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Abstract

Let Γ denote a bipartite Q-polynomial distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. We show that Γ has a certain equitable partition of its vertex set which involves $4d-4$ cells. We use this partition to show that the intersection numbers of Γ satisfy the following divisibility conditions:

$$c_{i+1} - 1 \text{ divides } c_i(c_i - 1) \text{ for } 2 \leq i \leq d-1,$$

$$b_{i-1} - 1 \text{ divides } b_i(b_i - 1) \text{ for } 1 \leq i \leq d-1.$$

Using these divisibility conditions we show Γ does not exist if $d = 4$.

1 Introduction

Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection numbers a_i, b_i, c_i (see Sections 2 and 4 for formal definitions). In [7], H. Lewis showed that the girth of Γ is at most 6. The present paper is part of an effort to classify the examples with girth 6. In order to motivate our results we give some comments on this case.

Assume for the moment that Γ has girth 6. Then $a_1 = a_2 = 0$ and $c_2 = 1$. By [8, Theorem 6.3], Γ is bipartite or almost bipartite. The classification of almost bipartite Q-polynomial distance-regular graphs with girth 6 is given in [6], so it remains to consider the case in which Γ is bipartite.

For the rest of this introduction we assume Γ is bipartite with girth 6. We present a result which we hope will lead to a classification. Let $V\Gamma$ denote the vertex set of Γ and fix vertices $x, y \in V\Gamma$ such that $\partial(x, y) = 2$. For all integers i, j $(0 \leq i, j \leq d)$ we define $D^i_j = D^i_j(x, y)$ by

$$D^i_j = \{w \in V\Gamma \mid \partial(x, w) = i \text{ and } \partial(y, w) = j\}.$$

Observe that $D^i_j = \emptyset$ if $|i - j| > 2$. Since Γ is bipartite, we also have $D^i_j = \emptyset$ whenever $i + j$ is odd. Let z be the unique common neighbour of x and y. For $1 \leq i \leq d$ and for $w \in D^i_i$ we find from the triangle inequality that $\partial(w, z) \in \{i - 1, i + 1\}$; we define

$$D^i_i(1) = \{w \in D^i_i \mid \partial(w, z) = i - 1\} \text{ and } D^i_i(0) = \{w \in D^i_i \mid \partial(w, z) = i + 1\}.$$

We show that the sets D^{i-2}_i, D^i_{i-2} $(2 \leq i \leq d)$, $D^i_i(0)$ $(2 \leq i \leq d - 1)$ and $D^i_i(1)$ $(1 \leq i \leq d)$ are nonempty. Using Terwilliger's "balanced set" characterization of the Q-polynomial property we show that the partition of $V\Gamma$ into the above sets is equitable. Moreover, the corresponding parameters of this partition are independent of the choice of vertices x, y.

Using the fact that the corresponding parameters are nonnegative integers, we show that the intersection numbers of Γ satisfy the following divisibility conditions:

$$c_{i+1} - 1 \text{ divides } c_i(c_i - 1) \text{ for } 2 \leq i \leq d - 1,$$

$$b_{i-1} - 1 \text{ divides } b_i(b_i - 1) \text{ for } 1 \leq i \leq d - 1.$$

We use these conditions to show that there does not exist a bipartite Q-polynomial distance-regular graph with valency $k \geq 3$, $c_2 = 1$ and diameter 4.

Our paper is organized as follows. In Sections 2 and 4 we set up some necessary tools for the proof of our main results. In Section 3 we describe a partition of the vertex set of Γ and in Section 5 we show that this partition is equitable. We look at the case $d = 4$ in Section 6.

2 Preliminaries

In this section, we review some definitions and basic concepts. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Throughout this paper, Γ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $V\Gamma$, edge set $E\Gamma$, path length distance function ∂, and diameter $d := \max\{\partial(x,y) | x, y \in V\Gamma\}$. For a vertex $x \in V\Gamma$ define $\Gamma_i(x)$ to be the set of vertices at distance i from x. We abbreviate $\Gamma(x) := \Gamma_1(x)$. The graph Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le d)$, and all $x, y \in V\Gamma$ with $\partial(x,y) = h$, the number

$$p^h_{ij} := |\{z \,|\, z \in V\Gamma,\ \partial(x,z) = i,\ \partial(y,z) = j\}| \tag{1}$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1i-1}$ for $1 \le i \le d$, $a_i := p^i_{1i}$ for $0 \le i \le d$, $b_i := p^i_{1i+1}$ for $0 \le i \le d-1$, $k_i := p^0_{ii}$ for $0 \le i \le d$, and $c_0 = b_d = 0$. We observe $a_0 = 0,\ c_1 = 1$. Moreover, for $0 \le i \le d$ we have

$$c_i + a_i + b_i = k, \tag{2}$$

where $k := k_1$. Observe that for h, i, j $(0 \le h, i, j \le d)$, $p^h_{ij} = 0$ (resp. $p^h_{ij} \ne 0$) if one of h, i, j is greater than (resp. equal to) the sum of the other two. Observe also that Γ is bipartite if and only if $a_i = 0$ for $0 \le i \le d$. In this case $b_i + c_i = k$ for $0 \le i \le d$ and $p^h_{ij} = 0$ unless $i + j + h$ is even.

