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Preprint series, Vol. 42 (2004), 920

INVARIANT SUBSPACES, DUALITY, AND COVERS OF THE PETERSEN GRAPH

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ISSN 1318-4865

May 17, 2004

Ljubljana, May 17, 2004

Invariant Subspaces, Duality, and Covers of the Petersen Graph

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May 14, 2004

Abstract

A general method for finding elementary abelian regular covering projections of finite connected graphs is applied to the Petersen graph. As a result, a complete list of pairwise non-isomophic elementary abelian covers admitting a lift of a vertex-transitive group of automorphisms is given. The resulting graphs are explicitly described in terms of voltage assignments.

1 Introduction

Covering techniques have long been known as a powerful tool in topology, group theory, and graph theory. They are of particular importance when symmetry properties of objects are analysed. For example, in his seminal work [4], Djoković used graph covers (and in particular, lifts of automorphisms along covering projections) to study s-arc transitivity of graphs. Among other, he showed that if an s-arc transitive group of automorphisms lifts along a regular covering projection, then the covering graph is at least s-arc transitive. Using this fact he constructed the first infinite family of 5-arc transitive cubic graphs arising as elementary abelian regular covers of Tutte's 8-cage. (A regular covering projection is called *elementary abelian* if the group of covering transformations is elementary abelian.) Elementary abelian covering projections are fairly frequently encountered and have been studied by several authors (see for example [2, 8, 20]). For instance, let a connected graph X admit a solvable group G of automorphisms. Then a minimal normal subgroup N of G is elementary abelian. Further, if the vertex-stabiliser N_v is trivial for every vertex v, then X is a regular cover (with N as the group of covering transformation) over the corresponding quotient graph X_N (which admits a solvable group of automorphisms isomorphic to G/N). This fact suggests an inductive approach to graphs admitting solvable automorphism groups (for applications see [1, 12, 15]).

The problem of lifting automorphisms has first been adressed in the context of topological spaces, and is well understood at a very general level. However, in a more specific setting of graphs, these general results do not provide satisfactory techniques to cope with concrete examples and specific question arising when studying symmetry properties from a combinatorial point of view.

*The authors were supported in part by 'Ministrstvo za šolstvo, znanost in šport RS', projects Z1-3124 and Z1-4186. The second author would like to thank the US Department of State and the Fulbright Scholar Program who sponsored his visit to Northern Arizona University in spring 2004.

A number of papers dealing with this problem have been published (see for instance [5, 6, 14, 17, 18, 19], to mention just a few that appeared recently). In terms of content they range from general considerations to more concrete applications in the field of graphs and maps on surfaces, dealing with enumeration, constructions of infinite families of graphs with specific symmetry properties, etc.

A thorough treatment of lifting graph automorphisms, together with a combinatorial approach in terms of voltage assignments, was given by Nedela, Škoviera and the frist author in [11]. Along these lines, an indept analysis of lifting automorphisms along elementary abelian regular covering projections was undertaken by Marušič and the authors in [13], where it was shown that the problem can be described and solved in a purely algebraic way. In particular, the problem was reduced to finding invariant subspaces of matrix groups over prime fields, linearly representing groups of graph automorphisms.

The aim of this article is twofold. First, to shed new light on some of the results in [13], stressing the duality principle between invariant subspaces of matrices and their trasposes. And second, to find all vertex-transitive elementary abelian covers of the Petersen graph, a graph whose unique properties are a source of numerous exceptions and counterexamples throughout graph theory in general. We remark further the fact that all elementary abelian covers of the Petersen graph are 3-edge-colourable was used by the second author in order to prove, by induction, that every connected cubic graph admitting a vertex transitive solvable group of automorphisms (with the sole exception of the Petersen graph itself) is 3-edge-colourable (see [15]).

2 Preliminaries

Graphs and coverings A *graph* is an ordered pair $X = (V, \sim)$, where $V(X) = V$ is a nonempty set of *vertices* and $\sim$ is an irreflexive symmetic relation on V, called *adjacency. Edges* of X are unordered pairs $E(X) = \{uv \mid u \sim v\}$ of adjacent vertices while *arcs* are the corresponding ordered pairs $A(X) = \{(u, v) \mid u \sim v\}$. A *morphism* of graphs $Y \to X$ is a function $V(Y) \to V(X)$ mapping adjacent vertices to adjacent vertices, with composition denoted by $\circ$. In particular, the *automorphism group* $\mathrm{Aut}\, X \leq \mathrm{Sym}\, V(X)$ of a graph X is the subgroup of all adjacency preserving permutations of $V(X)$, with the product of permutations defined by $\alpha\beta = \alpha \circ \beta$. Let $f \colon Y \to X$ and $f' \colon Y' \to X'$. A pair of morphisms $g_X \colon X \to X'$ and $g_Y \colon Y \to Y'$ such that $g_X \circ f = f' \circ g_Y$ is a *morphism* $(g_X, g_Y) \colon f \to f'$. We also say that g_X *lifts* to g_Y (and that g_Y *projects* to g_X) along (f, f') – see the commutative diagram in Figure 1.

