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CHARACTERIZING r-PERFECT CODES IN DIRECT PRODUCTS OF CYCLES

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Abstract

An r-perfect code of a graph $G = (V, E)$ is a set $C \subseteq V$ such that the r-balls centered at vertices of C form a partition of V. It is proved that the direct product of C_m and C_n ($r \geq 1$, $m, n \geq 2r + 1$) contains an r-perfect code if and only if m and n are each multiple of $(r + 1)^2 + r^2$ and that the direct product of C_m, C_n, and C_ℓ ($r \geq 1$, $m, n, \ell \geq 2r + 1$) contains an r-perfect code if and only if m, n, and ℓ are each multiple of $r^3 + (r + 1)^3$. The corresponding r-codes are essentially unique. Also, r-perfect codes in $C_{2r} \times C_n$ ($r \geq 2$, $n \geq 2r$) are characterized.

Key words: perfect codes, direct products of graphs, cycles

AMS subject classification (2000): 05C69, 05C12

1 Introduction

The direct product of graphs is one of the four standard graph products [8]. This product is very natural and consequently known under many different names, for instance as the cardinal product, the Kronecker product and the categorical product. It is the product in the category of graphs [6] and has been considered from several points of view. For

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instance, McKenzie [18] proved that finite, nonbipartite, connected graphs have unique prime factor decomposition with respect to the direct product in the class of undirected graphs with loops, while Imrich [7] extended this result by showing that the factorization can be found in polynomial time.

In this paper we consider r-perfect codes in direct products of two and three cycles. (For a related problem of determining the r-domination numbers of direct products of two paths see [14, 15].) The study of 1-perfect codes in graphs was initiated by Biggs in 1993 [2]. We refer to the monograph [16] for a collection of results up to 1991 and to [3, 13] and references therein for some recent result in the area. Perfect codes in graphs arising from interconnection networks were studied in [17]. We note that many of these graphs are Cartesian products or similar graphs. Since $C_m \times C_n$ is a 4-regular graph and $C_m \times C_n \times C_\ell$ is 8-regular, it is worth to add that the 1-perfect code problem remains NP-complete on k-regular graphs (for any fixed $k \geq 3$) [16, Theorem 7.2.2].

Motivated by a problem of efficient resource placement Jha [9, 10, 11] studied partitions of the direct product of cycles into r-perfect codes and proved the following:

Theorem 1.1 (i) [9] *If $r \geq 1$ and m and n are each multiple of $(r+1)^2 + r^2$ then (each connected component of) $C_m \times C_n$ can be partitioned into r-perfect codes.*

(ii) [10] *If $r \geq 1$ and m, n, and ℓ are each multiple of $(r+1)^3 + r^3$ then (each connected component of) $C_m \times C_n \times C_\ell$ can be partitioned into r-perfect codes.*

(Similar constructions as the one for proving Theorem 1.1 has been used elsewhere, for instance in [4] in the case of perfect codes in the Lee metric and in [5] for tilings of integer lattices with spheres defined by the Manhattan metric.)

In this paper we complement Theorem 1.1 by proving that $C_m \times C_n$ ($r \geq 1$, $m, n \geq 2r+1$) contains an r-perfect code if and only if m and n are each multiple of $(r+1)^2 + r^2$ and that $G = C_m \times C_n \times C_\ell$ ($r \geq 1$, $m, n, \ell \geq 2r+1$) contains an r-perfect code if and only if m, n, and ℓ are each multiple of $r^3 + (r+1)^3$. Moreover, in these cases codes are essentially unique. In addition, we also characterize r-perfect codes for $C_{2r} \times C_n$, where $r \geq 2$ and $n \geq 2r$.

In the rest of the introduction we fix the terminology and notation. Then, in Section 2, we consider products of two cycles, and in the last section products of three cycles.

The *direct product* $G \times H$ of graphs G and H is the graph defined on the Cartesian product of the vertex sets of the factors. Two vertices (u_1, u_2) and (v_1, v_2) being adjacent when $u_1 v_1 \in E(G)$ and $u_2 v_2 \in E(H)$. This product of graphs is commutative and associative in a natural way. Moreover, the direct product of two graphs is connected if and only if both factors are connected and at least one of them is not bipartite [19], cf. also [8]. If both factors are connected and bipartite, then their direct product consists of two connected components. In the special case when $G = C_{2m} \times C_{2n}$ it is in addition well-known that the connected components of G are isomorphic.

For a graph $G = (V, E)$, the *distance* $d_G(u, v)$, or briefly $d(u, v)$, between vertices u and v, is defined as the number of edges on a shortest u,v-path. For a vertex $v \in V$ let $B_r(v) = \{u \in V \mid d(u, v) \leq r\}$ be the *r-ball centered at v*. In particular, $N[v] = B_1(v)$ and set $N(v) = N[v] \setminus \{v\}$.

A set $C \subseteq V$ is an *r-code* in G if $B_r(u) \cap B_r(v) = \emptyset$ for any two distinct vertices $u, v \in C$. In addition, a r-code C is called a *r-perfect code* if $\{B_r(u) \mid u \in C\}$ forms a partition of V.

Throughout the paper we will set $V(C_n) = \{0, \ldots, n-1\}$. Whenever applicable, the vertices of a cycle will be calculated modulo the number of its vertices. An explicit formula for the distance function in the direct product was first given by Kim in [12], but for our purposes the following approach from [1] is more useful.

Lemma 1.2 *Let $X = G \times H$ and let $(a,x), (b,y)$ be vertices of X. Then $d_X((a,x),(b,y))$ is the smallest d such that there is an a,b-walk of length d in G and an x,y-walk of length d in H. In particular, if such walks do not exist, then (a,x) and (b,y) are in different connected components of X.*

2 Products of two cycles

In this section we complement Theorem 1.1 (i). For this sake, some preparation is needed.