By [2, page 127] we have

$$k_i = (b_0 b_1 \cdots b_{i-1})/(c_1 c_2 \cdots c_i) \quad (0 \le i \le d). \tag{3}$$

The following formulae will be useful.

Lemma 2.1 ([2, Lemma 4.1.7]) *Let Γ denote a distance-regular graph with diameter $d \ge 3$. Then the following* (i)–(iii) *hold.*

(i) $p^1_{i-1,i} = (b_1 b_2 \cdots b_{i-1})/(c_1 c_2 \cdots c_{i-1}) \quad (1 \le i \le d)$;

(ii) $p^2_{i-2,i} = (b_2 b_3 \cdots b_{i-1})/(c_1 c_2 \cdots c_{i-2}) \quad (2 \leq i \leq d)$;

(iii) $p^2_{ii} = (b_2 b_3 \cdots b_{i-1})(c_i b_{i-1} + a_i^2 + c_{i+1} b_i - k - a_1 a_i)/(c_1 c_2 \cdots c_i) \quad (2 \leq i \leq d-1)$.

∎

From now on we assume Γ is distance-regular with diameter $d \geq 3$. For $0 \leq i \leq d$, the *distance matrix* A_i has rows and columns indexed by $V\Gamma$, and x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ & \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in V\Gamma). \tag{4}$$

The matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative semi-simple $\mathbb{R}$-algebra M, known as the *Bose-Mesner algebra*, see for example Godsil [5, Lemma 11.2.2]. The algebra M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |V\Gamma|^{-1} J, \tag{5}$$

$$E_i E_j = \delta_{ij} E_i \text{ for } 0 \leq i, j \leq d, \tag{6}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{7}$$

$$E_i^t = E_i \text{ for } 0 \leq i \leq d, \tag{8}$$

see Godsil [5, Theorem 12.2.1]. The $E_0, E_1, \ldots, E_d$ are known as the *primitive idempotents* of Γ, and E_0 is the *trivial* idempotent.

Set $A := A_1$, and for $0 \leq i \leq d$ define a real number θ_i by

$$A = \sum_{i=0}^{d} \theta_i E_i. \tag{9}$$

Then $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq d$, and $\theta_0 = k$. The scalars $\theta_0, \theta_1, \ldots, \theta_d$ are distinct, since A generates M [1, p. 197]. The $\theta_0, \theta_1, \ldots, \theta_d$ are known as the *eigenvalues* of Γ.

For notational convenience, we identify $V\Gamma$ with the standard orthonormal basis in the Euclidean space $(V, \langle\, , \rangle)$, where $V = \mathbb{R}^{|V\Gamma|}$ (column vectors), and where $\langle\, , \rangle$ is the dot product

$$\langle u, v \rangle = u^t v \quad (u, v \in V).$$

Let θ denote an eigenvalue of Γ, and let E denote the associated primitive idempotent. For $0 \leq i \leq d$ define a real number θ_i^* by

$$E = |V\Gamma|^{-1} \sum_{i=0}^{d} \theta_i^* A_i.$$

We call the sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ the *dual eigenvalue sequence* associated with θ, E. We say the sequence is *trivial* whenever $E = E_0$ (in which case $\theta_0^* = \theta_1^* = \cdots = \theta_d^* = 1$). In the following lemma, we cite a well known result about primitive idempotents.

Lemma 2.2 (Terwilliger [11, Lemma 1.1]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$, let E denote a primitive idempotent of Γ, and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Then for $0 \leq i \leq d$ and for all $x, y \in V\Gamma$ with $\partial(x, y) = i$ we have $\langle Ex, Ey \rangle = |V\Gamma|^{-1}\theta_i^*$.* ▮

An *equitable partition* of a graph is a partition $\pi = \{C_1, C_2, \ldots, C_s\}$ of its vertex set into nonempty cells such that for all integers i, j $(1 \leq i, j \leq s)$ the number c_{ij} of neighbours, which a vertex in the cell C_i has in the cell C_j, is independent of the choice of the vertex in C_i. We call the c_{ij} the *corresponding parameters.*

3 The partition - part I

Let Γ denote a bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. In this section we describe a certain partition of the vertex set $V\Gamma$.