$$\begin{array}{ccc} Y & \xrightarrow{\;g_Y\;} & Y' \\ f\big\downarrow & & \big\downarrow f' \\ X & \xrightarrow{\;g_X\;} & X' \end{array}$$

Figure 1.

If g_X and g_Y are isomorphisms, then $(g_X, g_Y) \colon f \to f'$ is called an *isomorphism*, and $f' = g_X \circ f \circ g_Y^{-1}$ is denoted by f^{g_X, g_Y}. In particular, an isomorphism is called an *equivalence* when $X = X'$ and $g_X = \mathrm{id}$. A pair of automorphisms $(g_X, g_Y) \colon f \to f$ is an *automorphism* of f. If $f \colon Y \to X$ is a morphism such that every element of a group $G \leq \mathrm{Aut}\, X$ lifts along (f, f) (*along* f for short) we say that f is G*-admissible*. The collection of all lifts of all elements of G constitutes a group $\tilde{G} \leq \mathrm{Aut}\, Y$, called the *lift of* G. If f is G-admissible for a vertex-transitive group G, then f is *vertex-transitive*. By $\mathrm{Ker}\, f$ we denote the lift of the trivial group along f, that is, $\mathrm{Ker}\, f = \{\alpha \in \mathrm{Aut} Y \mid f \circ \alpha = f\}$.

Let X be a connected graph. Recall that a permutation group is *semiregular* if all its vertex stabilisers are trivial. A surjective morphism $\wp\colon \tilde{X} \to X$ is called a *regular covering projection* if there exists a semiregular subgroup $\mathrm{CT}(\wp) \le \mathrm{Aut}\, \tilde{X}$, (called the *group of covering transformations*) whoes vertex orbits coincide with the sets $\wp^{-1}(v)$, $v \in V(X)$ (called *fibres*). Note that if the *covering graph* $\tilde{X}$ is connected, then $\mathrm{Ker}\, \wp = \mathrm{CT}(\wp)$. Furhter, if $\mathrm{CT}(\wp)$ is isomorphic to an elementary abelian p-group, then the covering projection is called *p-elementary abelian.*

Regular covering projections are usually studied up to equivalence, or possibly, up to isomorphism. When considering G-admissible covers the following proposition will be frequently referred to. Its proof is straightforward and is omitted.

Proposition 2.1 *Let $\wp\colon \tilde{X} \to X$ be a regular covering projection, let $\alpha\colon X \to X'$, $\tilde{\alpha}\colon \tilde{X} \to \tilde{X}'$ be graph isomorphisms, and let $G \le \mathrm{Aut}\, X$. Then $\wp$ is G-admissible if and only if $\wp^{\alpha,\tilde{\alpha}}\colon \tilde{X}' \to X'$ is $(\alpha\, G\, \alpha^{-1})$-admissible. In particular, the following holds.*

(a) *If $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$ are isomorphic regular covering projections, then $\wp$ is G-admissible if and only if $\wp'$ is G'-admissible for some $(\mathrm{Aut}\, X)$-conjugate G' of G. Moreover, if $\wp$ and $\wp'$ are equivalent, then $\wp$ is G-admissible if and only if $\wp'$ is G-admissible.*

(b) *If $\wp\colon \tilde{X} \to X$ is G-admissible and G' is conjugate to G in $\mathrm{Aut}\, X$, then there exists a regular covering projection $\wp'\colon \tilde{X}' \to X$ isomorphic to $\wp$ such that $\wp'$ is G'-admissible.* ■