Lemma 2.1 *Let $r \geq 1$ and $n \geq m \geq 2r+1$ and let P be an r-perfect code of connected component of $C_m \times C_n$. Assume that $(i,j) \in P$. Let $s = 2r+1$ and set*

$$R_1 = \{(i+s, j+1), (i-1, j+s), (i-s, j-1), (i+1, j-s)\},$$
$$R_2 = \{(i+1, j+s), (i-s, j+1), (i-1, j-s), (i+s, j-1)\}.$$

If $P \cap R_k \neq \emptyset$, then $R_k \subseteq P$ $(1 \leq k \leq 2)$.

Proof. (The sets R_1 and R_2 are schematically shown on Fig. 1 for the case $r = 2$.) By symmetry it suffices to prove the lemma for R_1. Using symmetry again we may assume that $(i+s, j+1) \in P$. Let

$$[ij]^\bullet = \{(i+s, j+1+2k), (i+s-2k, j+s);\ 0 \leq k \leq (s-1)/2\},$$
$${}^\bullet[ij] = \{(i-1-2k, j+s), (i-s, j+s-2k);\ 0 \leq k \leq (s-1)/2\},$$
$${}_\bullet[ij] = \{(i-s, j-1-2k), (i-s+2k, j-s);\ 0 \leq k \leq (s-1)/2\},$$
$$[ij]_\bullet = \{(i+1+2k, j-s), (i+s, j-s+2k);\ 0 \leq k \leq (s-1)/2\}.$$

Since $(i,j) \in P$, P does not contain any vertex of $B_{2r}(i,j)$, hence the vertices $(i \pm (r+1), j \pm (r+1))$ are not in P. As these vertices do not belong to $B_r(i,j)$, P must contain exactly one vertex of each of the sets $[ij]^\bullet$, ${}^\bullet[ij]$, ${}_\bullet[ij]$ and $[ij]_\bullet$. Since $(i+s, j+1) \in [ij]^\bullet$, P does not contain any other vertex of $[ij]^\bullet$. Consider now the vertex $(i+r-1, j+r+1)$. This vertex can only lie in $B_r(i-1, j+s)$, where $(i-1, j+s) \in {}^\bullet[ij]$. Hence $(i-1, j+s) \in P$. If we repeat analogous arguments for the vertices $(i-r-1, j+r-1)$ and $(i-r+1, j-r-1)$, we get $(i-s, j-1) \in P$ and $(i+1, j-s) \in P$. We conclude that $R_1 \subseteq P$. □

Lemma 2.2 *Under the assumptions of Lemma 2.1, either $R_1 \subseteq P$ and $R_2 \cap P = \emptyset$, or $R_2 \subseteq P$ and $R_1 \cap P = \emptyset$.*

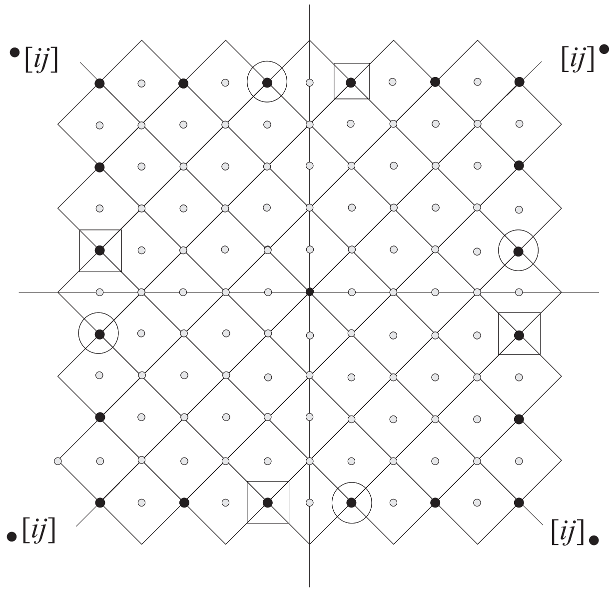

Figure 1: R_1 is denoted by circles and R_2 by squares

Proof. Suppose that $R_1 \cap P \neq \emptyset$ and $R_2 \cap P \neq \emptyset$. Then by Lemma 2.1, $R_1 \subseteq P$ and $R_2 \subseteq P$. But since in $R_1 \cup R_2$ we have vertices at distance two, this in not possible.

Suppose now that $R_2 \cap P = \emptyset$. Then $(i+s, j-1) \notin P$ and the vertex $(i+r+1, j+r-1)$ must lie in one of the r-balls $B_r(i+s, j+1+2k)$, $0 \leq k \leq (s-3)/2$. If we consider the vertex $(i+r-1, j+r+1)$ as in the proof of Lemma 2.1, we get that $(i-1, j+s) \in P$. By the same lemma, $R_1 \subseteq P$. The other case is analogous. □

Lemma 2.3 *Under the assumptions of Lemma 2.1 and* $R_1 \subseteq P$*, we have* $\{(i+2, j-2s), (i+2s, j+2), (i-2, j+2s), (i-2s, j-2)\} \subseteq P$.

Proof. Since $(i+1, j-s) \in R_1 \subseteq P$, Lemma 2.2 implies that P contains exactly one of the following sets

$$\{(i+1+s, j-s+1), (i,j), (i+1-s, j-s-1), (i+2, j-2s)\},$$
$$\{(i+2, j), (i+1-s, j-s+1), (i, j-2s), (i+1+s, j-s-1)\}.$$

On the other hand, since $(i,j) \in P$, Lemma 2.1 implies that P contains the first set. Hence $(i+2, j-2s) \in P$.

