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Abstract. We investigate some topological properties of a normal functor H introduced earlier by Radul which is some functorial compactification of the Hartman-Mycielski construction HM. We prove that the pair (HX, HMY) is homeomorphic to the pair (Q, σ) for each non-degenerated metrizable compactum X and each dense σ-compact subset Y.

1. Introduction

The general theory of functors acting on the category $\mathcal{C}omp$ of compact Hausdorff spaces (compacta) and continuous mappings was founded by Shchepin [Sh]. He described some elementary properties of such functors and defined the notion of the normal functor which has become very fruitful. The classes of all normal and weakly normal functors include many classical constructions: the hyperspace *exp*, the space of probability measures P, the superextension λ, the space of hyperspaces of inclusion G, and many other functors (cf. [FZ] and [TZ]).

Let X be a space and d an admissible metric on X bounded by 1. By $HM(X)$ we shall denote the space of all maps from $[0,1)$ to the space X such that $f|[t_i, t_{i+1}) \equiv$ *const*, for some $0 = t_0 \le \dots \le t_n = 1$, with respect to the following metric

$$d_{HM}(f,g) = \int_0^1 d(f(t), g(t))dt, \qquad f, g \in HM(X).$$

The construction of $HM(X)$ is known as the *Hartman-Mycielski construction* [HM]. Recall, that the Hilbert cube is denoted by Q, and the following subspace of Q

$$\big\{(a_n)_{n=1}^{\infty} \in Q \mid a_k = 0 \text{ for all but finitely many } k\big\}$$

is denoted by σ. Telejko has shown in [Te] that for any non-degenerated separable metrizable σ-compact strongly countable-dimensional space X the space HMX is homeomorphic to σ.

For every $Z \in \mathcal{C}omp$ consider

$$HM_n(Z) = \Big\{f \in HM(Z) \mid \text{there exist } 0 = t_1 < \dots < t_{n+1} = 1 \\ \text{with } f|[t_i, t_{i+1}) \equiv z_i \in Z, i = 1, \dots, n\Big\}.$$

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Let $\mathcal{U}$ be the unique uniformity of Z. For every $U \in \mathcal{U}$ and $\varepsilon > 0$, let

$$< \alpha, U, \varepsilon >= \{\beta \in HM_n(Z) \mid m\{t \in [0,1) \mid (\alpha(t), \beta(t')) \notin U\} < \varepsilon\}.$$

The sets $< \alpha, U, \varepsilon >$ form a base of a compact Hausdorff topology in HM_nZ. Given a map $f : X \to Y$ in $\mathcal{C}omp$, define a map $HM_nX \to HM_nY$ by the formula $HM_nF(\alpha) = f \circ \alpha$. Then HM_n is a normal functor in $\mathcal{C}omp$ (cf. [TZ; 2.5.2]).

For $X \in \mathcal{C}omp$ we consider the space HMX with the topology described above. In general, HMX is not compact. Zarichnyi has asked if there exists a normal functor in $\mathcal{C}omp$ which contains all functors HM_n as subfunctors (see [TZ]). Such a functor H was constructed in [Ra].

We investigate some topological properties of the space HX which is some natural compactification of HMX. The main results of this paper are:

Theorem 1.1. *HX is homeomorphic to the Hilbert cube for each non-degenerated metrizable compactum X.*

Theorem 1.2. *The pair (HX, HMY) is homeomorphic to the pair (Q, σ) for each non-degenerated metrizable compactum X and each dense σ-compact subset Y.*

2. Construction of H

Let $X \in \mathcal{C}omp$. By CX we denote the Banach space of all continuous functions $\varphi : X \to \mathbb{R}$ with the usual sup-norm: $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}$. We denote the segment $[0,1]$ by I.