Regular covering projections, combinatorially Let X be a connected graph and N an (abstract) group, called the *voltage group.* Assign to each arc (u, v) of X a *voltage* $\zeta(u, v) \in N$ so that $\zeta(v, u) = (\zeta(u, v))^{-1}$. Let $\mathrm{Cov}(X; \zeta)$ be the *derived graph* with vertex set $V \times N$ and adjacency relation defined by $(u, a) \sim (v, a\,\zeta(u, v))$ for $u \sim v$ in X. Then the projection onto the first coordinate is a regular covering projection $\wp_\zeta\colon \mathrm{Cov}(X; \zeta) \to X$, where $\mathrm{CT}(\wp_\zeta)$ arises from the action of N via left multiplication on itself. Moreover, it can be shown that each regular covering projection $\wp\colon \tilde{X} \to X$ is equivalent to $\wp_\zeta\colon \mathrm{Cov}(X, \zeta) \to X$ for some voltage assignment ζ valued in $N \cong \mathrm{CT}(\wp)$. Voltage assignments ζ and ζ' are called *equivalent* (*isomorphic*) if the derived covering projections $\wp_\zeta$ and $\wp_{\zeta'}$ are equivalent (isomorphic). For an extensive treatment of graph coverings we refer the reader to [7].

Now let ζ be a voltage assignment valued in an elementary abelian group $\mathbb{Z}_p^d$. The extension of the voltage assignment to all walks in X (defined by $\zeta(v_0, v_1, \ldots, v_n) = \zeta(v_0, v_1) + \zeta(v_1, v_2) \ldots + \zeta(v_{n-1}, v_n)$) induces a $\mathbb{Z}_p$-linear mapping $\bar{\zeta}\colon H_1(X; \mathbb{Z}_p) \to \mathbb{Z}_p^d$. Conversely, given a $\mathbb{Z}_p$-linear mapping $f\colon H_1(X; \mathbb{Z}_p) \to \mathbb{Z}_p^d$, there exists a voltage assignment ζ on X valued in $\mathbb{Z}_p^d$ such that $\bar{\zeta} = f$. Observe that the derived graph $\mathrm{Cov}(X; \zeta)$ is connected if and only if $\bar{\zeta}$ is surjective, and that, assumming connectedness, ζ and ζ' give rise to equivalent covering projections if and only if $\mathrm{Ker}\, \bar{\zeta} = \mathrm{Ker}\, \bar{\zeta}'$. Hence there is a bijective correspondence between linear subspaces of $H_1(X; \mathbb{Z}_p)$ and equivalence classes of p-elementary abelian regular covering projections of connected graphs.

Lifting automorphisms, combinatorially Let $\wp\colon \tilde{X} \to X$ be a G-admissible regular covering projection. By Proposition 2.1, if $\wp'\colon \tilde{X}' \to X$ is isomorphic to $\wp$, then $\wp'$ might not be G-admissible; however, if $\wp'$ is equivalent to $\wp$, then $\wp'$ is G-admissible. Hence to determine, for a given graph X and a group $G \le \mathrm{Aut}\, X$, all G-admissible regular covering projections $\tilde{X} \to X$ up to equivalence it suffices to find all nonequivalent voltage assignments ζ on X such that G lifts along $\wp_\zeta \colon \mathrm{Cov}(X;\zeta) \to X$. For a general discussion, see [11]. In the case of elementary abelian covers, the lifting criterion can be refined as described in the sequel, see [13] for a more extensive treatment. Denote by

$$G^{\#_h} = \{\alpha^{\#_h} \mid \alpha \in G\} \leq \mathrm{GL}(H_1(X;\mathbb{Z}_p))$$

the induced action of G on $H_1(X;\mathbb{Z}_p)$. Then the following holds (c.f. [13, Corollary 6.5]).

Theorem 2.2 *Let $\wp_\zeta \colon Cov(X;\zeta) \to X$ be an elementary abelian regular covering projection and let $G \leq \mathrm{Aut}\, X$. Then $\wp_\zeta$ is G-admissible if and only if $\mathrm{Ker}\, \bar{\zeta}$ is invariant under the action of $G^{\#_h}$. Hence for each prime p there is a $1-1$ correspondence between $G^{\#_h}$-invariant subspaces of $H_1(X;\mathbb{Z}_p)$ and equivalence classes of G-admissible p-elementary abelian regular covering projections of connected graphs.* ■

By Theorem 2.2, the problem of finding, up to equivalence, all G-admissible elementary abelian regular covers such that the derived graph is connected, is reduced to a purely algebraic question of finding invariant subspaces of linear groups.

