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ARGUMENT PRINCIPLE AND HOLOMORPHIC EXTENDIBILITY

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ABSTRACT Let D be a bounded domain in the complex plane whose boundary consists of finitely many pairwise disjoint simple closed curves. Give bD the standard orientation and let $A(D)$ be the algebra of all continuous functions on $\overline{D}$ which are holomorphic on D. In the paper we prove that a continuous function f on bD extends to a function in $A(D)$ if and only if for each $g \in A(D)$ such that $f+g \neq 0$ on bD the change of argument of $f+g$ along bD is nonnegative.

1. Introduction and the main result

Let $D \subset \mathbb{C}$ be a bounded domain whose boundary consists of finitely many pairwise disjoint simple closed curves. We give bD the standard orientation. Denote by $A(D)$ the algebra of all continuous functions on $\overline{D}$ which are holomorphic on D. Our main result is

THEOREM 1.1 *A continuous function f on bD extends to a function in $A(D)$ if and only if for each $g \in A(D)$ such that $f+g \neq 0$ on bD the change of argument of $f+g$ along bD is nonnegative.*

If the condition in the theorem holds for all g belonging to a dense subset of $A(D)$ then it holds for all $g \in A(D)$. The only if part of the theorem is an obvious consequence of the argument principle. In fact, if f admits an extension $\tilde{f} \in A(D)$ then the change of argument of $f+g$ along bD equals 2π times the number of zeros of $\tilde{f}+g$ in D.

In the special case when $D = \Delta = \{\zeta \in \mathbb{C}\colon\ |\zeta| < 1\}$ the theorem was proved first by J. Wermer [W] for smooth functions and later by the author [Gl] for continuous functions.

2. Preliminaries

Every bounded domain $D \subset \mathbb{C}$ whose boundary consists of finitely many pairwise disjoint simple closed curves is biholomorphically equivalent to a domain D' bounded by finitely many pairwise disjoint circles [Go]. Moreover, every biholomorphic map $\Phi\colon D \to D'$ extends to a homeomorphism $\tilde{\Phi}\colon \overline{D} \to \overline{D'}$ (see the proof in [CL, pp. 46-49] which works also for multiply connected domains, bounded by finitely many pairwise disjoint simple closed curves). Thus, with no loss of generality assume that D is bounded by finitely many pairwise disjoint circles.

In general, not every real-valued harmonic function u on D is the real part of a holomorphic function on D. If there is a harmonic function v on D such that $u+iv$ is holomorphic on D then we call v a *conjugate of* u. If D is simply connected then every harmonic function on D has a conjugate on D. Let f be a complex valued harmonic function on D. Write $f = p + iq$ with $p,\ q$ real. We will say that f has a conjugate on D if p has a conjugate r on D and q has a conjugate s on D, and we will call the function $r - is$ a conjugate of f on D. This happens if and only if $f = F + \overline{G}$ where F and G are holomorphic functions on D. In fact, if $P = p + ir$ and $Q = q + is$ then P and Q are holomorphic functions on D and $F = (P + iQ)/2$ and $G = (P - iQ)/2$.

Given a continuous function Φ on bD there is a continuous extension of Φ to $\overline{D}$ which is harmonic on D and which we will denote by $\mathcal{H}(\Phi)$; moreover, if $\Phi \in C^\infty(bD)$ then $\mathcal{H}(\Phi) \in C^\infty(\overline{D})$ [B, p.53]. $\mathcal{H}$ is a linear map from $C(bD)$ to the space of continuous functions on $\overline{D}$ which are harmonic on D. If D is simply connected then $\mathcal{H}(\Phi)$ has a conjugate harmonic function on D which is also in $C^\infty(\overline{D})$ provided that $\Phi \in C^\infty(bD)$ [B, p.91]. If Ω_1 and Ω_2 are bounded, simply connected domains with boundaries of class C^∞ then a biholomorphic map Φ from Ω_1 to Ω_2 extends to a smooth map from $\overline{\Omega_1}$ to $\overline{\Omega_2}$ [B, p.28]. Applying this locally along bD and using the preceding discussion we see that if f is a harmonic function on D that has a conjugate on D then the conjugate extends smoothly to $\overline{D}$ provided that f extends smoothly to $\overline{D}$. We summarize this in

