^n for some $n \in \omega \cup \{\infty\}$. Moreover, the last case occurs if and only if the algebraic dimension of Y is at most countable. If Y is homeomorphic to Q^∞ , then by Theorem 2, the operator T is homeomorphic to the universal Zarichnyi map $\mu : \mathbb{R}^\infty \rightarrow Q^\infty$.

If Y is homeomorphic to \mathbb{R}^n for some $n \in \omega \cup \{\infty\}$, then the algebraic dimension of Y is at most countable and Y carries the strongest linear topology, see [Ba₂]. In this case there is a linear continuous operator $S : Y \rightarrow X$ such that $T \circ S = \text{id}$ and the map $h : X \rightarrow Y \times \text{Ker } T$ defined by $h(x) = (T(x), x - S \circ T(x))$ for $x \in X$ is a linear homeomorphism (with inverse $h^{-1}(y, l) = S(y) + l$) such that $\text{pr} \circ h = T$, where $\text{pr} : Y \times \text{Ker } T \rightarrow Y$ is the projection. Hence the operator T is homeomorphic to the projection $\text{pr} : Y \times \text{Ker } T \rightarrow Y$. Since $\text{Ker } T$ is a linear topological space from the class \mathcal{K}_{fd}^∞ we can apply Corollary 1 (or [Ba₂]) to conclude that $\text{Ker } T$ is homeomorphic to \mathbb{R}^m for some $m \in \omega \cup \{\infty\}$. Therefore T is homeomorphic to the projection $\text{pr} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$. \square

Next, we show that linear operators classifying by Theorem 3 do exist.

Theorem 5. *For every linear topological space $Y \in \mathcal{K}^\infty$ there is a \mathcal{K}_{fd} -invertible linear operator $T : X \rightarrow Y$ from a linear topological space $X \in \mathcal{K}_{fd}^\infty$.*

Proof. Since $Y \in \mathcal{K}^\infty$, the space Y possesses a countable family $\{F_n : n \in \omega\}$ of compact subsets such that every compact subset of Y lies in some F_n . For every $n \in \omega$ denote by \mathcal{K}_n the subclass of \mathcal{K}_{fd} consisting of all at most n -dimensional compacta. Using the Dranishnikov Theorem [Dr], for every n we can find an \mathcal{K}_n -invertible map $f_n : K_n \rightarrow F_n$ of a finite-dimensional metrizable compactum K_n onto F_n . Now consider the discrete sum $K = \sqcup_{n \in \omega} K_n$ and the map $f = \sqcup_{n \in \omega} f_n : K \rightarrow Y$. It is clear that $K \in \mathcal{K}_{fd}^\infty$ and the map f is \mathcal{K}_{fd} -invertible. Let $L(K) \supset K$ be the free linear topological space over K and $T : L(K) \rightarrow Y$ unique linear operator extending the \mathcal{K}_{fd} -invertible map f . It is clear that the operator T is \mathcal{K}_{fd} -invertible. By [Ba₂], $L(K) \in \mathcal{K}_{fd}^\infty$. \square

Note that Theorems 4 and 5 allow us to give an alternative construction of the universal map of Zarichnyi.

Question. *Let $h : G \rightarrow H$ be a \mathcal{K}_{fd} -invertible continuous homomorphism between topological groups. Is h strongly \mathcal{K}_{fd} -universal if its kernel $h^{-1}(1)$ is strongly \mathcal{K}_{fd} -universal?*

THE UNIVERSAL ZARICHNYI MAP IS NOT SELF-SIMILAR

We define a space X to be *locally self-similar* if every point of X has a basis of neighborhoods homeomorphic to X . It is well-known that the spaces \mathbb{R}^∞ and Q^∞ (like many other model spaces of infinite-dimensional topology) are locally self-similar.

In [Za5] M. Zarichnyi extended the notion of the local self-similarity onto maps and asked if the map $\mu : \mathbb{R}^\infty \rightarrow Q^\infty$ is locally self-similar. He defined a map $\pi : X \rightarrow Y$ to be *locally self-similar* if for every point $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that the map $\pi|_V : V \rightarrow \pi(V)$ is homeomorphic to π . Observe that for locally self-similar spaces X, Y the projection $\text{pr} : X \times Y \rightarrow Y$ is locally self-similar.