However, finding an explicit voltage assignment ζ for each equivalence class of covering projections can be simplified by considering a dualized version of the above $1-1$ correspondence. Namely, invariance of $\mathrm{Ker}\, \bar{\zeta}$ for $\alpha^{\#_h} \in G^{\#_h}$ is equivalent to existence of $\alpha^{\#} \in \mathrm{GL}(\mathbb{Z}_p^d)$ making the left diagram in Figure 2 commutative. This, in turn, is equivalent to existence of a linear mapping $\alpha^{\#^*} \in \mathrm{GL}((\mathbb{Z}_p^d)^*)$ on the space of linear functionals making the right diagram in Figure 2 commutative. Finally, the latter is equivalent to $\mathrm{Im}\, \bar{\zeta}^*$ being invariant for the dual representation $G^{\#_h^*}$.

$$\begin{array}{ccc} H_1(X;\mathbb{Z}_p) & \xrightarrow{\alpha^{\#_h}} & H_1(X;\mathbb{Z}_p) \\ {\scriptstyle\bar{\zeta}}\downarrow & & \downarrow{\scriptstyle\bar{\zeta}} \\ \mathbb{Z}_p^d & \xrightarrow{\alpha^{\#}} & \mathbb{Z}_p^d \end{array} \qquad\qquad \begin{array}{ccc} H_1(X;\mathbb{Z}_p)^* & \xleftarrow{\alpha^{\#_h^*}} & H_1(X;\mathbb{Z}_p)^* \\ {\scriptstyle\bar{\zeta}^*}\uparrow & & \uparrow{\scriptstyle\bar{\zeta}^*} \\ (\mathbb{Z}_p^d)^* & \xleftarrow{\alpha^{\#^*}} & (\mathbb{Z}_p^d)^*. \end{array}$$

Figure 2.

Observe that $\bar{\zeta}$ is surjective if and ony if its dual mapping $\bar{\zeta}^*$ is injective. Consequently, the $1-1$ correspondence in Theorem 2.2 can be replaced by a $1-1$ correspondence

$$\Psi_G^* \colon \mathrm{Inv}(G^{\#_h^*}) \to \mathcal{C}_G^p(X) \tag{1}$$

between the set of all $G^{\#_h^*}$-invariant subspaces of $H_1(X;\mathbb{Z}_p)^*$ (dentoted by $\mathrm{Inv}(G^{\#_h^*})$) and the set of all equivalence classes of G-admissible p-elementary abelian regular covering projections of connected graphs (denoted by $\mathcal{C}_G^p(X)$). Once the set of $G^{\#_h^*}$-invariant subspaces $\mathrm{Inv}(G^{\#_h^*})$ is found, the set of corresponding voltage assignments (one for each equivalence class in $\mathcal{C}_G^p(X)$) can easily be computed by considering matrix representations of all linear mappings involved as follows.

For a spanning tree $\mathcal{T}$ of a graph X choose a set $\{x_1, \ldots, x_r\} \subseteq A(X)$ containing exactly one arc from each edge in $E(X \setminus \mathcal{T})$, and let $\mathcal{B}_\mathcal{T} = [C_1, C_2, \ldots, C_r]$ be the corresonding basis of $H_1(X;\mathbb{Z}_p)$. Next, let $M_G \leq \mathbb{Z}_p^{r,r}$ be the matrix-representation of $G^{\#_h}$ and $Z \in \mathbb{Z}_p^{r,d}$ the matrix representation of $\bar{\zeta}$ with respect to the basis $\mathcal{B}_\mathcal{T}$ of $H_1(X;\mathbb{Z}_p)$ and the standard basis Σ of the voltage group $\mathbb{Z}_p^d$. Then the group M_G^t (consisting of all transposes of matrices in M_G) and the transposed matrix Z^t represent the dual group $G^{\#_h^*}$ and the dual linear mapping $\bar{\zeta}^*$ with respect to the dual bases $\mathcal{B}_\mathcal{T}^*$ and Σ^*. Taking into account that the subspace $\mathrm{Im}\, \zeta^*$ is spanned by the columns of the matrix Z^t, we have actually proved part (a) of the following theorem. (Note that the condition $\mathrm{rank}(Z) = d$ below is equivalent to connectedness of the derived covering graph $\mathrm{Cov}(X;\zeta)$.)