LEMMA 2.1 *If $f \in C^\infty(bD)$ is such that $\mathcal{H}(f)$ has a conjugate on D then both $\mathcal{H}(f)$ and its conjugate extend smoothly to $\overline{D}$.*

Every harmonic function f on D is real-analytic on D so, if f is holomorphic on an open subset of D then it is holomorphic on D. We will need the following fact which can be found as an exercise in [R].

LEMMA 2.2 *Let f be a harmonic function on D such that $z \mapsto zf(z)$ is harmonic on a nonempty open set $U \subset D$. Then f is holomorphic on D.*

Proof. By the preceding discussion it is enough to prove that f is holomorphic on a disc $\Omega \subset D$. Since f is harmonic on D there is a disc $\Omega \subset U$ such that $f = P + \overline{Q}$ on Ω where P and Q are holomorphic functions on Ω. By our assumption, the function $z \mapsto zf(z) = zP(z) + z\overline{Q(z)}$ is harmonic on Ω so

$$\frac{\partial^2}{\partial z \partial \overline{z}}[zP(z) + z\overline{Q(z)}] = 0 \quad (z \in \Omega)$$

which implies that $\overline{Q'(z)} = 0$ $(z \in \Omega)$. Thus, Q is constant on Ω and consequently f is holomorphic on Ω. This completes the proof.

3. A new proof in the case of a disc

The proof in [Gl] does not generalize to multiply connected domains. In this section we give a new, different proof of the theorem in the case when D is a disc which we later generalize to multiply connected domains.

Write $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Throughout this section, $D = \Delta$, Denote by Z the identity function: $Z(z) = z$ $(z \in bD)$. Assume that $f \in C(b\Delta)$ does not extend to a function from the disc algebra $A(\Delta)$. Then there is an $a \in \Delta$ such that

$$\mathcal{H}[(Z - a)f](a) \neq 0. \tag{3.1}$$

To see this, suppose for a moment that $\mathcal{H}[(Z - a)f](a) = 0$ $(a \in \Delta)$. This implies that $\mathcal{H}(Zf)(a) = a\mathcal{H}(f)(a)$ $(a \in \Delta)$. In particular, the function $a \mapsto a\mathcal{H}(f)(a)$ is harmonic on Δ. Since $\mathcal{H}(f)$ is harmonic on D Lemma 2.2 implies that $\mathcal{H}(f)$ is holomorphic on Δ so f extends holomorphically through Δ, a contradiction. This proves that there is an $a \in \Delta$ such that (3.1) holds.

With no loss of generality, replacing f with $e^{i\omega}f$, $\omega \in \mathbb{R}$, if necessary, we may assume that

$$\Re\{\mathcal{H}[(Z-a)f](a)\} = \beta \neq 0. \tag{3.2}$$

For easier understanding we complete the proof first under the additional assumption that f is smooth. Suppose for a moment that f is smooth. The function $z \mapsto \Re\{\mathcal{H}[(Z-a)f](z)\} - \beta$ is continuous on $\overline{\Delta}$, harmonic on Δ and has smooth boundary values $\Re[(z-a)f(z)] - \beta$ $(z \in b\Delta)$. Hence by Lemma 2.1 there is a function $g \in A(\Delta)$ such that

$$\Re[g(z)] = \Re\{\mathcal{H}[(Z-a)f](z)\} - \beta \quad (z \in \overline{\Delta}).$$