The following result shows that the strongly \mathcal{K}_{fd} -universal Zarichnyi map is close to being locally self-similar.

Theorem 6. *If $T : X \rightarrow Y$ is a linear operator homeomorphic to the strongly \mathcal{K}_{fd} -universal Zarichnyi map $\mu : \mathbb{R}^\infty \rightarrow Q^\infty$, then for every nonempty open convex subset $U \subset X$ the map $T|_U : U \rightarrow T(U)$ is homeomorphic to μ .*

Proof. Fix any non-empty open convex subset $U \subset X$. Since X is homeomorphic to \mathbb{R}^∞ , U is homeomorphic to \mathbb{R}^∞ too. Next, $T(U)$ is homeomorphic to Q^∞ , being an open contractible subspace of Y , the topological copy of Q^∞ . By Uniqueness Theorem, to show that the map $T|_U : U \rightarrow T(U)$ is homeomorphic to μ it suffices to verify that $T|_U$ is a strongly \mathcal{K}_{fd} -universal map.

First we show that the map $T|_U : U \rightarrow T(U)$ is \mathcal{K}_{fd} -invertible. Fix any map $g : A \rightarrow T(U)$ from a compactum $A \in \mathcal{K}_{fd}$. Since the map T is \mathcal{K}_{fd} -invertible, there is a map $f : A \rightarrow X$ such that $T \circ f = g$. For every point $x \in K$ find a point $a_x \in \text{Ker } T$ such that $f(x) + a_x \in U$. Next, let $U(x) = f^{-1}(U - a_x)$. Using the compactness of A find a finite subcover $\{U(x_1), \dots, U(x_n)\}$ of the open cover $\{U(x) : x \in A\}$ and let $\{\lambda_i : A \rightarrow [0, 1]\}_{i=1}^n$ be a partition of unity such that $\lambda_i^{-1}(0, 1] \subset U(x_i)$ for every $i \leq n$. Consider the map $\alpha : A \rightarrow \text{Ker } T$ defined by $\alpha(x) = \sum_{i=1}^n \lambda_i(x) a_{x_i}$ for $x \in K$. It can be shown that the map $h = f + \alpha : A \rightarrow X$ has the properties: $T \circ h = T \circ f = g$ and $h(A) \subset U$. Therefore, the map $T|_U$ is \mathcal{K}_{fd} -invertible.

To show that it is strongly \mathcal{K}_{fd} -universal, fix an embedding $f : B \rightarrow U$ of a closed subset B of a space $A \in \mathcal{K}_{fd}$ and a map $g : A \rightarrow T(U)$ such that $T \circ f = g$. Since the map $T|_U : U \rightarrow T(U)$ is \mathcal{K}_{fd} -invertible, there is a map $h_1 : K \rightarrow U$ such that $T \circ h_1 = g$. Next, using the strong \mathcal{K}_{fd} -universality of the operator T , find a map $h_2 : A \rightarrow X$ such that $h_2|_B = f$ and $T \circ h_2 = g$. Let $\lambda : A \rightarrow [0, 1]$ be a continuous map such that $\lambda(B) = \{0\}$ and $\lambda(A \setminus W) = \{1\}$, where $W = h^{-1}(U)$. Next, consider the map $h : A \rightarrow X$ defined by $h(a) = \lambda(a)h_1(a) + (1 - \lambda(a))h_2(a)$ for $a \in A$. It is easy to see that $T \circ h = g$ and $h(A) \subset U$.

Let $q : A \rightarrow A/B$ be the quotient map. Since the quotient space A/B is finite-dimensional, there is an embedding $i : A/B \rightarrow [0, 1]^n$ for some $n \in \omega$ such that $i(\{B\}) = 0^n$. The space $h(A)$ belongs to the class \mathcal{K}_{fd} , being a compact subset of the space $X \in \mathcal{K}_{fd}^\infty$. Then by the strong \mathcal{K}_{fd} -universality of the operator T , there is

an embedding $e : h(A) \times [0, 1]^n \rightarrow X$ such that $e(x, 0^n) = x$ and $T \circ e(x, t) = T(x)$ for every $x \in h(A)$ and $t \in [0, 1]^n$. Since $h(A) \subset U$, there exists $\varepsilon > 0$ such that $e(h(A) \times [0, \varepsilon]^n) \subset U$. Finally consider the map $\bar{f} : A \rightarrow X$ defined by $\bar{f}(a) = e(h(a), \varepsilon \cdot i \circ q(a))$ for $a \in A$. It can be easily shown that $\bar{f}|_B = f$, $\bar{f}(A) \subset U$ and $T \circ \bar{f} = g$. Therefore, the map $T|_U$ is strongly \mathcal{K}_{fd} -universal. The space U , being an open subset of the space $X \in \mathcal{K}_{fd}^\infty$, belongs to the class \mathcal{K}_{fd}^∞ . If the space $T(U)$ is homeomorphic to Y , then by Uniqueness Theorem the maps $T|_U$ and T are homeomorphic. \square