Definition 3.1 *Let Γ denote a bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. Fix vertices $x, y \in V\Gamma$ such that $\partial(x, y) = 2$. For all integers i, j we define $D_j^i = D_j^i(x, y)$ by*

$$D_j^i = \{w \in V\Gamma \mid \partial(x, w) = i \;\; and \;\; \partial(y, w) = j\}.$$

We observe $D_j^i = \emptyset$ unless $0 \leq i, j \leq d$. Moreover $|D_j^i| = p_{ij}^2$ for $0 \leq i, j \leq d$.

We say Γ is 2-*homogeneous* in the sense of Nomura [9] whenever (i) for all $x, y \in V\Gamma$ such that $\partial(x, y) = 2$, the partition of $V\Gamma$ given by Definition 3.1 is equitable; and (ii) the corresponding parameters do not depend on x, y.

Lemma 3.2 *With reference to Definition 3.1, the following* (i), (ii) *hold for $0 \leq i, j \leq d$.*

(i) *If $|i - j| > 2$ then $D_j^i = \emptyset$.*

(ii) *If $i+j$ is odd then $D^i_j = \emptyset$.*

PROOF. (i) If $|i-j| > 2$ then $p^2_{ij} = 0$ so $D^i_j = \emptyset$.
(ii) If $i+j$ is odd then $p^2_{ij} = 0$ so $D^i_j = \emptyset$. ∎

Lemma 3.3 *With reference to Definition 3.1, the following* (i), (ii) *hold for $2 \le i \le d$.*

(i) $|D^i_{i-2}| = |D^{i-2}_i| = (b_2 b_3 \cdots b_{i-1})/(c_1 c_2 \cdots c_{i-2})$;

(ii) $D^i_{i-2} \neq \emptyset$, $D^{i-2}_i \neq \emptyset$.

PROOF. (i), (ii) Immediate from Lemma 2.1(ii) and since $|D^i_{i-2}| = |D^{i-2}_i| = p^2_{i-2,i}$. ∎

Since Γ is bipartite, we easily obtain the following lemma.

Lemma 3.4 *With reference to Definition 3.1, there are no edges inside the set D^i_j for $0 \le i, j \le d$.* ∎

Lemma 3.5 *With reference to Definition 3.1, let $z \in D^1_1$. Then for $1 \le i \le d$ and for $w \in D^i_i$ we have $\partial(w,z) \in \{i-1, i+1\}$.*

PROOF. From the triangle inequality we find $i-1 \le \partial(w,z) \le i+1$. Now if $\partial(w,z) = i$, then we have a cycle of an odd length in Γ, a contradiction. ∎

Definition 3.6 *Let Γ denote a bipartite distance-regular graph with diameter $d \ge 3$, valency $k \ge 3$ and intersection number $c_2 = 1$. Fix vertices $x, y \in V\Gamma$ such that $\partial(x,y) = 2$ and let z denote the unique common neighbour of x and y. For $0 \le i, j \le d$ let the sets D^i_j be as defined in Definition 3.1. For $1 \le i \le d$ we define $D^i_i(1) = D^i_i(1)(x,y)$ and $D^i_i(0) = D^i_i(0)(x,y)$ by*

$$D^i_i(1) = \{w \in D^i_i \mid \partial(w,z) = i-1\} \quad \text{and} \quad D^i_i(0) = \{w \in D^i_i \mid \partial(w,z) = i+1\}.$$

We observe D^i_i is the disjoint union of $D^i_i(0)$ and $D^i_i(1)$.

We have a comment.

Lemma 3.7 *With reference to Definition 3.6, the following* (i), (ii) *hold.*

(i) $D^1_1(1) = \{z\}$ *and* $D^1_1(0) = \emptyset$.

(ii) $D_d^d(1) = D_d^d$ *and* $D_d^d(0) = \emptyset$.

Proof. Immediate from Definition 3.6. ∎

Lemma 3.8 *With reference to Definition 3.6, the following* (i), (ii) *hold.*

(i) $|D_i^i(1)| = p^1_{i-1,i} - p^2_{i-2,i}$ $(2 \leq i \leq d)$;

(ii) $|D_i^i(0)| = p^2_{ii} - p^1_{i-1,i} + p^2_{i-2,i}$ $(2 \leq i \leq d-1)$.