Considering the remain three vertices of R_1, we similarly get

$$\{(i+2s, j+2), (i-2, j+2s), (i-2s, j-2)\} \subseteq P.$$

□

Let P be an arbitrary r-perfect code of a connected component of $G = C_m \times C_n$ and $(i,j) \in P$. By Lemma 2.2, either $R_1 \subseteq P$ or $R_2 \subseteq P$. Suppose $R_1 \subseteq P$. Consider the sets $P_0 = \{(i-t, j+ts) \mid t \in \mathbb{N}\}$ and $Q_0 = \{(i+qs, j+q) \mid q \in \mathbb{N}\}$. Let ℓ_1 be the smallest integer such that $(i+\ell_1 s, j+\ell_1) \in P_0$ and ℓ_2 the smallest integer such that $(i-\ell_2, j+\ell_2 s) \in Q_0$. Define $P_k = \{(i-t+ks, j+ts+k) \mid t \in \mathbb{N}\}$ for $k = 1, \ldots, \ell_1 - 1$ and $Q_k = \{(i-k+qs, j+q+ks) \mid t \in \mathbb{N}\}$ for $k = 1, \ldots, \ell_2 - 1$. By Lemma 2.3 $P_k \subseteq P$ and $Q_k \subseteq P$. Then

$$P = \bigcup_{k=0}^{\ell_1 - 1} P_k = \bigcup_{k=0}^{\ell_2 - 1} Q_k \,.$$

Since $|P_k| = pm$ and $|Q_k| = qn$ we infer that $|P| = \ell_1 pm = \ell_2 qn$. On the other hand $|B_r(u)| = (r+1)^2 + r^2$, cf. [11, Lemma 2.1], hence $|P| = \frac{mn}{(r+1)^2+r^2}$. We conclude that m and n are each multiple of $(r+1)^2 + r^2$. Note that the above set P is uniquely determined by its vertices (i,j) and $(i+s, j+1)$. We denote this set by $P_{ij}(i+s, j+1)$.

In the case when $R_2 \subseteq P$ we argue analogously that m and n are each multiple of $(r+1)^2 + r^2$. In this case P is uniquely determined by the vertices (i,j) and $(i+1, j+s)$. We denote this set by $P_{ij}(i+1, j+s)$.

Suppose that $P = P_{ij}(i+s, j+1)$ or $P = P_{ij}(i+1, j+s)$. Then it follows from Lemma 1.2 that $d(x,y) \geq 2r+1$ for any vertices $x, y \in C_m \times C_n$. (We do not give details as this also follows directly from Theorem 1.1. In conclusion, we have proved the following result.

Theorem 2.4 *Let $r \geq 1$, $n \geq m \geq 2r+1$, and $G = C_m \times C_n$. Then connected component G contains an r-perfect code P if and only if m and n are each multiple of $(r+1)^2 + r^2$. Moreover, if $(i,j) \in P$ then either $P = P_{ij}(i+s, j+1)$ or $P = P_{ij}(i+1, j+s)$.*

Combining Theorems 2.4 and 1.1 (i) we also have:

Corollary 2.5 *Let $r \geq 1$, $n \geq m \geq 2r+1$, and $G = C_m \times C_n$. Then (each connected component of) G can be partitioned into r-perfect codes if and only if m and n are each multiple of $(r+1)^2 + r^2$.*

Corollary 2.6 *Let $r \geq 1$ and $n \geq 2r+1$. Then $C_{2r+1} \times C_n$ contains no r-perfect code.*

Proof. By Theorem 2.4, $2r+1$ must be a multiple of $(r+1)^2 + r^2$ if $C_{2r+1} \times C_n$ would contain an r-perfect code. □

The conditions $r \geq 1$ and $n \geq m \geq 2r+1$ in the above results assure that for any vertex (i,j) of $G = C_m \times C_n$ we have $|B_r(i,j)| = (r+1)^2 + r^2$. This is no longer true if one of the n and m is smaller because then in $B_r(i,j)$ we have vertices that can be reached by paths of length at most r going in both directions around the "torus". The first such case is when $r \geq 2$ and $n \geq 2r$. In this case we have:

Proposition 2.7 *Let $r \geq 2$ and $n \geq 2r$. Then $C_{2r} \times C_n$ contains an r-perfect code precisely in the following two cases:*

(i) $n = 2r$,

(ii) $n > 2r$ *and* $n = \ell(2r+1)$*, for some* $\ell \in \mathbb{N}$.

Proof. Let $n = 2r$. Then $C_{2r} \times C_{2r}$ consists of two isomorphic components H_1 and H_2. Select any vertices $u_1 \in H_1$ and $u_2 \in H_2$. Then, since $|B_r(u_i)| = 2r^2 = |V(H_i)|$, $C_{2r} \times C_{2r}$ contains an r-perfect code.

Suppose $n > 2r$ and let P be an r-perfect code of $C_{2r} \times C_n$. We first claim that n is a multiple of $2r+1$. Let $|P| = k$ and let B be a arbitrary r-ball of G. Then $|B| = r(2r+1)$ and hence $2rn = kr(2r+1)$. Since $n \in \mathbb{N}$ it follows that k is even, and hence it follows that n is a multiple of $2r+1$.