By (3.2) we have $\Re g(a) = \Re\{\mathcal{H}[(Z-a)f](a)\} - \beta = 0$ so by adding an imaginary constant to g if necessary we may assume that $g(0) = 0$ so $g(z) = (z-a)h(z)$ $(z \in \overline{\Delta})$ where $h \in A(\Delta)$. Consider the function $z \mapsto G(z) = (z-a)f(z) - g(z)$ $(z \in b\Delta)$. We have $\Re[G(z)] = \beta$ $(z \in b\Delta)$ which, since $\beta \neq 0$, implies that $G \neq 0$ on $b\Delta$ and that the change of argument of G around $b\Delta$ equals zero. Since $a \in \Delta$ the change of argument of $z \mapsto (z-a)$ around $b\Delta$ equals 2π. Since $G(z) = (z-a)[f(z) - h(z)]$ $(z \in b\Delta)$ it follows that $f - h \neq 0$ on $b\Delta$ and that the change of argument of $f - h$ around $b\Delta$ is negative. Since $h \in A(\Delta)$ this completes the proof in the special case when f is smooth.

In general, we have to approximate f by smooth functions as follows:
Let f_1 be a smooth function on $b\Delta$ such that

$$|(z-a)[f(z) - f_1(z)]| < |\beta|/4 \quad (z \in b\Delta). \tag{3.3}$$

Write $\beta_1 = \Re\{\mathcal{H}[(Z-a)f_1](a)\}$. By (3.3) and by the maximum principle for the real harmonic function $\Re\{\mathcal{H}[(Z-a)(f_1 - f)]\}$ we have

$$|\beta_1 - \beta| < |\beta|/4. \tag{3.4}$$

The function $z \mapsto \Re\{\mathcal{H}[(Z-a)f_1](z)\} - \beta_1$ is continuous on $\overline{D}$, harmonic on Δ and has smooth boundary values $\Re[(z-a)f_1(z)] - \beta_1$ $(z \in b\Delta)$. Hence by Lemma 2.1 there is a function $g_1 \in A(\Delta)$ such that

$$\Re[g_1(z)] = \Re\{\mathcal{H}[(Z-a)f_1](z)\} - \beta_1 \ (z \in \overline{\Delta}). \tag{3.5}$$

Clearly $\Re[g_1(a)] = \Re\{\mathcal{H}[(Z-a)f_1](a)\} - \beta_1 = 0$ so by adding an imaginary constant to g_1 if necessary we may assume that $g_1(a) = 0$ so that $g_1(z) = (z-a)h(z)$ $(z \in \overline{\Delta})$ where $h \in A(\Delta)$.

Consider the function $z \mapsto G(z) = (z-a)f(z) - g_1(z)$ $(z \in b\Delta)$. By (3.5) we have $\Re G(z) = \beta + \Re[(z-a)(f(z) - f_1(z))] + (\beta_1 - \beta)$ $(z \in b\Delta)$ which, by (3.3) and (3.4) implies that

$$|\Re G(z) - \beta| < |\beta|/4 + |\beta|/4 = |\beta|/2 \quad (z \in b\Delta)$$

so

$$\beta - |\beta|/2 < \Re G(z) < \beta + |\beta|/2 \ (z \in \Delta),$$

which, since $\beta \neq 0$, implies that $G \neq 0$ on $b\Delta$ and that the change of argument of G around $b\Delta$ is zero. We now repeat the reasoning from the proof in the smooth case to conclude that $f - h \neq 0$ on $b\Delta$ and that the change of argument of $f - h$ around $b\Delta$ is negative. Since $h \in A(\Delta)$ this completes the proof.

4. Harmonic functions and their conjugates

The main problem in generalizing the proof in Section 3 to multiply connected domains D is that in general, a harmonic function on D has no conjugate on D.

We have assumed that $D \subset\subset \mathbb{C}$ is a domain bounded by pairwise disjoint circles. Denote these circles by $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ where Γ_n is the boundary of the unbounded component of $\mathbb{C} \setminus \overline{D}$. For each $k, 1 \leq k \leq n$, the *harmonic measure function* ω_k is the continuous function on $\overline{D}$, harmonic on D which satisfies $\omega_k \equiv 1$ on Γ_k and $\omega_k \equiv 0$ on $\Gamma_j,\ 1 \leq j \leq n, j \neq k$. By the preceding discussion each $\omega_k, 1 \leq k \leq n$, is smooth on $\overline{D}$. We have $\sum_{k=1}^{n} \omega_k \equiv 1$ on $\overline{D}$.