For $X \in \mathcal{C}omp$ let us define the uniformity of HMX. For each $\varphi \in C(X)$ and $a, b \in [0,1]$ with $a < b$ we define the function $\varphi_{(a,b)} : HMX \to \mathbb{R}$ by the following formula

$$\varphi_{(a,b)} = \frac{1}{(b-a)} \int_a^b \varphi \circ \alpha(t) dt.$$

Define

$$S_{HM}(X) = \{\varphi_{(a,b)} \mid \varphi \in C(X) \text{ and } (a,b) \subset [0,1)\}.$$

For $\varphi_1, \ldots, \varphi_n \in S_{HM}(X)$ define a pseudometric $\rho_{\varphi_1,\ldots,\varphi_n}$ on HMX by the formula

$$\rho_{\varphi_1,\ldots,\varphi_n}(f,g) = \max\{|\varphi_i(f) - \varphi_i(g)| \mid i \in \{1, \ldots, n\}\},$$

where $f, g \in HMX$. The family of pseudometrics

$$\mathcal{P} = \{\rho_{\varphi_1,\ldots,\varphi_n} \mid n \in \mathbb{N}, \text{ where } \varphi_1, \ldots, \varphi_n \in S_{HM}(X)\},$$

defines a totally bounded uniformity $\mathcal{U}_{HMX}$ of HMX (see [Ra]).

For each compactum X we consider the uniform space $(HX, \mathcal{U}_{HX})$ which is the completion of $(HMX, \mathcal{U}_{HMX})$ and the topological space HX with the topology induced by the uniformity $\mathcal{U}_{HX}$. Since $\mathcal{U}_{HMX}$ is totally bounded, the space HX is compact.

Let $f : X \to Y$ be a continuous map. Define the map $HMf : HMX \to HMY$ by the formula $HMf(\alpha) = f \circ \alpha$, for all $\alpha \in HMX$. It was shown in [Ra] that the map $HMf : (HMX, \mathcal{U}_{HMX}) \to (HMY, \mathcal{U}_{HMY})$ is uniformly continuous. Hence there exists the continuous map $Hf : HX \to HY$ such that $Hf|HMX = HMf$. It is easy to see that $H : \mathcal{C}omp \to \mathcal{C}omp$ is a covariant functor and HM_n is a subfunctor of H for each $n \in \mathbb{N}$.

3. Preliminaries

All spaces are assumed to be metrizable. We begin this section with the investigation of certain structure of equiconnectivity on HX for some compactum X. Define the map $e_1 : HMX \times HMX \times I \to HMX$ by the condition that $e_1(\alpha_1, \alpha_2, t)(l)$ is equal to $\alpha_1(l)$ if $l < t$ and $\alpha_2(l)$ in the opposite case for $\alpha_1, \alpha_2 \in HMX$, $t \in I$ and $l \in [0, 1)$. We consider HMX with the uniformity $\mathcal{U}_{HMX}$ and I with the natural metric.

Lemma 3.1. *The map $e_1 : HMX \times HMX \times I \to HMX$ is uniformly continuous.*

Proof. Let us consider any $U \in \mathcal{U}_{HMX}$. We can suppose that

$$U = \{(\alpha, \beta) \in HMX \times HMX \mid |\varphi_{(0,1)}(\alpha) - \varphi_{(0,1)}(\beta)| < \delta\},$$

for some $\delta > 0$ and $\varphi \in C(X)$. The proof of the general case is the same.

Put $c = \max_{x\in X} |\varphi(x)|$. Choose $n \in \mathbb{N}$ such that $1/n < \frac{\delta}{2c}$ and put $a_i = i/n$ for $i \in \{0, \dots, n\}$. Consider an element V of uniformity $\mathcal{U}_{HMX}$ defined as follows

$$V = \{(\alpha, \beta) \in HMX \times HMX \mid |\varphi_{(a_i,a_{i+1})}(\alpha) - \varphi_{(a_i,a_{i+1})}(\beta)| < \frac{\delta}{2n^2},$$

for each $i \in \{0, \dots, n-1\}\}$. Put $E = \{(l, s) \in I \times I \mid |l - s| < \frac{1}{2n}\}$.