It remains to show that $X = C_{2r} \times C_n$ contains an r-perfect code whenever $n = \ell(2r+1)$ for some $\ell \in \mathbb{N}$. Set

$$P = \bigcup_{t=0}^{\ell-1} \Big\{(r-1, t(2r+1)),\ (r, t(2r+1))\Big\}\,.$$

Lemma 1.2 implies that $d_X((r-1, p(2r+1)), (r, q(2r+1)) = \ell(2r+1)$ whenever $p = q$ and at least $2r+1$ whenever $p \neq q$. Since $|B_r(x)| = r(2r+1)$ for any vertex x of X, we have $\sum_{x \in P} |B_r(x)| = 2\ell r(2r+1) = |X|$. □

We note that in the case $X = C_{2r} \times C_{\ell(2r+1)}$ it is also easy to obtain a partition of $V(X)$ into r-perfect codes.

3 Products of three cycles

We now continue with the direct product of three cycles. For a description of r-perfect codes in such graphs we introduce the following notation. Let $x = (i, j, k)$ be an arbitrary vertex of $C_m \times C_n \times C_\ell$, let $r \geq 1$ and $s = 2r+1$. For $a_1, a_2, a_3 \in \{\oplus, \ominus, +, -\}$ let $x^{a_1 a_2 a_3}$ be the vertex $(i + i', j + j', k + k')$, where $i' = s, -s, 1, -1$ if $a_1 = \oplus, \ominus, +, -$, respectively, and j', k' are defined analogously. For instance, $x^{\oplus ++} = (i+s, j+1, k+1)$ and $x^{-\ominus +} = (i-1, j-s, k+1)$. In addition, let

$$X^{\oplus **} = \{x^{\oplus ++}, x^{\oplus +-}, x^{\oplus -+}, x^{\oplus --}\}\,,$$

and define the sets $X^{\ominus **}, X^{*\oplus *}, X^{*\ominus *}, X^{**\oplus}$, and $X^{**\ominus}$ analogously. For instance, $X^{*\ominus *} = \{x^{+\ominus +}, x^{+\ominus -}, x^{-\ominus +}, x^{-\ominus -}\}$.

Lemma 3.1 *Let* P *be an* r*-perfect code of* $C_m \times C_n \times C_\ell$ $(m, n, \ell \geq 2r+1)$*, and let* $x = (i, j, k) \in P$*. Then*

$$|X^{\oplus **} \cap P| = |X^{\ominus **} \cap P| = |X^{*\oplus *} \cap P| = |X^{*\ominus *} \cap P| = |X^{**\oplus} \cap P| = |X^{**\ominus} \cap P| = 1.$$

Proof. Let r be an odd number, $s = 2r+1$, and consider the vertex $T_1 = (i+r+1, j, k)$. Since P is an r-perfect code and $T_1 \notin B_r(x)$, there must be an r-ball B_1 with center $(x_1, y_1, z_1) \in P$ such that $T_1 \in B_1$ and $B_1 \cap B_r(x) = \emptyset$. Therefore $x_1 = i+s$, $j-r \leq y_1 \leq j+r$ and $k-r \leq z_1 \leq k+r$. Suppose that $y_1 \geq j+1$. We want to prove that then $y_1 = j+1$. Assume the contrary, then $j+1 < y_1 \leq j+r$ (note that in this case $r \geq 3$). Now consider the vertices $T_2 = (i+r-1, j+r+1, k)$, $T_3 = (i-r-1, j+r-1, k)$, $T_4 = (i-r+1, j-r-1, k)$, and $T_5 = (i+r+1, j-r+1, k)$. Since $T_i \notin B_r(x) \cup B_1$ for $i = 2,3,4,5$ and since every r-ball containing two of them intersects $B_r(x)$, there are four disjoint r-balls B_i with centers $(x_i, y_i, z_i) \in P$ such that $T_i \in B_i$ and $B_i \cap (B_r(x) \cup B_1) = \emptyset$ for $i = 2,3,4,5$. Since $B_2 \cap (B_r(x) \cup B_1) = \emptyset$ and $T_2 \in B_2$, we find that $x_2 = i-1$, $y_2 = j+s$ and $k-r \leq z_2 \leq k+r$. Using similar arguments for vertices T_3, T_4, and T_5 we infer that $x_3 = i-s$, $y_3 = j-1$, $x_4 = i+1$, $y_4 = j-s$, $x_5 = i+s$, $y_5 = j+1$, and $k-r \leq z_3, z_4, z_5 \leq k+r$. But then $T_1 \in B_5$ so we conclude that $y_1 = j+1$.

Suppose now that $y_1 \leq j-1$ and consider vertices $T_2 = (i+r-1, j-r-1, k)$, $T_3 = (i-r-1, j-r+1, k)$, $T_4 = (i-r+1, j+r+1, k)$, and $T_5 = (i+r+1, j+r-1, k)$. Using similar arguments as above we find that $y_1 = j-1$. Since r is odd, $y_1 \neq j$, thus $y_1 = j+1$ or $y_1 = j-1$. By symmetry we conclude $z_1 = k+1$ or $z_1 = k-1$, hence $(x_1, y_1, z_1) \in X^{\oplus **}$. Because the vertices of $X^{\oplus **}$ are pairwise at distance two, P contains exactly one of them.

Arguments for r even are similar and left to the reader. $\square$

Lemma 3.2 *Let P be an r-perfect code of $C_m \times C_n \times C_\ell$ $(m, n, \ell \geq 2r+1)$, let $x = (i, j, k) \in P$, and let:*

$$R_1 = \{x^{\oplus ++}, x^{-\oplus +}\},\ R_2 = \{x^{\oplus ++}, x^{-\oplus -}\},$$
$$R_3 = \{x^{\oplus +-}, x^{-\oplus +}\},\ R_4 = \{x^{\oplus +-}, x^{-\oplus -}\},$$
$$R_5 = \{x^{\oplus -+}, x^{+\oplus +}\},\ R_6 = \{x^{\oplus -+}, x^{+\oplus -}\},$$
$$R_7 = \{x^{\oplus --}, x^{+\oplus +}\},\ R_8 = \{x^{\oplus --}, x^{+\oplus -}\}.$$

Then there is exactly one $i \in \{1, \ldots, 8\}$ such that $R_i \subseteq P$. In addition, P is uniquely determined by $R_i \cup \{x\}$.