For each $k,\ 1 \leq k \leq n-1$, let γ_k be a circle with the same center as Γ_k and with a a slightly larger radius, and let γ_n be a circle with the same center as Γ_n and with a slightly smaller radius so that the circles $\gamma_k,\ 1 \leq k \leq n$ bound a domain D', slightly smaller than D, whose closure is contained in D. We give each γ_k the orientation induced by the standard orientation of bD'.

Let u be a real-valued harmonic function on D. A conjugate v of u has to satisfy the Cauchy-Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$

This system is always solvable for v locally. It is solvable for v on D if and only if

$$\int_{\gamma_k} \Big(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\Big) = 0 \quad (1 \leq k \leq n-1). \tag{4.1}$$

Since

$$2\int_{\gamma_k} \frac{\partial u}{\partial z} dz = i\int_{\gamma_k} \Big(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\Big) \quad (1 \leq k \leq n-1)$$

(4.1) holds if and only if

$$\int_{\gamma_k} \frac{\partial u}{\partial z} dz = 0 \ \ (1 \leq k \leq n-1). \tag{4.2}$$

If u is a real valued harmonic function on D then there are real constants $c_1, c_2, \cdots, c_{n-1}$ such that the harmonic function $u + \sum_{j=1}^{n-1} c_j\omega_j$ has a conjugate on D. For this to happen we must have

$$\int_{\gamma_k} \frac{\partial u}{\partial z}\Big[u + \sum_{j=1}^{n-1} c_j\omega_j\Big] dz = 0 \ \ (1 \leq k \leq n-1),$$

that is,

$$\sum_{j=1}^{n-1} c_j \int_{\gamma_k} \frac{\partial \omega_j}{\partial z} dz = -\int_{\gamma_k} \frac{\partial u}{\partial z} dz \tag{4.3}$$

The system (4.3) has a unique solution since the matrix

$$\Big[\int_{\gamma_k}\frac{\partial\omega_j}{\partial z}dz\Big]_{1\le j,k\le n-1}=\Big[\int_{\Gamma_k}\frac{\partial\omega_j}{\partial z}dz\Big]_{1\le j,k\le n-1}$$

is known to be nonsingular [B, p.82]. (In the last equality we used the fact that all functions $\frac{\partial\omega_j}{\partial z}$ are smooth on $\overline{D}$ and holomorphic on D.) The Poisson formula implies that given $\varepsilon>0$ and a compact set $K\subset D$ there is a $\delta>0$ such that $|\frac{\partial u}{\partial z}|<\varepsilon$ on K whenever u is a real harmonic function on D such that $|u|<\delta$ on D. Thus, given $\varepsilon>0$ there is a $\delta>0$ such that

$$\Big|\int_{\gamma_k}\frac{\partial u_1}{\partial z}dz-\int_{\gamma_k}\frac{\partial u}{\partial z}dz\Big|<\varepsilon\quad(1\le k\le n-1)$$

provided that u_1 is a real harmonic function on D satisfying $|u_1(z)-u(z)|<\delta$ $(z\in D)$.

The preceding discussion gives

LEMMA 4.1 *Given a harmonic function f on D there is a unique (n-1)-tuple $c_1(f),\cdots,c_{n-1}(f)$ of complex numbers such that $f+\sum_{j=1}^{n-1}c_j(f)\omega_j$ has a conjugate on D. These numbers depend continuously on f in the sup norm on D.*

5. Proof of Theorem 1.1, Part 1

Let D be a domain bounded by pairwise disjoint circles $\Gamma_1,\cdots,\Gamma_n$ where Γ_n is the boundary of the unbounded component of $\mathbb{C}\setminus\overline{D}$. Section 3 contains the proof of Theorem 1.1 in the case $n=1$ so assume that $n\ge 2$.