Let us consider any $((\alpha_1, \beta_1), (\alpha_2, \beta_2), (l, s)) \in V \times V \times E$. Then we have

$$|\int_{a_i}^{a_{i+1}} (\varphi \circ \alpha_1(t) - \varphi \circ \alpha_2(t))dt| =$$

$$= n\frac{1}{a_{i+1} - a_i}|\int_{a_i}^{a_{i+1}} (\varphi \circ \alpha_1(t) - \varphi \circ \alpha_2(t))dt| =$$

$$= n|\varphi_{(a_i,a_{i+1})}(\alpha_1) - \varphi_{(a_i,a_{i+1})}(\alpha_2)| < n\frac{\delta}{2n^2} = \frac{\delta}{2n},$$

for each $i \in \{0, \dots, n-1\}$. We have the same for β_1 and β_2. Since $|t_1 - t_2| < \frac{1}{2n}$, there exists $i_0 \in \{0, \dots, n-1\}$ such that $t_1, t_2 \in [a_{i_0}, a_{i_0+1}]$. Then we have

$$|\varphi_{(0,1)}(e_1(\alpha_1, \beta_1, t_1)) - \varphi_{(0,1)}(e_1(\alpha_2, \beta_2, t_2))| =$$

$$= |\int_0^1 (\varphi \circ e_1(\alpha_1, \beta_1, t_1)(t) - \varphi \circ e_1(\alpha_2, \beta_2, t_2)(t))dt| \le$$

$$\le \sum_{i=0}^{i_0-1} |\int_{a_i}^{a_{i+1}} (\varphi \circ \alpha_1(t) - \varphi \circ \alpha_2(t))dt| + c(a_{i_0+1} - a_{i_0})+$$

$$+\sum_{i=i_0}^{n-1} |\int_{a_i}^{a_{i+1}} (\varphi \circ \beta_1(t) - \varphi \circ \beta_2(t))dt| < \frac{n-1}{2n}\delta + \frac{\delta}{2} < \delta.$$

Therefore $(e_1(\alpha_1, \beta_1, t_1), e_1(\alpha_2, \beta_2, t_2)) \in V$. Thus the lemma is proved.

Hence there exists the extension of e_1 to the continuous map $e : HX \times HX \times I \to HX$. It is easy to check that $e(\alpha, \alpha, t) = \alpha$, for each $\alpha \in HX$. The next lemma can be proved by a direct verification and its proof is left to the reader.

Lemma 3.2. *For each $\varphi \in C(X)$ and $\alpha, \beta \in HX$ such that $\varphi_{(b,c)}(\alpha) > a$, $\varphi_{(b,c)}(\beta) > a$ (resp. $\varphi_{(b,c)}(\alpha) < a$, $\varphi_{(b,c)}(\beta) < a$), for some $a, b, c \in \mathbb{R}$, we have $e(\alpha, \beta, t) > a$ (resp. $e(\alpha, \beta, t) > a$) for each $t \in I$.*

Let (Z, e) be an equiconnected space where $e : Z \times Z \times I \to Z$ is the map which defines the structure of equiconnectedness. A subset $A \subset Z$ is called *e-convex* if $e(a, b, t) \in A$, for each a, $b \in A$ and $t \in I$. We say that the space Z is *locally equiconnected* if there exists a base of e-connected subsets.

Corollary 3.3. *The space (HX, e) is locally equiconnected.*

The next corollary follows from the previous results and a result of Cauty [Ca].

Corollary 3.4. *HX is absolute retract for each metrizable compactum X.*

4. Proofs

We will need the following version of Torunczyk Characterization Theorem (see for example [BRZ]):

Theorem 4.1. *Let X be a metrizable AR-compactum. Then X is homeomorphic to the Hilbert cube Q if and only if there exist homotopies H_1, H_2:$X \times I \to X$ such that $H_1(X \times (0; 1]) \cap H_2(X \times (0; 1]) = \emptyset$.*

Proof of Theorem 1.1. Choose any x_1, $x_2 \in X$ such that $x_1 \neq x_2$. Define α_0, $\beta_0 \in HX$ by $\alpha_0(t) = x_1$ and $\beta_0(t) = x_2$, for each $t \in [0; 1)$. Define H_1, H_2 by the formulae $H_1(\alpha, t) = e(\alpha_0, \alpha, t)$ and $H_2(\beta, t) = e(\beta_0, \beta, t)$. Let us show that for each t_1, $t_2 > 0$ and each α, $\beta \in HX$ we have $H_1(\alpha, t_1) \neq H_2(\beta, t_2)$.