Proof. By Lemma 3.1 there is exactly one of the vertices of $X^{\oplus **}$ contained in P. Without loss of generality assume $x^{\oplus ++} \in P$. Since P is an r-perfect code, $x^{+\oplus +}, x^{+\oplus -} \notin P$. Therefore P contains one of the vertices $x^{-\oplus +}$ and $x^{-\oplus -}$. If $x^{-\oplus +} \in P$, then using Lemma 3.1 again, $x^{-\oplus -} \notin P$. Therefore $R_1 \subseteq P$ and R_i is not contained in P for $i \neq 1$. In the case $x^{-\oplus -} \in P$, we have $R_i \subseteq P$ if and only if $i = 2$.

We next wish to show that P is uniquely determined by $R_i \cup \{x\}$. Without loss of generality assume $R_1 \subseteq P$. Then $x^{\ominus ++}, x^{\ominus +-} \notin P$ since both are at distance less then s from $x^{-\oplus +}$, thus P contains $x^{\ominus -+}$ or $x^{\ominus --}$. Suppose that $x^{\ominus -+} \in P$. Considering the vertex $T = (i, j, k+r+1)$ if r is odd (and $T = (i+1, j+1, k+r+1)$ if r is even) we can see that any r-ball containing T intersects either $B_r(x^{\oplus ++})$ or $B_r(x^{\ominus -+})$, thus $x^{\ominus -+} \notin P$, and therefore $x^{\ominus --} \in P$. Analogously we find that $x^{+\ominus -}, x^{--\oplus}, x^{++\ominus} \in P$. We have thus uniquely determined which vertex of each of the sets $X^{\oplus**}, X^{\ominus**}, X^{*\oplus*}, X^{*\ominus*}, X^{**\oplus}$ and $X^{**\ominus}$ is contained in P.

In next step we wish to determine those vertices of P laying closest to vertices of $(X^{*\oplus*} \cup X^{\ominus**}) \cap P$. More precisely, if $x_1 = x^{-\oplus+}$ and $x_2 = x^{\ominus--}$ then we wish to determine the vertex of $X_1^{\ominus**}$ and the vertex of $X_2^{*\oplus*}$ contained in P. It is easy to see that $X_1^{\ominus**} \cap X_2^{*\oplus*} = \{x_1^{\ominus--}\} = \{x_2^{-\oplus+}\}$ and by Lemma 3.1 P contains exactly one vertex of, say $X_1^{\ominus**}$. We argue that $x_1^{\ominus--} = x_2^{-\oplus+} \in P$. For $r \geq 2$ this is straightforward, since otherwise P contains no vertex of $X_2^{*\oplus*}$, because the vertices of $X_1^{\ominus**} \cup X_2^{*\oplus*}$ are pairwise at distance less or equal 4. For $r = 1$ the argument is more complicated. Suppose $r = 1$ and $x_2^{-\oplus+} \notin P$. By Lemma 3.1, P contains exactly one of the vertices of $X_2^{*\oplus*}$. Since $x_2^{+\oplus+}$ and $x_2^{+\oplus-}$ are at distance 2 from x, we infer $x_2^{+\oplus+}, x_2^{+\oplus-} \notin P$, thus $x_2^{-\oplus-} \in P$. But then we also have $x_2^{+\ominus+} \in P$, which is equal to $x_3^{\ominus-+}$. Since $x = x_3^{-\oplus+} \in P$, we further have $x_3^{\oplus+-} \in P$. Let $x_0 = x^{\oplus++}$, then $x_0^{-\ominus-}, x_0^{-\ominus+} \notin P$, since they are at distance 2 from x and $x_0^{+\ominus-} \notin P$, since it is at distance 2 from $x_3^{\oplus+-}$. Thus $x_0^{+\ominus+} \in P$, which is a contradiction, since then the vertex $(i+4, j-2, k)$ has all eight neighbors in 1-balls centered in $x_0^{+\ominus+}$ and $x_3^{\oplus+-}$. Thus for any $r \geq 1$ we have $x_1^{\ominus--} = x_2^{-\oplus+} \in P$.

Considering $x_2 = x^{\ominus--}$ and $x_3 = x^{+\ominus-}$, we find $X_2^{*\ominus*} \cap X_3^{\ominus**} = \{x_2^{+\ominus+}, x_2^{+\ominus-}\} = \{x_3^{\ominus-+}, x_3^{\ominus--}\}$, and so P contains $x_2^{+\ominus+}$ or $x_2^{+\ominus-}$. Suppose that $x_2^{+\ominus+} \in P$ and consider $T_1 = (i-s, j-1, k+r)$ (or $T_1 = (i-s+1, j, k+r)$ if r is even). Since any r-ball containing T_1 intersects either $B_r(x_2^{+\ominus+})$ or $B_r(x_2^{-\oplus+})$, we find $x_2^{+\ominus+} \notin P$ and therefore $x_2^{+\ominus-} \in P$. Analogously we check that $x_1^{\oplus++}$ and $x_3^{\oplus++}$ are in P. So far we know that

$$\{(i-s, j-1, k-1), (i, j, k), (i+s, j+1, k+1)\} \subseteq P$$

and

$$\{(i+1, j-s, k-1), (i, j, k), (i-1, j+s, k+1)\} \subseteq P$$

as well as

$$\{(i-1, j-1, k+s), (i, j, k), (i+1, j+1, k-s)\} \subseteq P\,.$$

If we proceed this way we get the following description of P:

$$P = \{(i + \alpha_1 s - \alpha_2 - \alpha_3, j + \alpha_1 + \alpha_2 s - \alpha_3, k + \alpha_1 + \alpha_2 + \alpha_3 s) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}_0\}\,.$$

It follows that P is uniquely determined by $R_1 \cup \{x\}$. □

The elements and the corresponding r-balls (for $r = 2$) of the code P are shown on Fig. 2. The third component of the center of each r-ball is written in the upper right corner of the corresponding ball.