Theorem 2.3 [13, Proposition 6.3, Corollary 6.5] *Let $\mathcal{T}$ be a spanning tree of a connected graph X and let the set $\{x_1, x_2, \ldots, x_r\} \subseteq A(X)$ contain exacty one arc from each cotree edge. Let $\zeta\colon A(X) \to \mathbb{Z}_p^{d,1}$ be a voltage assignment on X which is trivial on $\mathcal{T}$, and let the matrix $Z \in \mathbb{Z}_p^{d,r}$ with columns*

$$\zeta(x_1), \zeta(x_2), \ldots, \zeta(x_r)$$

have rank d. Then the following holds.

- **(a)** *A group $G \leq AutX$ lifts along $\wp_\zeta\colon Cov(X;\zeta) \to X$ if and only if the columns of Z^t form a basis of a M_G^t-invariant d-dimensional subspace $\mathcal{S}(\zeta) \leq \mathbb{Z}_p^{r,1}$.*
- **(b)** *If $\zeta'\colon A(X) \to \mathbb{Z}_p^{d,1}$ is another voltage assignment satisfying* **(a)**, *then $\wp_{\zeta'}$ is equivalent to $\wp_\zeta$ if and only if $\mathcal{S}(\zeta) = \mathcal{S}(\zeta')$. Moreover, $\wp_{\zeta'}$ is isomorphic to $\wp_\zeta$ if and only if there exists an automorphism $\alpha \in \mathrm{Aut}\,X$ such that the matrix M_α^t maps $\mathcal{S}(\zeta')$ onto $\mathcal{S}(\zeta)$.* ■

By Theorem 2.3 one can find, up to equivalence, all G-admissible p-elementary abelian regular covering projections of a graph X (such that the respective covering graph is connected) – by first finding a basis for each invariant subspace of the dual representation M_G^t, writing components of the respective base vectors in rows, and then reading off the voltages of cotree arcs as columns. Moreover, the choice of a spanning tree as well as choosing a basis for an invariant subspace is irrelevant as long as we consider covering projections up to equivalence. Also, Theorem 2.3 enables us to further reduce the obtained (non-equivalent) covering projections up to isomorphism.

Finding vertex-transitive covers Obviously, a regular covering projection is vertex-transitive if and only if it is admissible for some minimal vertex-transitive group of automorphisms. Moreover, in order to find, up to isomorphism, all vertex-transitive covering projections of X it suffices to take one minimal vertex-transitive group from each conjugacy class in $\mathrm{Aut}\,X$ (see Proposition 2.1).

In the case of vertex-transitive elementary abelian covers, all invariant subspaces of the dual representaion M_H^t (see Theorem 2.3) need to be calculated for each minimal vertex-transitive subgroup $H \leq \mathrm{Aut}\,X$. To reduce the obtained projections up to isomorphism one has to consider the action of $M_{\mathrm{Aut}\,X}^t$ on the set of all of these subspaces, and then take one representative subspace from each orbit. (Note that $M_{\mathrm{Aut}\,X}^t$ might not act on the invariant subspaces of a chosen subgroup H, but it does act on the invariant subspaces of all its conjugate subgroups.) We illustrate this procedure on a concrete example, namely, the Petersen graph.

3 Vertex-transitive covers of the Petersen graph

Let $V^{(2)}$ denote the set of all two-element subsets of a set V. For the purpose of this paper we define the *Petersen graph* $\mathcal{P}$ as the graph with vertex set $V(\mathcal{P}) = \mathbb{Z}_5^{(2)}$ and adjacency relation $u \sim v$ given by $u \cap v = \emptyset$. It is generally known that the full automorphism group $\mathrm{Aut}\,\mathcal{P}$ of the Petersen graph is isomorphic to the symmetric group $S_5 = \mathrm{Sym}\,\mathbb{Z}_5$, acting in a natural way on $\mathbb{Z}_5^{(2)}$ (in the sequel, automorphisms of the Petersen graph are actually identified with the corresponding permutations of $\mathbb{Z}_5$). In order to describe all vertex-transitive subgroups of $\mathrm{Aut}\,\mathcal{P}$ we have chosen the following three generators:

$$\begin{aligned}\rho &= (0,1,2,3,4),\\ \tau &= (1,2,4,3),\\ \alpha &= (2,3,4).\end{aligned}$$