Thus the local self-similarity of the Zarichnyi map μ would be proven if we would find a linear operator between locally convex spaces, homeomorphic to μ . Unfortunately, no such an operator exists. This is so because each locally convex space $X \in \mathcal{K}_{fd}^\infty$ has at most countable Hamel basis, see [Ba₂]. Consequently, each linear image of X also has at most countable Hamel basis and thus can not be homeomorphic to Q^∞ . But this is not a unique reason why we can not find a linear operator between locally convex spaces, homeomorphic to the universal Zarichnyi map μ .

Theorem 6. *Each open \mathcal{K}_{fd} -invertible map $f : \mathbb{R}^\infty \rightarrow Q^\infty$ is not self-similar.*

Let us define a map $\pi : X \rightarrow Y$ to be *locally \mathcal{K}_{fd} -invertible* if for every point $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that the map $\pi|_V : V \rightarrow \pi(V)$ is \mathcal{K}_{fd} -invertible. It is clear that each locally self-similar \mathcal{K}_{fd} -invertible map is locally \mathcal{K}_{fd} -invertible.

We define a space X to be *almost finite-dimensional* if there is $n \in \mathbb{N}$ such that $\dim F \leq n$ for every finite-dimensional closed subset F of X . Let us note that there exist infinite-dimensional almost finite-dimensional compact spaces, see [En₂, 5.2.23].

Theorem 6 will be derived from

Lemma 3. *Each compactum K admitting a surjective locally \mathcal{K}_{fd} -invertible map $f : X \rightarrow K$ from a space $X \in \mathcal{K}_{fd}^\infty$, is almost finite-dimensional.*

To prove this lemma we need

Lemma 4. *For every compact space K and a closed subset $A \subset K \times Q$ with $\dim A < \dim K$ there is a closed subset $F \subset K \times Q$ such that $F \cap A = \emptyset$ but $F \cap s(K) \neq \emptyset$ for every section $s : K \rightarrow K \times Q$ of the projection $\text{pr} : K \times Q \rightarrow K$.*

Proof. Find any $n \in \omega$ with $\dim A \leq n < \dim K$. By Hurewicz-Wallman Theorem [En₂, 1.9.3], there exists a map $f : L \rightarrow S^n$ from a closed subset L of K into the n -dimensional sphere which has no continuous extension $\bar{f} : K \rightarrow S^n$.

Since S^n is an ANR, we can extend f to a continuous map $\tilde{f} : \bar{O}(L) \rightarrow S^n$ defined on the closure of an open neighborhood $O(L)$ of L in K . Let $B = L \times Q$ and $U = O(L) \times Q$. Since $\dim A \leq n$, we can apply Hurewicz-Wallman Theorem again to find a continuous map $p : A \rightarrow S^n$ such that $p|_{A \cap \bar{U}} = \tilde{f} \circ \text{pr}|_{A \cap \bar{U}}$. Next, since S^n is an ANR, there is a continuous map $\bar{p} : V \rightarrow S^n$ defined on an open neighborhood V of the closed set $A \cap \bar{U}$ in $K \times Q$ such that $\bar{p}|_A = p$ and $\bar{p}|_{\bar{U}} = \tilde{f} \circ \text{pr}|_{\bar{U}}$. Consider the open set $O(A) = O(B) \cup (V \setminus B)$, where $O(B) = \{x \in U : d(\bar{p}(x), \tilde{f} \circ \text{pr}(x)) < 1\}$ and d stands for the standard Euclidean metric of $\mathbb{R}^{n+1} \supset S^n$.