Proof. (i) A vertex w is in $D_i^i(1)$ if and only if w is at distance i from x and $i-1$ from z, but not at distance $i-2$ from y. Since there is precisely $p^1_{i-1,i}$ vertices which are at distance i from x and $i-1$ from z, and precisely $p^2_{i-2,i}$ vertices which are at distance i from x and $i-2$ from y, the result follows.
(ii) Observe that D_i^i is a disjoint union of $D_i^i(1)$ and $D_i^i(0)$ and that $|D_i^i| = p^2_{ii}$. The result follows. ∎

Corollary 3.9 *With reference to Definition 3.6, the following* (i), (ii) *hold.*

(i)
$$|D_i^i(1)| = \frac{(b_{i-1}-1)b_2\cdots b_{i-1}}{c_1\cdots c_{i-1}} \quad (2 \leq i \leq d);$$

(ii)
$$|D_i^i(0)| = \frac{(c_{i+1}-1)b_2\cdots b_i}{c_1\cdots c_i} \quad (2 \leq i \leq d-1).$$

Proof. (i), (ii) Combine Lemma 3.8 and Lemma 2.1. ∎

Lemma 3.10 *With reference to Definition 3.6, the following* (i)–(iii) *hold.*

(i) *For* $2 \leq i \leq d$, *there is no edge between* $D_i^i(0)$ *and* $D_i^i(1)$.

(ii) *For* $2 \leq i \leq d-1$, *there is no edge between* $D_i^i(0)$ *and* $D_{i-1}^{i-1}(1)$.

(iii) *There is no edge between* $D_2^2(1)$ *and* $D_1^3 \cup D_3^1$.

Proof. (i) Immediate from Lemma 3.4.
(ii) By the definition of the set $D_i^i(0)$.
(iii) Let $w \in D_2^2(1)$. Then z is the common neighbour of x and w and of y and w. Since $c_2 = 1$, w has no neighbours in $D_1^3 \cup D_3^1$. ∎

With reference to Definition 3.6, Lemma 3.2, Lemma 3.7 and Lemma 3.10, we visualize $D_i^{i-2}, D_{i-2}^i, D_i^i(0)$ and $D_i^i(1)$ in Figure 1.

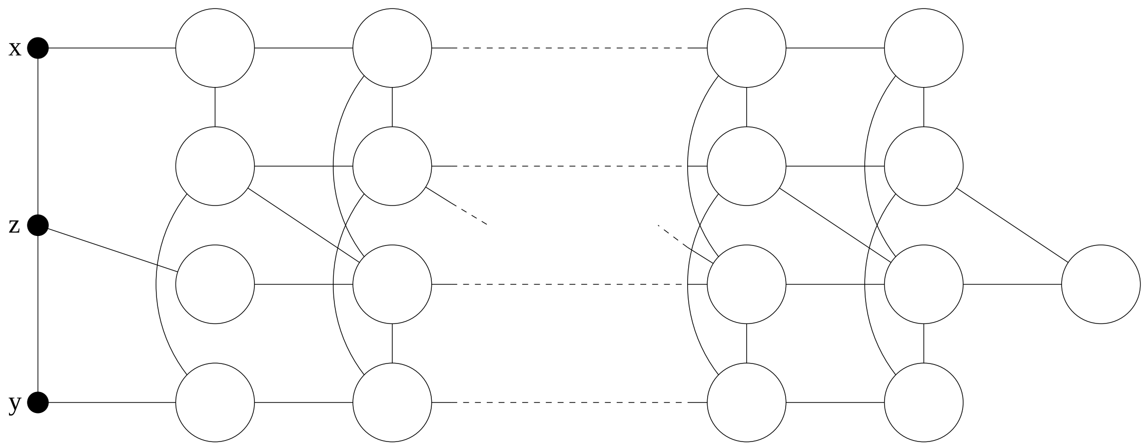

Figure 1: The partition. The circles in the first line represent the sets D_i^{i-2} $(2 \leq i \leq d)$, the circles in the second line represent the sets $D_i^i(0)$ $(2 \leq i \leq d-1)$, the circles in the third line represent the sets $D_i^i(1)$ $(2 \leq i \leq d)$ and the circles in the fourth line represent the sets D_{i-2}^i $(2 \leq i \leq d)$. Observe that $\Gamma_i(x) = D_{i-2}^i \cup D_i^i(1) \cup D_i^i(0) \cup D_{i+2}^i$ and $\Gamma_i(y) = D_i^{i-2} \cup D_i^i(0) \cup D_i^i(1) \cup D_i^{i+2}$.