It can be readily verified that each proper vertex-transitive subgroup of $\mathrm{Aut}\,\mathcal{P}$ is conjugate in $\mathrm{Aut}\,\mathcal{P}$ either to the group $\langle\rho,\tau\rangle$ or to the group $\langle\rho,\alpha\rangle$. Hence both of them are also maximal among proper subgroups of $\mathrm{Aut}\,\mathcal{P}$. Note that $\langle\rho,\tau\rangle$ is isomorphic to the affine group $\mathrm{AGL}(1,5) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ with presentation $\langle\rho,\tau \mid \rho^5 = \tau^4 = 1,\ \tau\rho\tau^{-1} = \rho^2\rangle$, whereas $\langle\rho,\alpha\rangle$ is isomorphic to the alternating group A_5. Thus, in order to determine all vertex-transitive elementary abelian covering projections up to isomorphism, it suffices (by Proposition 2.1) to find those which are $\langle\rho,\tau\rangle$-admissible or $\langle\rho,\alpha\rangle$-admissible. By Theorem 2.3, this is equivalent to finding all invariant subspaces of the representations $\langle\rho,\tau\rangle^{\#^*_h}$ and $\langle\rho,\alpha\rangle^{\#^*_h}$, and reducing them further by the action of $(\mathrm{Aut}\,\mathcal{P})^{\#^*_h}$.

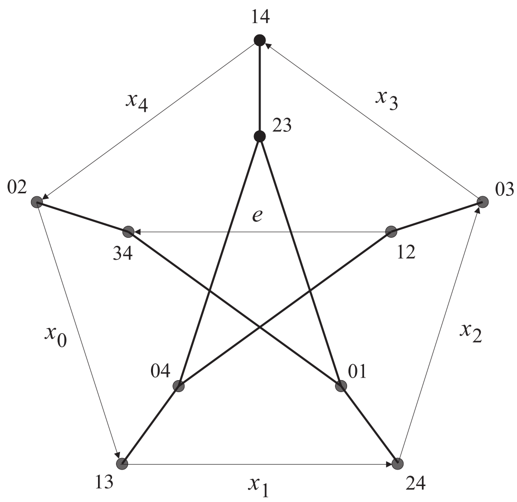

Figure 3

Let $\mathcal{T}$ be the spanning tree consisting of all the *spokes* $\{\{i,i+2\},\{i+3,i+4\}\}$, $i \in \mathbb{Z}_5$, and four *inner edges* $\{\{i,i+1\},\{i+2,i+3\}\}$, $i \in \mathbb{Z}_5 \setminus \{1\}$, (see Figure 3). Moreover, let

$$\vec{e} = (\{1,2\},\{3,4\}) \quad \text{and} \quad \vec{x}_i = (\{i,i+2\},\{i+1,i+3\}),\ i \in \mathbb{Z}_5, \tag{2}$$

be six cotree arcs (one from each cotree edge). With this notation, the following theorem holds.

Theorem 3.1 *Each vertex-transitive p-elementary abelian covering projection of the Petersen graph (with the covering graph being connected) is isomorphic to a derived covering projection associated with one of the pairwise non-isomorphic voltage assignemnts given in Tables 4 and 5.*

Table 4: Vertex-transitive p-elementary abelian covers of the Petersen graph, $p \neq 5$

	inv. subsp.	$\zeta(\vec{e})$	$\zeta(\vec{x}_0)$	$\zeta(\vec{x}_1)$	$\zeta(\vec{x}_2)$	$\zeta(\vec{x}_3)$	$\zeta(\vec{x}_4)$	admissible for	existence condition
1	$\langle u_1 \rangle$	$(3-\iota)$	(-2)	$(1-\iota)$	(-2)	$(1-\iota)$	$(1-\iota)$	$\mathrm{AGL}(1,5)$	$p \equiv 1 \pmod 4$; $\iota^2 = -1$
2	$\langle u_2 \rangle$	$(3+\iota)$	(-2)	$(1+\iota)$	(-2)	$(1+\iota)$	$(1+\iota)$	$\mathrm{AGL}(1,5)$	$p \equiv 1 \pmod 4$; $\iota^2 = -1$
3	$\langle v_1 \rangle$	(1)	(0)	(1)	(0)	(1)	(1)	$\mathrm{Aut}\,\mathcal{P}$	$p = 2$
4	$\langle v_2 \rangle$	(0)	(1)	(1)	(1)	(1)	(1)	A_5	$p = 2$
5	K_0	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	if $p = 2$: $\mathrm{Aut}\,\mathcal{P}$; otherwise: $\mathrm{AGL}(1,5)$	none
6	K_1^A	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}\nu_2\\-1\\\nu_1\end{pmatrix}$	$\begin{pmatrix}0\\-\nu_1\\-\nu_1\end{pmatrix}$	$\begin{pmatrix}1\\\nu_1\\-1\end{pmatrix}$	A_5	$p \equiv \pm 1 \pmod 5$; ν_1, ν_2 roots of $\nu^2 + \nu - 1 = 0$
7	K	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}$	$\mathrm{AGL}(1,5)$	none
8	$\langle u_1, K \rangle$	$\begin{pmatrix}3-\iota\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1-\iota\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1-\iota\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1-\iota\\-1\\-1\\-1\\-1\end{pmatrix}$	$\mathrm{AGL}(1,5)$	$p \equiv 1 \pmod 4$; $\iota^2 = -1$
9	$\langle u_2, K \rangle$	$\begin{pmatrix}3+\iota\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1+\iota\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1+\iota\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1+\iota\\-1\\-1\\-1\\-1\end{pmatrix}$	$\mathrm{AGL}(1,5)$	$p \equiv 1 \pmod 4$; $\iota^2 = -1$
10	$\langle v_1, K \rangle$	$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$	$\mathrm{AGL}(1,5)$	$p = 2$
11	$\mathbb{Z}_5^6$	$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$	$\mathrm{Aut}\,\mathcal{P}$	none