Let f be a continuous function on bD which does not extend holomorphically through D. Define

$$A(a,f)=\mathcal{H}[(Z-a)f](a)=\mathcal{H}(Zf)(a)-a\mathcal{H}(f)(a)\quad(a\in\overline{D}).$$

Since $\mathcal{H}(f)$ is not holomorphic Lemma 2.2 implies that

$$\{a\in D\colon\ A(a,f)=0\}\ \text{ is a closed, nowhere dense subset of }\ D. \tag{5.1}$$

There are constants $c_k(f)$, $1\le k\le n-1$, such that

$$\mathcal{H}(f)(z)+\sum_{k=1}^{n-1}c_k(f)\omega_k(z)=P_f(z)+\overline{Q_f(z)}\quad(z\in D) \tag{5.2}$$

where P_f and Q_f are holomorphic functions on D. Similarly, there are constants $d_k(f)$, $1\le k\le n-1$, such that

$$\mathcal{H}(Zf)(z)+\sum_{k=1}^{n-1}d_k(f)\omega_k(z)=R_f(z)+\overline{S_f(z)}\quad(z\in D) \tag{5.3}$$

where R_f and S_f are holomorphic functions on D. We know that the constants $c_k(f)$ and $d_k(f)$, $1\le k\le n-1$, are determined uniquely and, by the maximum principle for harmonic functions depend continuously on $f\in C(bD)$. We have

$$c_k(e^{i\omega}f)=e^{i\omega}c_k(f),\quad d_k(e^{i\omega}f)=e^{i\omega}d_k(f)\quad(1\le k\le n-1,\ \omega\in\mathbb{R}) \tag{5.4}$$

and

$$A(a, e^{i\omega} f) = e^{i\omega} A(a, f) \quad (\omega \in \mathbb{R}, \ a \in D). \tag{5.5}$$

Define

$$\Phi_{a,f}(z) = \sum_{j=1}^{n-1} [d_j(f) - ac_j(f)].[\omega_j(z) - \omega_j(a)] - A(a, f) \quad (z \in D).$$

For each $a \in D$ the function $\Phi_{a,f}$ is smooth on $\overline{D}$ and harmonic on D. By the preceding discussion for each $a \in D$ the harmonic function

$$z \mapsto \mathcal{H}[(Z - a)f](z) + \Phi_{a,f}(z) \quad (z \in D)$$

vanishes at a and has a conjugate on D, that is, it is of the form $F_{a,f} + \overline{G_{a,f}}$ where $F_{a,f}$ and $G_{a,f}$ are holomorphic on D.

6. The function $\Phi_{a,f}$

Note that for each $a \in D$ the function $\Phi_{a,f}$ is smooth on $\overline{D}$, harmonic on D and constant on each component $\Gamma_j, \ 1 \leq j \leq n$, of bD.

LEMMA 6.1 *There is an $a \in D$ such that $\Phi_{a,f}(z) \neq 0$ $(z \in bD)$.*

Proof. Recall that for each $a \in D$ the function $\Phi_{a,f}|\Gamma_k$ is constant for each $k, \ 1 \leq k \leq n$; we have to prove that for some $a \in D$ these constants are all different from 0. We shall prove that

$$\left.\begin{array}{l}\text{for each } k, \ 1 \leq k \leq n, \ \text{the set } \{a \in D\colon \ \Phi_{a,f}|\Gamma_k = 0\} \\ \text{is a closed subset of } D \text{ with empty interior.}\end{array}\right\} \tag{6.1}$$

Assume that we have done this. Then $\cup_{k=1}^n \{a \in D\colon \ \Phi_{a,f}|\Gamma_k = 0\}$ is a closed subset of D with empty interior which implies that there is an open dense subset of D of those a for which $\Phi_{a,f}(z) \neq 0$ $(z \in bD)$ which will complete the proof. It remains to prove (6.1).