Lemma 3.11 *With reference to Definition 3.6, the following* (i)–(iii) *hold.*

(i) *For each integer i $(2 \leq i \leq d)$, each $w \in D_{i-2}^i$ (resp. D_i^{i-2}) is adjacent to*
- (a) *precisely* c_{i-2} *vertices in* D_{i-3}^{i-1} *(resp.* D_{i-1}^{i-3}*),*
- (b) *precisely* b_i *vertices in* D_{i-1}^{i+1} *(resp.* D_{i+1}^{i-1}*),*
- (c) *precisely* $c_{i-1} - c_{i-2}$ *vertices in* $D_{i-1}^{i-1}(1)$*,*
- (d) *precisely* $c_i - c_{i-1}$ *vertices in* $D_{i-1}^{i-1}(0)$*,*

and no other vertices in $V\Gamma$*.*

(ii) *For each integer i $(2 \leq i \leq d-1)$, each $w \in D_i^i(0)$ is adjacent to*
- (a) *precisely* b_{i+1} *vertices in* $D_{i+1}^{i+1}(0)$*,*
- (b) *precisely* $c_i - |\Gamma(w) \cap D_{i-1}^{i-1}(0)|$ *vertices in* D_{i-1}^{i+1}*,*
- (c) *precisely* $c_i - |\Gamma(w) \cap D_{i-1}^{i-1}(0)|$ *vertices in* D_{i+1}^{i-1}*,*
- (d) *precisely* $b_i + |\Gamma(w) \cap D_{i-1}^{i-1}(0)| - c_i - b_{i+1}$ *vertices in* $D_{i+1}^{i+1}(1)$*,*
- (e) *precisely* $|\Gamma(w) \cap D_{i-1}^{i-1}(0)|$ *vertices in* $D_{i-1}^{i-1}(0)$*,*

and no other vertices in $V\Gamma$*.*

(iii) *For each integer i $(1 \leq i \leq d)$, each $w \in D_i^i(1)$ is adjacent to*

(a) *precisely*	c_{i-1}	*vertices in* $D^{i-1}_{i-1}(1)$,
(b) *precisely*	$b_i - \lvert\Gamma(w)\cap D^{i+1}_{i+1}(1)\rvert$	*vertices in* D^{i+1}_{i-1},
(c) *precisely*	$b_i - \lvert\Gamma(w)\cap D^{i+1}_{i+1}(1)\rvert$	*vertices in* D^{i-1}_{i+1},
(d) *precisely*	$c_i + \lvert\Gamma(w)\cap D^{i+1}_{i+1}(1)\rvert$ $-b_i - c_{i-1}$	*vertices in* $D^{i-1}_{i-1}(0)$,
(e) *precisely*	$\lvert\Gamma(w)\cap D^{i+1}_{i+1}(1)\rvert$	*vertices in* $D^{i+1}_{i+1}(1)$,

and no other vertices in $V\Gamma$.

PROOF. (i) The proof of (a) and (b) is a routine. We now prove the case (c). Observe that w is at distance $i-1$ from z, so there is exactly $c_{i-1}c_{i-2}\cdots c_1$ paths of length $i-1$ from w to z. Since w has exactly c_{i-2} neighbours in D^{i-1}_{i-3} (resp. D^{i-3}_{i-1}), $c_{i-2}c_{i-2}c_{i-3}\cdots c_1$ of these paths pass through D^{i-1}_{i-3} (resp. D^{i-3}_{i-1}). The remaining $(c_{i-1}-c_{i-2})c_{i-2}\cdots c_1$ paths must pass through $D^{i-1}_{i-1}(1)$. Let $v \in D^{i-1}_{i-1}(1)$. Since $\partial(v,z)=i-2$, there is exactly $c_{i-2}\cdots c_1$ paths of length $i-2$ between v and z. Hence w has exactly $c_{i-1}-c_{i-2}$ neighbours in $D^{i-1}_{i-1}(1)$. The case (d) is now trivial, since Γ is regular with valency $k=b_i+c_i$.

(ii) Since, by Definition 3.6, $\partial(w,z)=i+1$, w must have exactly b_{i+1} neighbours in $\Gamma_{i+2}(z)$. But $\Gamma_{i+2}(z)\cap\Gamma(w)\subseteq D^{i+1}_{i+1}(0)$, so (a) follows. The proof of (b), (c), (d) and (e) is now a routine.

(iii) Since $\partial(w,z)=i-1$, w must have exactly c_{i-1} neighbours in $\Gamma_{i-2}(z)$. But $\Gamma_{i-2}(z)\cap\Gamma(w)\subseteq D^{i-1}_{i-1}(1)$, so (a) follows. The proof of (b), (c), (d) and (e) is now a routine. ∎

4 Q-polynomial property

Let Γ denote a distance-regular graph with diameter $d\geq 3$. The *Krein parameters* q^h_{ij} $(0\leq h,i,j\leq d)$ of Γ are defined by

$$E_i \circ E_j = |V\Gamma|^{-1}\sum_{h=0}^{d} q^h_{ij}E_h \quad (0\leq i,j\leq d), \tag{10}$$

where $\circ$ denotes entrywise multiplication. We say Γ is *Q-polynomial* (with respect to the given ordering $E_0,E_1,\ldots,E_d$ of the primitive idempotents), whenever for all distinct integers i,j $(0\leq i,j\leq d)$,

$$q^1_{ij}\neq 0 \ \text{ if and only if } \ |i-j|=1.$$

Let E denote a nontrivial primitive idempotent of Γ. We say Γ is *Q-polynomial with respect to E* whenever there exists an ordering $E_0, E_1 = E, \ldots, E_d$ of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial. We have the following useful lemmas about the Q-polynomial property.