Table 5: Vertex-transitive 5-elementary abelian covers of the Petersen graph

row	invariant subspace	$\zeta(\vec{e})$	$\zeta(\vec{x}_0)$	$\zeta(\vec{x}_1)$	$\zeta(\vec{x}_2)$	$\zeta(\vec{x}_3)$	$\zeta(\vec{x}_4)$	admissible for
1	$W_0(\infty)$	(1)	(3)	(4)	(3)	(4)	(4)	$\mathrm{AGL}(1,5)$
2	$W_0(0)$	(0)	(1)	(1)	(1)	(1)	(1)	$\mathrm{AGL}(1,5)$
3	$W_1(\infty)$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\mathrm{AGL}(1,5)$
4	$W_1(0)$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}4\\2\end{pmatrix}$	$\begin{pmatrix}3\\3\end{pmatrix}$	$\begin{pmatrix}2\\4\end{pmatrix}$	$\mathrm{AGL}(1,5)$
5	$W_2(\infty)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}3\\4\\2\end{pmatrix}$	$\begin{pmatrix}3\\3\\3\end{pmatrix}$	$\begin{pmatrix}2\\2\\4\end{pmatrix}$	$\mathrm{AGL}(1,5)$
6	$W_2(0)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\2\\3\end{pmatrix}$	$\begin{pmatrix}3\\2\\1\end{pmatrix}$	$\mathrm{AGL}(1,5)$
7	$W_2(1)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}4\\4\\2\end{pmatrix}$	$\begin{pmatrix}1\\3\\3\end{pmatrix}$	$\begin{pmatrix}3\\2\\4\end{pmatrix}$	$\mathrm{AGL}(1,5)$
8	$W_2(2)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\4\\2\end{pmatrix}$	$\begin{pmatrix}2\\3\\3\end{pmatrix}$	$\begin{pmatrix}0\\2\\4\end{pmatrix}$	$\mathrm{AGL}(1,5)$
9	$W_2(3)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\4\\2\end{pmatrix}$	$\begin{pmatrix}4\\3\\3\end{pmatrix}$	$\begin{pmatrix}4\\2\\4\end{pmatrix}$	$\mathrm{AGL}(1,5)$
10	$W_2(4)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}2\\4\\2\end{pmatrix}$	$\begin{pmatrix}0\\3\\3\end{pmatrix}$	$\begin{pmatrix}1\\2\\4\end{pmatrix}$	$\mathrm{Aut}\,\mathcal{P}$
11	$W_3(\infty)$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}4\\1\\2\\3\end{pmatrix}$	$\begin{pmatrix}4\\3\\2\\1\end{pmatrix}$	$\mathrm{AGL}(1,5)$
12	$W_3(0)$	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}4\\4\\4\\4\end{pmatrix}$	$\mathrm{AGL}(1,5)$
13	$W_4(\infty)$	$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}3\\4\\4\\4\\4\end{pmatrix}$	$\mathrm{AGL}(1,5)$
14	$W_4(4)$	$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}2\\4\\4\\4\\4\end{pmatrix}$	$\mathrm{AGL}(1,5)$
15	$\mathbb{Z}_5^6$	$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$	$\mathrm{Aut}\,\mathcal{P}$