Let $1 \leq k \leq n-1$. On Γ_k the function $\Phi_{a,f}$ is equal to the constant $-A(a,f) + \sum_{j=1, j\neq k}^{n-1} [d_j(f) - ac_j(f)].[-\omega_j(a)] + [d_k(f) - ac_k(f)].[1 - \omega_k(a)]$. Since $a \mapsto A(a,f)$ is continuous on D it follows that $\{a \in D\colon \ \Phi_{a,f}|\Gamma_k = 0\}$ is a closed subset of D. Suppose that it contains a disc U. Then

$$A(a, f) = \sum_{j=1, j\neq k}^{n-1} [d_j(f) - ac_j(f)].[-\omega_j(a)] + [d_k(f) - ac_k(f)].[1 - \omega_k(a)] \tag{6.2}$$

for all $a \in U$. Since both sides of (6.2) are real-analytic in a on D it follows that (6.2) holds for all $a \in D$. Since both sides of (6.2) are continuous in a on $\overline{D}$ it follows that (6.2) holds for all $a \in bD$. However, $A(a, f) = 0$ $(a \in bD)$ so

$$\sum_{j=1, j\neq k}^{n-1} [d_j(f) - ac_j(f)].[\omega_j(a)] = [d_k(f) - ac_k(f)].[1 - \omega_k(a)] \quad (a \in bD). \tag{6.3}$$

If $a \in \Gamma_j,\ 1 \le j \le n-1, j \ne k$, then $\omega_j(a) = 1$ and $\omega_i(a) = 0$ for all $i,\ 1 \le i \le n,\ i \ne j$, so (6.3) implies that

$$d_j(f) - ac_j(f) = d_k(f) - ac_k(f) \quad (a \in \Gamma_j, 1 \le j \le n-1, j \ne k). \tag{6.4}$$

If $a \in \Gamma_n$ then $\omega_j(a) = 0$ for all $j,\ 1 \le j \le n-1$, including k, so (6.3) gives

$$d_k(f) - ac_k(f) = 0 \quad (a \in \Gamma_n). \tag{6.5}$$

Now, (6.5) implies that $d_k(f) = c_k(f) = 0$ which, by (6.4) gives $d_j(f) = c_j(f) = 0$ $(1 \le j \le n-1,\ j \ne k)$ so, by (6.2) it follows that $A(a, f) = 0$ for every $a \in D$ which contradicts (5.1). This proves that $\{a \in D\colon\ \Phi_{a,f}|\Gamma_k = 0\}$ has empty interior for each $k,\ 1 \le k \le n-1$.

Let $k = n$. We have

$$\Phi_{a,f}|\Gamma_n = -A(a, f) + \sum_{j=1}^{n-1}[d_j(f) - ac_j(f)].[-\omega_j(a)]$$

As before, the continuity of $a \mapsto A(a, f)$ on D implies that the set $\{a \in D\colon\ \Phi_{a,f}|\Gamma_n = 0\}$ is closed. Suppose that it has nonempty interior. As before, we get

$$0 = A(a, f) = \sum_{j=1}^{n-1}[d_j(f) - ac_j(f)].[-\omega_j(a)] \quad (a \in bD) \tag{6.6}$$

It follows that $d_j(f) - ac_j(f) = 0$ $(a \in \Gamma_j,\ 1 \le j \le n-1\}$ which implies that $d_j(f) = c_j(f) = 0$ $(1 \le j \le n-1)$ so again $A(a, f) \equiv 0$ $(a \in D)$ which contradicts (5.1). This completes the proof.

7. Proof Theorem 1.1, Part 2

By Lemma 6.1 there is an $a \in D$ such that the constants $\Phi_{a,f}|\Gamma_k,\ 1 \le k \le n$, are all different from 0. By (5.4) and (5.5) we have $\Phi_{a,e^{i\omega}f} = e^{i\omega}\Phi_{a,f}$ $(\omega \in \mathbb{R})$ so replacing f by $e^{i\omega}f$ if necessary we may assume with no loss of generality that

$$(\Re\Phi_{a,f})|\Gamma_k = \beta_k \ne 0 \quad (1 \le k \le n). \tag{7.1}$$

To make the proof easier to understand we first complete it under the assumption that f is smooth. Assume that f is smooth. In this case, by Lemma 2.1,

$$\mathcal{H}[(Z-a)f](z) + \Phi_{a,f}(z) = F_{a,f} + \overline{G_{a,f}(z)} \quad (z \in \overline{D})$$

where $F_{a,f}$ and $G_{a,f}$ belong to $A(D)$ so

$$\Re\{\mathcal{H}[(Z-a)f](z) + \Phi_{a,f}(z)\} = \Re[g(z)] \quad (z \in \overline{D})$$

where $g = (F_{a,f} + G_{a,f})/2 \in A(D)$. Clearly $\Re[g(a)] = 0$ so by adding an imaginary constant to g if necessary we may assume that $g(0) = 0$ so $g(z) = (z-a)h(z)$ $(z \in \overline{D})$ where $h \in A(D)$.