We claim that the closed set $F = (K \times Q) \setminus O(A)$ misses A but meets the image $s(K)$ of every any section $s : K \rightarrow K \times Q$ of the projection pr . Assuming the

converse, we would find a section $s : K \rightarrow K \times Q$ of the projection pr such that $s(Q) \cap F = \emptyset$. Then $s(K) \subset O(A)$ and we can consider the map $g = \bar{p} \circ s : K \rightarrow S^n$. Observe that $s(L) \subset B \cap O(A) = O(B)$ and hence $d(g(x), f(x)) = (d(\bar{p} \circ s(x), \bar{f} \circ \text{pr} \circ s(x))) < 1$ for all $x \in L$ which yields that the maps $g|L$ and f are homotopic. Since $g|L$ has the extension $g : K \rightarrow S^n$, we may apply the Borsuk Extension Theorem [En₂, 1.9.7] to conclude that the map f has a continuous extension $\bar{f} : K \rightarrow S^n$, which contradicts to the choice of f . \square

Proof of Lemma 3. Suppose a compactum K admits an open surjective locally \mathcal{K}_{fd} -invertible map $\pi : X \rightarrow K$ of a space $X \in \mathcal{K}_{fd}^\infty$. Assume that the compactum K is not almost finite-dimensional. The compactum K , being a countable union of metrizable compacta, is metrizable. By the compactness of K there is a point $y \in K$ having no almost infinite dimensional neighborhood in K . Fix a countable base $\{U_n : n \in \omega\}$ of neighborhoods of y in K and for every $n \in \omega$ find a finite dimensional compactum $K_n \subset U_n$ with $\dim K_n > n$.

Fix any point $x \in X$ with $\pi(x) = y$ and let $\{X_n : n \in \omega\}$ be an increasing collection of finite-dimensional compact subsets of X generating its topology. Without loss of generality, $x \in X_0$ and $\dim X_n \leq n$ for every $n \in \omega$. The space X , having a countable network of the topology, admits an injective continuous map $i : X \rightarrow Q$ into the Hilbert cube. Now consider the injective map $e = (\pi, i) : X \rightarrow K \times Q$ defined by $e(x) = (f(x), i(x))$ for $x \in X$. For every $n \in \omega$ let $A_n = i(X_n) \cap (K_n \times Q)$. Since $\dim A_n \leq \dim X_n \leq n < \dim K_n$, we can apply Lemma 4 to find a closed subset $F_n \subset K_n \times Q$ such that $A_n \cap F_n = \emptyset$ but $F_n \cap s(K_n) \neq \emptyset$ for every section $s : K_n \rightarrow K_n \times Q$ of the projection $\text{pr} : K_n \times Q \rightarrow K_n$.

Consider the set $F = \bigcup_{k \in \omega} e^{-1}(F_k)$. Since each set F_k is closed in $K \times Q$ and $X_n \cap F = X_n \cap (\bigcup_{k=0}^{n-1} e^{-1}(F_k))$ for each $n \in \omega$, we get that F is a closed subset of X . Next, since $X_0 \cap F = \emptyset$, the set $U = X \setminus F$ is an open neighborhood of the point x in X .

Let us show that for every open neighborhood $V \subset U$ of x the map $\pi|V : V \rightarrow \pi(V)$ is not \mathcal{K}_{fd} -invertible. Indeed, since the map π is open, $\pi(V)$ is an open neighborhood of the point $\pi(x) = y$ and thus $f(V) \supset K_n$ for some $n \in \omega$. Assuming that the map $\pi|V$ is \mathcal{K}_{fd} -invertible we would find a map $g : K_n \rightarrow V \subset U$ such that $\pi \circ g = \text{id}$. Then the map $s = e \circ g : K_n \rightarrow K \times Q$ has the properties: $\text{pr} \circ s = \text{id}$ and $s(K_n) \cap F_n = \emptyset$, a contradiction with the choice of the set F_n . \square

Proof of Theorem 6. Assume that $f : \mathbb{R}^\infty \rightarrow Q^\infty$ is an open \mathcal{K}_{fd} -invertible locally self-similar map. Then f is locally \mathcal{K}_{fd} -invertible. Let $K \subset Q^\infty$ be a topological copy of the Hilbert cube and $X = f^{-1}(K)$. Then $X \in \mathcal{K}_{fd}^\infty$ and $f|K : X \rightarrow K$ is an open surjective locally \mathcal{K}_{fd} -invertible map, a contradiction with Lemma 3. \square

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