Until now we have been studying the local structure of an r-perfect code. In order to determine the lengths of the cycles in mind, we shall now consider the code on a larger scale.

Lemma 3.3 *Let P be an r-perfect code of $C_m \times C_n \times C_\ell$ $(m, n, \ell \geq 2r+1)$ and $R_1 \cup \{(i, j, k)\} \subseteq P$. Then for some $t \in \mathbb{N}$ we have*

$$(i-1, j+s, k+1) = (i-1+2t((r+1)^3 + r^3), j+s, k+1)\,.$$

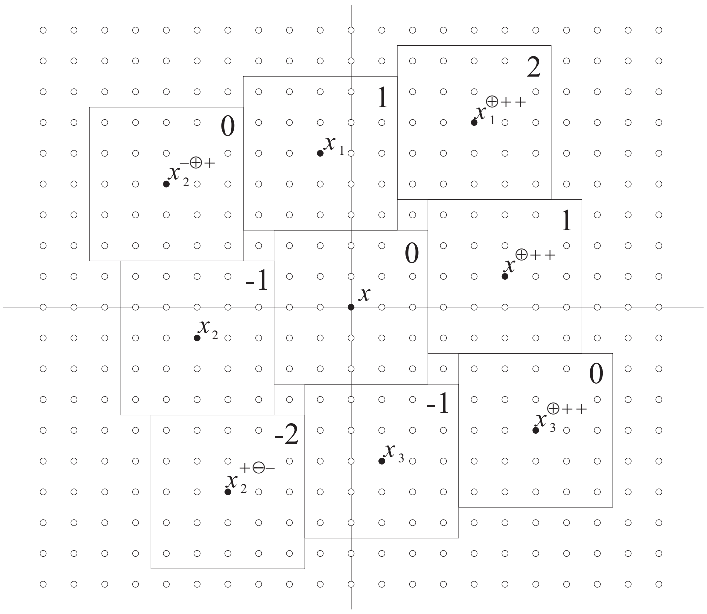

Figure 2: 2-balls of code P

Proof. Let

$$W_h = \{(i + h(s^2 + s + 2) + ds, j + 2h + d, k + d) \mid d = 0, \ldots, s + 1\}$$

for $h = 0, \ldots, r$. Note that the r-balls centered in vertices of W_h form a sequence of neighboring r-balls. We use induction to prove $W_h \subseteq P$ for $h = 0, \ldots, r$. One can see that $W_0 \subseteq P$. Assume $W_h \subseteq P$. Since the code P is uniquely determined by the set $R_1 \cup \{(i, j, k)\}$ and

$$(i + h(s^2 + s + 2) + s^2 + s, j + 2h + s + 1, k + s + 1) \in W_h \subseteq P\,,$$

there must be, by Lemma 3.2, $(i + h(s^2 + s + 2) + s^2 + s + 1, j + 2h + s + 2, k + 1)$ and $(i + h(s^2 + s + 2) + s^2 + s + 2, j + 2h + 2, k)$ in P. But the latter is contained in W_{h+1} (set $d = 0$), hence $W_{h+1} \subseteq P$. Thus $W_h \subseteq P$ for $h = 0, \ldots, r$. In the special case when $h = r$ and $d = 1$, we have

$$(i - 1 + 2((r + 1)^3 + r^3), j + s, k + 1) \in P\,.$$

Suppose that

$$(i - 1 + 2((r + 1)^3 + r^3), j + s, k + 1) \neq (i - 1, j + s, k + 1)\,,$$

for otherwise we are done. Then consider the vertex

$$(i + 2((r + 1)^3 + r^3), j, k) \in P\,.$$

Now we will construct a similar family of sets as above, with the starting vertex $(i+2((r+1)^3+r^3), j, k)$, and call them W_h^1. More generally, we define W_h^t for every $t \in \mathbb{N}_0$ as follows:

$$W_h^t = \{(i + 2t((r+1)^3 + r^3) + h(s^2 + s + 2) + ds, j + 2h + d, k + d) \mid d = 0, \dots, s+1\}$$

for $h = 0, \dots, r$. For instance, for the case $r = 2$ the sets W_0^0, W_1^0, and W_2^0 are schematically shown on Fig. 3. Next to each vertex from these sets their third components are written and the 2-balls are indicated. For a better visualization of these balls two copies of the interval $[0, 34]$ are drawn—c.f. Fig. 4 for a more realistic picture of these balls.