Lemma 4.1 (Brouwer et al. [2, Thm. 8.1.1]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Suppose Γ is Q-polynomial with respect to E. Then $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are mutually distinct.*

Lemma 4.2 (Terwilliger [11, Thm. 3.3]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Then the following* (i), (ii) *are equivalent.*

(i) *Γ is Q-polynomial with respect to E.*

(ii) *$\theta_0^* \neq \theta_i^*$ for $1 \leq i \leq d$; for all integers h, i, j $(1 \leq h \leq d)$, $(0 \leq i, j \leq d)$ and for all vertices $x, y \in V\Gamma$ with $\partial(x, y) = h$ the following hold:*

$$\sum_{\substack{z \in V\Gamma \\ \partial(x,z)=i \\ \partial(y,z)=j}} Ez - \sum_{\substack{z \in V\Gamma \\ \partial(x,z)=j \\ \partial(y,z)=i}} Ez \in span\{Ex - Ey\}.$$

Suppose (i), (ii) *hold. Then for all integers h, i, j $(1 \leq h \leq d)$, $(0 \leq i, j \leq d)$ and for all $x, y \in V\Gamma$ such that $\partial(x, y) = h$,*

$$\sum_{\substack{z \in V\Gamma \\ \partial(x,z)=i \\ \partial(y,z)=j}} Ez - \sum_{\substack{z \in V\Gamma \\ \partial(x,z)=j \\ \partial(y,z)=i}} Ez = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*}(Ex - Ey). \tag{11}$$

∎

5 The partition - part II

In this section we assume Γ is Q-polynomial. We show the partition from Section 3 is equitable, and that the corresponding parameters are independent of x, y.

Lemma 5.1 *Let Γ denote a bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ denote the corresponding dual eigenvalue sequence. Assume Γ is Q-polynomial with respect to E. Then with reference to Definition 3.6, the following* (i), (ii) *hold.*

(i) *For $2 \leq i \leq d-1$ and for $w \in D^i_i(0)$,*

$$|\Gamma(w) \cap D^{i-1}_{i-1}(0)| = c_i \frac{(\theta^*_0 - \theta^*_i)(\theta^*_3 - \theta^*_{i+1}) - (\theta^*_1 - \theta^*_{i-1})(\theta^*_2 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i-1} - \theta^*_{i+1})}.$$

(ii) *For $1 \leq i \leq d-1$ and for $w \in D^i_i(1)$,*

$$|\Gamma(w) \cap D^{i+1}_{i+1}(1)| = b_i \frac{(\theta^*_0 - \theta^*_i)(\theta^*_3 - \theta^*_{i-1}) - (\theta^*_1 - \theta^*_{i+1})(\theta^*_2 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i+1} - \theta^*_{i-1})}.$$

PROOF. (i) We abbreviate $\tau = |\Gamma(w) \cap D^{i-1}_{i-1}(0)|$ and $\eta = |\Gamma(w) \cap D^{i-1}_{i+1}|$. We observe $\tau + \eta = c_i$. By Lemma 4.2 we find

$$\sum_{\substack{v \in V\Gamma \\ \partial(x,v)=i-1 \\ \partial(w,v)=1}} Ev - \sum_{\substack{v \in V\Gamma \\ \partial(x,v)=1 \\ \partial(w,v)=i-1}} Ev = c_i \frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(Ex - Ew). \qquad (12)$$

Observe that $\{v \in V\Gamma \mid \partial(x,v) = 1,\ \partial(w,v) = i-1\} \subseteq D^1_3$. Taking the inner product of (12) with Ey using Lemma 2.2, we get (after multiplying by $|V\Gamma|$)

$$\tau\theta^*_{i-1} + \eta\theta^*_{i+1} - c_i\theta^*_3 = c_i \frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(\theta^*_2 - \theta^*_i).$$

Evaluating the above line using $\eta = c_i - \tau$ and $\theta^*_{i-1} \neq \theta^*_{i+1}$ we obtain

$$\tau = c_i \frac{(\theta^*_0 - \theta^*_i)(\theta^*_3 - \theta^*_{i+1}) - (\theta^*_1 - \theta^*_{i-1})(\theta^*_2 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i-1} - \theta^*_{i+1})}.$$

The assertion now follows.