Choose any $\varphi \in C(X)$ such that $\varphi(x_1) = 0$ and $\varphi(x_2) = 1$. Put $\varepsilon = min\{t_1, t_2\}$. It follows by Lemma 3.1 that there exist α_1, $\beta_1 \in HMX$ such that

$$|\tilde{\varphi}_{(0;\varepsilon)}(e(\alpha_0, \alpha, \varepsilon)) - \tilde{\varphi}_{(0;\varepsilon)}(e(\alpha_0, \alpha_1, \varepsilon))| < \varepsilon/3$$

and

$$|\tilde{\varphi}_{(0;\varepsilon)}(e(\beta_0, \beta, \varepsilon)) - \tilde{\varphi}_{(0;\varepsilon)}(e(\beta_0, \beta_1, \varepsilon))| < \varepsilon/3.$$

Then we have

$$|\tilde{\varphi}_{(0;\varepsilon)}(e(\alpha_0, \alpha, t_1)) - \tilde{\varphi}_{(0;\varepsilon)}(e(\beta_0, \beta, t_2))| \geq$$

$$\geq |\tilde{\varphi}_{(0;\varepsilon)}(e(\alpha_0, \alpha_1, t_1)) - \tilde{\varphi}_{(0;\varepsilon)}(e(\beta_0, \beta_1, t_2))| - \frac{2\varepsilon}{3} =$$

$$|\frac{1}{\varepsilon}\int_0^\varepsilon \varphi \circ \alpha_0(t)dt - \frac{1}{\varepsilon}\int_0^\varepsilon \varphi \circ \beta_0(t)dt| - \frac{2\varepsilon}{3} =$$

$$= |\frac{1}{\varepsilon} \cdot \varepsilon \cdot 1 - \frac{1}{\varepsilon} \cdot \varepsilon \cdot 0| - \frac{2\varepsilon}{3} \neq 0.$$

Hence, $H_1(\alpha, t_1) \neq H_2(\beta, t_2)$. The theorem is thus proved.

Proof of Theorem 1.2. Since HX is equiconnected and locally equiconnected, and HMY is a dense e-convex subset, homeomorphic to σ (cf. [Te]), Theorem 1.2 follows by [TRZ;3.1.7] and Lemma 4.1 below.

Lemma 4.1. *Let (X, e) be an equiconnected and locally equiconnected space and A is dense e-convex subset. Then A is homotopically dense in X.*

Proof. Both X and A are absolute retracts by [Ca]. Hence it suffices to prove that for each cover $\mathcal{U}$ of X there exists a map $f : X \to A$ which is $\mathcal{U}$-close to id_X.

We may assume that all covers and bases are countable. Let $\mathcal{B}$ be a base of X consisting of e-convex subsets. We may also assume that $\mathcal{U} \subset \mathcal{B}$ and $\mathcal{U}$ are locally finite. Choose any locally finite cover $\mathcal{V}$ such that $\mathcal{U} \subset \mathcal{B}$ and $\mathcal{V}$ star-refines $\mathcal{U}$. Let $\{p_V : X \to [0;1] \mid V \in \mathcal{V}\}$ be a partition of unity for the cover $\mathcal{V}$.

Let us fix some order $\mathcal{V} = \{V_1, V_2, \dots\}$. Since A is dense in X, we can choose $a_i \in V_i \cap A$ for each $i \in \mathbb{N}$. Define the function $f : X \to X$ as follows. Consider any $x \in X$. Since $\mathcal{V}$ is locally finite, there exists $n \in \mathbb{N}$ such that $p_{V_i}(x) = 0$ for each $i > n$. Define the finite sequence $\{x_1, \dots, x_n\}$ by induction. Put $x_1 = a_1$. Suppose that we have already defined x_j for each $j \leq i - 1$ where $1 < i \leq n$. Define

$$x_i = e(x_{i-1}, a_i, \frac{\sum_{l=1}^{i-1} p_{V_i}(x)}{\sum_{l=1}^{i} p_{V_i}(x)}).$$

Put $f(x) = x_n$. One can check that the function f is continuous and $\mathcal{U}$-close to id_X. Since A is e-convex, it follows that $f(X) \subset A$. The lemma is thus proved.

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