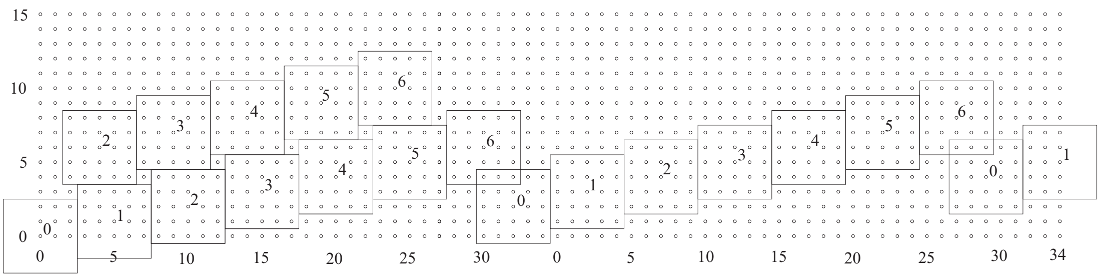

Figure 3: Sets W_h^t

Note that $W_h^0 = W_h$ for $h = 0, \dots, r$. We use induction again to prove $W_h^t \subseteq P$ for $t \in \mathbb{N}_0$. We already know that $W_h^0 \subseteq P$. Assume that $W_h^t \subseteq P$ for $h = 0, \dots, r$. Then for $h = r$ and $d = 1$ we have

$$(i + 2t((r+1)^3 + r^3) + r(s^2 + s + 2) + s, j + 2r + 1, k + 1) \in P\,.$$

It follows that

$$(i + 2t((r+1)^3 + r^3) + r(s^2 + s + 2) + s + 1, j, k) \in P\,,$$

which is equal to

$$(i + 2(t+1)((r+1)^3 + r^3), j, k) \in W_0^{t+1}\,,$$

hence $W_h^{t+1} \subseteq P$ for $h = 0, \dots, r$. Thus we have $W_h^t \subseteq P$ for $h = 0, \dots, r$ and $t \in \mathbb{N}_0$. Since $W_r^{t-1} \subseteq P$ for $t \in \mathbb{N}$ we get, if we choose $d = 1$,

$$(i - 1 + 2t((r+1)^3 + r^3), j + s, k + 1) \in P$$

for $t \in \mathbb{N}_0$. Since the code P is finite there exists an integer $t \in \mathbb{N}$ such that

$$(i - 1 + 2t((r+1)^3 + r^3), j + s, k + 1) = (i - 1, j + s, k + 1)\,.$$

□

Considering the sets W_h^t we can see that the vertex $(i-1, j+s, k+1)$, which is a nearest vertex in P positioned left to (i, j, k), can be reached by going in the right direction from (i,j,k), taking steps of length s and passing through the cycle C_m once or more times. Furthermore, the path that is traversed by going in the right direction is of length $2t((r+1)^3+r^3)$. We have thus arrived to the final question: How many times does this path, starting in (i,j,k) and going in the right direction, pass through the cycle C_m before reaching $(i-1,j+s,k+1)$? In Lemma 3.5 we shall see that not more then twice. Since the length of this path is a multiple of m we either have $2t((r+1)^3+r^3)=m$ or $2t((r+1)^3+r^3)=2m$. In any case m is a multiple of $((r+1)^3+r^3)$.

The next lemma follows immediately from Lemma 1.2.

Lemma 3.4 *Let (x_1,x_2,x_3) be a vertex of $C_m \times C_n \times C_\ell$. Then*

$$d((x_1,x_2,x_3),(x_1+2\ell_1,x_2+2\ell_2,x_3+2\ell_3)) \le \max\{2\ell_1,2\ell_2,2\ell_3\}$$

for any $\ell_1,\ell_2,\ell_3 \in \mathbb{N}_0$.

Lemma 3.5 *Let P be an r-perfect code of $C_m \times C_n \times C_\ell$ ($m,n,\ell \ge 2r+1$) and $R_1 \cup \{(i,j,k)\} \subseteq P$. Let t_0 be the smallest integer such that*

$$(i-1,j+s,k+1) = (i-1+2t_0((r+1)^3+r^3),j+s,k+1)\,.$$

Then $2t_0((r+1)^3+r^3) \le 2m$.

Proof. Assume on the contrary that $2t_0((r+1)^3+r^3) > 2m$. Denote by $W_h^t(d)$ the vertex

$$W_h^t(d) = (i+2t((r+1)^3+r^3)+h(s^2+s+2)+ds,j+2h+d,k+d)\,,$$

so that $W_h^t = \{W_h^t(d) \mid d=0,\ldots,s+1\}$, cf. Lemma 3.3. Let

$$X = \bigcup_{t=0}^{t_0-1}\left(\bigcup_{h=0}^{r-1} W_h^t \bigcup W_r^t(0) \bigcup W_r^t(1)\right).$$

On Fig 4 the elements of the set X are marked with black dots.

Note that the set X is constructed in such a way, that for any $u \le 2m$ there is a vertex $(u_1,u_2,u_3) \in X$, such that $u \le u_1 \le u+2r$. The vertex $W_r^{t_0-1}(1)$ with maximum first component among all vertices of X, is equal to $(i-1+2t_0((r+1)^3+r^3),j+s,k+1)$. Let $T_1 = W_{h_1}^{t_1}(d_1)$ be an element of X, such that

$$m \le 2t_1((r+1)^3+r^3)+h_1(s^2+s+2)+d_1s \le m+2r\,.$$

Thus $T_1 = (i+p,j+2h_1+d_1,k+d_1)$ for some $0 \le p \le 2r$. Suppose that $h_1 \le \frac{r-1}{2}$ and $d_1 \le r+1$. Then consider the vertex $T_2 = W_{2h_1}^{2t_1}(2d_1)$. Note that $T_2 = (i+2p,j+4h_1+2d_1,k+2d_1)$ and $T_2 \in X$. By Lemma 3.4