(ii) We abbreviate $\tau_1 = |\Gamma(w) \cap D^{i+1}_{i+1}(1)|$ and $\eta_1 = |\Gamma(w) \cap D^{i+1}_{i-1}|$. We observe $\tau_1 + \eta_1 = b_i$. By Lemma 4.2 we find

$$\sum_{\substack{v \in V\Gamma \\ \partial(x,v)=i+1 \\ \partial(w,v)=1}} Ev - \sum_{\substack{v \in V\Gamma \\ \partial(x,v)=1 \\ \partial(w,v)=i+1}} Ev = b_i \frac{\theta^*_{i+1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(Ex - Ew). \qquad (13)$$

We continue as in the case (i) above to obtain the desired result. ∎

Our next goal is to show $D^i_i(0) \neq \emptyset$ for $2 \leq i \leq d-1$ and $D^i_i(1) \neq \emptyset$ for $1 \leq i \leq d$. We will need the following result.

Lemma 5.2 *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. Then*

(i) $2 \leq c_3 \leq \cdots \leq c_d$;

(ii) $b_0 \geq b_1 \geq \cdots \geq b_{d-1} \geq 2$.

PROOF. (i) By [7, Theorem 28], we have $c_3 \geq 2$, and by [2, Proposition 4.1.6.(i)] we have $c_i \geq c_{i-1}$ for $4 \leq i \leq d$. The result follows.
(ii) By [2, Proposition 4.1.6.(i)] we have $b_{i-1} \geq b_i$ for $1 \leq i \leq d-1$. We show $b_{d-1} \geq 2$. Assume for the moment $b_{d-1} = 1$. Observe that in this case $p^d_{2,d} = 0$. We define δ by

$$2\delta = \max\{i + j - d \mid 0 \leq i, j \leq d,\ p^d_{ij} \neq 0\}.$$

Since Γ is bipartite, we find by [3, Lemma 8.3] that $\delta = 0$. Fix a vertex $u \in V\Gamma$ and define graph Γ^2_d by

$$V\Gamma^2_d = \Gamma_d(u), \quad E\Gamma^2_d = \{(v, w) \in V\Gamma^2_d \times V\Gamma^2_d \mid \partial_\Gamma(v, w) = 2\}.$$

By [3, Theorem 9.2], Γ^2_d is a connected graph with diameter 0. Thus $|\Gamma_d(u)| = k_d = 1$ and, by [2, Proposition 5.1.1], Γ is antipodal 2-cover. By [4, Theorem 42], Γ is 2-homogeneous. But then, by [10, Theorem 1.2], $c_2 > 1$, a contradiction. We conclude $b_{d-1} \geq 2$ and the result follows. ∎

Lemma 5.3 *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. With reference to Definition 3.6, the following* (i), (ii) *hold.*

(i) *For* $2 \leq i \leq d-1$, $D^i_i(0) \neq \emptyset$.

(ii) *For* $1 \leq i \leq d$, $D^i_i(1) \neq \emptyset$.

PROOF. (i) Immediate from Corollary 3.9(ii) and since $c_{i+1} > 1$ by Lemma 5.2(i).
(ii) Immediate from Corollary 3.9(i) and since $b_{i-1} > 1$ by Lemma 5.2(ii). ∎

Theorem 5.4 *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. Then with reference to Definition 3.6, the partition of $V\Gamma$ into the sets D_i^{i-2}, D_{i-2}^i $(2 \leq i \leq d)$, $D_i^i(0)$ $(2 \leq i \leq d-1)$ and $D_i^i(1)$ $(1 \leq i \leq d)$ is equitable. Moreover the corresponding parameters are independent of x, y.*

PROOF. Immediate from Lemma 3.3(ii), Lemma 3.11, Lemma 5.1 and Lemma 5.3. ∎

We are now ready to find divisibility conditions, which must be satisfied by the intersection numbers of Γ. We first prove a second version of Lemma 5.1.