Consider the function $z \mapsto G(z) = (z-a)f(z) - g(z)$ $(z \in bD)$. We have $\Re[G(z)] = -\Re\Phi_{a,f}(z)$ $(z \in bD)$. By (7.1) for each k, $1 \le k \le n$, the expression on the right is a nonzero constant on Γ_k which implies that for each k, $1 \le k \le n$, the change of argument of $z \mapsto G(z) = (z-a)[f(z) - h(z)]$ along Γ_k equals 0. Since $a \in D$ the change of argument of $z \mapsto (z-a)$ along each Γ_k, $1 \le k \le n-1$, is zero, and the change of argument of $z \mapsto (z-a)$ along Γ_n is 2π. Thus, the change of argument of $z \mapsto f(z) - h(z)$ along Γ_n equals -2π. So, the change of argument of $z \mapsto f(z) - h(z)$ along bD is negative. Since $h \in A(D)$ this completes the proof in the case when f is smooth.

In the case of general f we have to approximate f by smooth functions. We already know that the constants $c_k(f)$ and $d_k(f)$ depend continuously on $f \in C(bD)$. Further, for our fixed $a \in D$ the maximum priciple for harmonic functions implies that $A(a,f) = \mathcal{H}[(Z-a)f](a)$ also depends continuously on $f \in C(bD)$. It follows that Φ_{a,f_1} is uniformly arbitrarily close to $\Phi_{a,f}$ on $\overline{D}$ provided that $f_1 \in C(bB)$ is sufficiently close to f. Fix ε,

$$0 < \varepsilon < (1/4)\min\{|\beta_k|\colon\ 1 \le k \le n\} \tag{7.3}$$

and let f_1 be a smooth function on bD which is so close to f that

$$|\Phi_{a,f_1}(z) - \Phi_{a,f}(z)| < \varepsilon \quad (z \in \overline{D}) \tag{7.4}$$

and

$$|(z-a)[f_1(z) - f(z)]| < \varepsilon \quad (z \in bD). \tag{7.5}$$

As before, since f_1 is smooth, Lemma 2.1 applies to show that

$$\Re\{\mathcal{H}[(Z-a)f_1](z) + \Phi_{a,f_1}(z)\} = \Re[g(z)] \quad (z \in \overline{D}) \tag{7.6}$$

where $g \in A(D)$ satisfies $g(a) = 0$. Consider the function $z \mapsto G(z) = (z-a)f(z) - g(z)$. We have $\Re[G(z)] = \Re\{(z-a)[f(z) - f_1(z)]\} + \Re[\Phi_{a,f}(z)] + \Re[\Phi_{a,f_1}(z) - \Phi_{a,f}(z)]$ so by (7.4) and (7.5) it follows that

$$\left|\Re[G(z)] - \Re[\Phi_{a,f}(z)]\right| < 2\varepsilon \quad (z \in bD)$$

so by (7.1) it follows that

$$\left|\Re[G(z)] - \beta_k\right| < 2\varepsilon \quad (z \in \Gamma_k,\ 1 \le k \le n),$$

which, by (7.3), implies that

$$\beta_k - |\beta_k|/2 < \Re[G(z)] < \beta_k + |\beta_k|/2 \quad (z \in \Gamma_k,\ 1 \le k \le n).$$

Since $\beta_k \ne 0$ $(1 \le k \le n)$, it follows that for each k, $1 \le k \le n$, the change of argument of G along Γ_k is zero. Now we conclude the proof as in the smooth case.

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