$$d(T_2,(i,j,k)) \le \max\{2p,4h_1+2d_1,2d_1\} \le 4r\,.$$

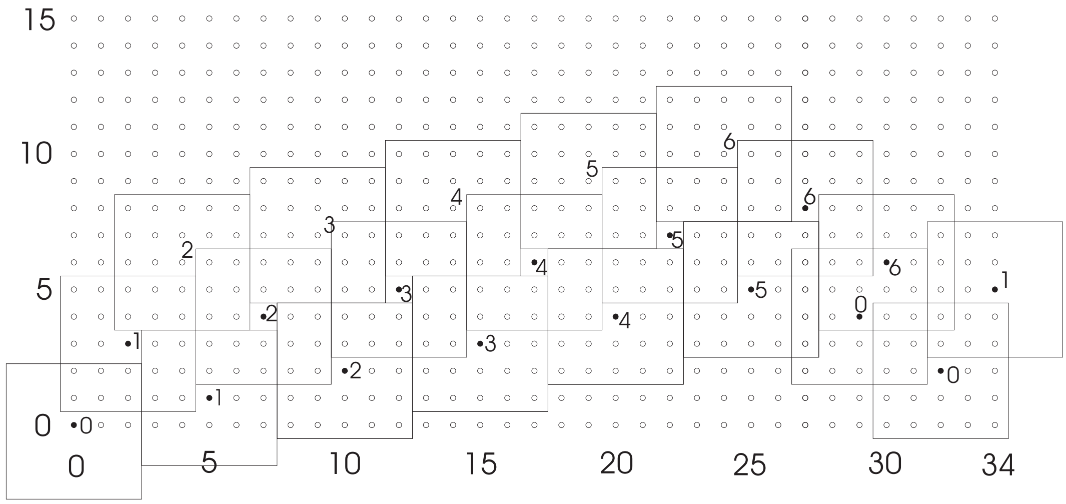

Figure 4: Set X

There are exactly two vertices in P satisfying above conditions, namely (i, j, k) and $(i + s, j + 1, k + 1)$. Thus T_2 equals to one of these two vertices. But since $2p$ is even and s is odd, we find $T_2 = (i, j, k)$. Since

$$(i, j, k) = (i + 4t_1((r + 1)^3 + r^3) + 2h_1(s^2 + s + 2) + 2d_1 s, j + 4h_1 + 2d_1, k + 2d_1)\,,$$

we find $d_1 = h_1 = 0$ and $p = 0$, which is a contradiction, since then

$$(i - 1, j + s, k + 1) = (i - 1 + 4t_1((r + 1)^3 + r^3), j + s, k + 1)$$

and

$$2m = 4t_1((r + 1)^3 + r^3) \geq 2t_0((r + 1)^3 + r^3)\,.$$

Suppose next $h_1 > \frac{r-1}{2}$ and $d_1 \leq r + 1$. Consider the vertex

$$T_3 = W^{2t_1+1}_{2h_1-r}(2d_1 - 1) \in X\,.$$

Then the first component of T_3 equals to

$$i + 2(2t_1 + 1)((r + 1)^3 + r^3) + (2h_1 - r)(s^2 + s + 2) + (2d_1 - 1)s$$

which is equal to

$$i + [4t_1((r + 1)^3 + r^3) + 2h_1(s^2 + s + 2) + 2d_1 s] + 1\,.$$

Thus

$$T_3 = (i + 2p + 1, j + 2(2h_1 - r) + 2d_1 - 1, k + 2d_1 - 1)$$

and hence by Lemma 3.4

$$d(T_3,(i,j,k)) \le \max\{2p, 4h_1 - 2r + 2d_1, 2d_1\} + 1 \le 4r + 1\,.$$

The only vertex satisfying these conditions is $(i+s, j+1, k+1)$. Therefore $d_1 = 1$, $h_1 = r/2$, and $p = r$. Thus we have

$$(i-1, j+s, k+1) = (i + [4t_1((r+1)^3 + r^3) + r(s^2+s+2) + 2s] - s, j+s, k+1)\,,$$

which is in turn equal to

$$(i - 1 + (4t_1+2)((r+1)^3 + r^3), j+s, k+1)\,.$$

Finally we have

$$2m = (4t_1+2)((r+1)^3+r^3) \ge 2t_0((r+1)^3+r^3)\,,$$

which is also a contradiction.

Using Lemma 3.4 and considering the vertex $W^{2t_1}_{2h_1+1}(2d_1 - 2r - 2)$ we analogously argue in the case $h_1 \le \frac{r-1}{2}$ and $d_1 > r+1$. Finally, for the case $h_1 > \frac{r-1}{2}$ and $d_1 > r+1$ consider the vertex $W^{2t_1+1}_{2h_1-r+1}(2d_1 - 2r - 3)$. □

Lemmas 3.3 and 3.5 are formulated and proved for the case when $R_1 \cup \{(i,j,k)\} \subseteq P$. By the symmetry both results also hold for all cases $R_i \cup \{(i,j,k)\} \subseteq P$, $(i = 2, \ldots, 8)$. Therefore, combining these two lemmas we get:

Corollary 3.6 *Let $r \ge 1$, $m,n,\ell \ge 2r+1$, and let P be an r-perfect code of $C_m \times C_n \times C_\ell$. Then m is a multiple of $r^3 + (r+1)^3$.*

By the commutativity of the direct product the result of Corollary 3.6 can also be applied to the other two cycles of the product. We have thus arrived to the main result of this section:

Theorem 3.7 *Let $r \ge 1$, $m,n,\ell \ge 2r+1$, and $G = C_m \times C_n \times C_\ell$. Then each connected component of G contains an r-perfect code if and only if m, n, and ℓ are each multiple of $r^3 + (r+1)^3$.*

Finally, combining Theorem 3.7 with Theorem 1.1 (ii) we can also state:

Corollary 3.8 *Let $r \ge 1$, $m,n,\ell \ge 2r+1$, and $G = C_m \times C_n \times C_\ell$. Then each connected component of G can be partitioned into r-perfect codes if and only if m, n, and ℓ are each multiple of $r^3 + (r+1)^3$.*

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