Lemma 5.5 *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. Then with reference to Definition 3.6 the following* (i), (ii) *hold.*

(i) *For $2 \leq i \leq d-1$ and for $w \in D_i^i(0)$,*

$$|\Gamma(w) \cap D_{i-1}^{i-1}(0)| = \frac{c_i(c_i - 1)}{c_{i+1} - 1}.$$

(ii) *For $1 \leq i \leq d-1$ and for $w \in D_i^i(1)$,*

$$|\Gamma(w) \cap D_{i+1}^{i+1}(1)| = \frac{b_i(b_i - 1)}{b_{i-1} - 1}.$$

PROOF. (i) Observe that the assertion holds for $i = 2$, so assume $i \geq 3$. By Lemma 3.11(ii), every vertex in $D_{i-1}^{i-1}(0)$ has precisely b_i neighbours in $D_i^i(0)$. By Lemma 5.1, every vertex in $D_i^i(0)$ has a constant number τ of neighbours in $D_{i-1}^{i-1}(0)$. So we have

$$|D_{i-1}^{i-1}(0)| b_i = |D_i^i(0)| \tau.$$

Calculating τ using Corollary 3.9(ii) we obtain $\tau = c_i(c_i - 1)/(c_{i+1} - 1)$.
(ii) Observe that the assertion holds for $i = 1$, so assume $i \geq 2$. By Lemma 3.11(iii), every vertex in $D_{i+1}^{i+1}(1)$ has precisely c_i neighbours in $D_i^i(1)$. By Lemma 5.1, every vertex in $D_i^i(1)$ has a constant number τ_1 of neighbours in $D_{i+1}^{i+1}(1)$. So we have

$$|D_i^i(1)| \tau_1 = |D_{i+1}^{i+1}(1)| c_i.$$

Calculating τ_1 using Corollary 3.9(i) we obtain $\tau_1 = b_i(b_i - 1)/(b_{i-1} - 1)$. ∎

Theorem 5.6 *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. Then the following* (i), (ii) *hold.*

(i) $c_{i+1} - 1$ *divides* $c_i(c_i - 1)$ *for* $2 \leq i \leq d-1$;

(ii) $b_{i-1} - 1$ *divides* $b_i(b_i - 1)$ *for* $1 \leq i \leq d-1$.

PROOF. Immediate from Lemma 5.5. ∎

Corollary 5.7 *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $c_2 = 1$. Then $k-2$ divides $(c_3 - 1)(c_3 - 2)$. In particular, either $c_3 = 2$ or $(c_3 - 1)(c_3 - 2) \geq k - 2$.*

PROOF. Apply Theorem 5.6(ii) with $i = 3$, $b_2 = k - 1$ and $b_3 = k - c_3$. ∎

6 Case $d = 4$

In this section we show there does not exist a Q-polynomial bipartite distance-regular graph with diameter $d = 4$, valency $k \geq 3$ and intersection number $c_2 = 1$.

Theorem 6.1 *There does not exist a Q-polynomial bipartite distance-regular graph with diameter $d = 4$, valency $k \geq 3$ and intersection number $c_2 = 1$.*

PROOF. Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $d = 4$, valency $k \geq 3$ and intersection number $c_2 = 1$. Abbreviate $b = k-1$ and $c = c_3-1$. Since $c_4 = k$ we obtain from Theorem 5.6(i) $c(c+1) = \alpha b$ for some nonnegative integer α. Further, since $b_2 = k - 1$, we obtain from Theorem 5.6(ii) that $k - 2$ divides $b_3(b_3 - 1) = (k - 2 - c_3 + 2)(k - 2 - c_3 + 1)$, hence $c(c - 1) = \beta(b - 1)$ for some nonnegative integer β. If $c = 1$, then $\alpha b = 2$, implying $k = 3$. But then $b_3 = 1$, contradicting Lemma 5.2(ii). So we have

$$\frac{c+1}{c-1} = \frac{\alpha}{\beta}\frac{b}{b-1}.$$

Since the sequence $\frac{n+1}{n-1}$ $(2 \leq n)$ is decreasing and since $2 \leq c \leq b-1$, we obtain $\frac{c+1}{c-1} \geq \frac{b}{b-2} > \frac{b}{b-1}$. So $\frac{\alpha}{\beta} > 1$, and we have $\beta = \alpha - a$ for some positive integer a.

If we subtract the equations $c^2 + c = \alpha b$ and $c^2 - c = (\alpha - a)(b-1)$, we obtain $2c = \alpha + ab - a$.

Suppose first that $a \geq 2$. Then we have $c = \frac{\alpha}{2} + \frac{a}{2}(b-1) \geq 1 + b - 1 = b$, a contradiction. Thus $a = 1$ and $c = (\alpha + b - 1)/2$. So we have $c^2 + c = c(c+1) = (\alpha + b - 1)(\alpha + b + 1)/4 = \alpha b$, or $(\alpha - b - 1)(\alpha - b + 1) = 0$. If $\alpha = b + 1$ then $c = b$, a contradiction. Hence $\alpha = b - 1$, implying $c_3 = k - 1$. But then $b_3 = 1$, contradicting Lemma 5.2(ii). We conclude Γ does not exist. ∎

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