4 The Proof

In this section we carry out the procedure described in Section 2 to prove Theorem 3.1. Let $\mathcal{B}$ be the ordered basis of $H_1(\mathcal{P};\mathbb{Z}_p)$ associated with the spanning tree $\mathcal{T}$ and the six cotree arcs as in (2). Abusing the notation we shall use the symbols $\vec{e}$ and $\vec{x}_i$, $i = 0,\ldots,4$ (in that order) to denote both, the arcs of $\mathcal{P}$ and the corresponding cycles in $\mathcal{B}$. Let R, T and A be the transposes of the matrices which represent the linear transformations $\rho^{\#_h}$, $\tau^{\#_h}$ and $\alpha^{\#_h}$ relative to $\mathcal{B}$, respectively. Recall that the rows of these matrices are obtained by letting the automorphisms ρ,τ and α act on $\mathcal{B}$. For example, the permutation τ maps the cycle

$$(\{0,2\},\{1,3\},\{0,4\},\{2,3\},\{0,1\},\{3,4\},\{0,2\}),$$

corresponding to $\vec{x}_0$, to the cycle

$$(\{0,4\},\{1,2\},\{0,3\},\{1,4\},\{0,2\},\{1,3\},\{0,4\}).$$

Since the latter is the sum of the base cycles corresponding to $\vec{x}_3$, $\vec{x}_4$ and $\vec{x}_0$, the second row of T is $(0,1,0,0,1,1)$. By similar computations we get

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

The rest of the procedure in purely algebraic and amounts to finding invariant subspaces of matrix groups $\langle R,T\rangle$ and $\langle R,A\rangle$. Clearly, the full space $\mathbb{Z}_p^6$ is $\langle R,T,A\rangle$-invariant, and gives rise to $(\mathrm{Aut}\,\mathcal{P})$-admissible "homological" covering projections (see rows 11 and 15 of Tables 4 and 5).

We remark that a careful choice of a spanning tree (and thus a basis of the homology group) can simplify computations to some extent - but not significantly. In our case, we could have chosen $\mathcal{T}$ more optimally (for example, by letting $\mathcal{T}$ contain the spoke $\{\{1,4\},\{2,3\}\}$ and all the edges of the inner and outer cycles with the exception of $\{\{1,2\},\{3,4\}\}$ and $\{\{1,3\},\{2,4\}\}$). However, our deliberate non-optimal choice of $\mathcal{T}$ was done in order to show the robustness of the method.

4.1 Finding invariant subspaces – general remarks

We start by recalling some general facts from linear algebra (see for example [9]). Let $A \in \mathbb{F}^{n,n}$ be an $n\times n$ matrix over a field $\mathbb{F}$, acting as a linear transformation $\underline{x} \mapsto A\underline{x}$ on the column vector space $\mathbb{F}^{n,1}$. Let $\kappa_A(x) = f_1(x)^{n_1} f_2(x)^{n_2} \ldots f_k(x)^{n_k}$ be the characteristic polynomial and $m_A(x) = f_1(x)^{s_1} f_2(x)^{s_2} \ldots f_k(x)^{s_k}$ the minimal polynomial of A where $f_j(x)$, $j = 1,\ldots,k$, are pairwise distinct irreducible factors over $\mathbb{F}$. Then $\mathbb{F}^{n,1}$ can be written as a direct sum of the A-invariant subspaces

$$\mathbb{F}^{n,1} = \mathrm{Ker}\, f_1(A)^{s_1} \oplus \mathrm{Ker}\, f_2(A)^{s_2} \oplus \ldots \oplus \mathrm{Ker}\, f_k(A)^{s_k}.$$

Moreover, all A-invariant subspaces can be found by first considering the invariant subspaces of $\mathrm{Ker}\, f_j(A)^{s_j}$, $j = 1,\ldots,k$, and then taking direct sums of some of these. In particular, the minimal ones are just the minimal A-invariant subspaces of $\mathrm{Ker}\, f_j(A)^{s_j}$, $j = 1,\ldots,k$. Now the subspace $\mathrm{Ker}\, f_j(A)^{s_j}$ has dimension $d_j n_j$, where $d_j = \deg\, f_j(x)$ is the degree of the polynomial $f_j(x)$. Its minimal A-invariant subspaces are cyclic of the form $\langle v, Av, \ldots, A^{d_j-1}v\rangle$, where $v \in \operatorname{Ker} f_j(A)$, and each such defines an increasing sequence of length at most s_j of nested invariant subspaces (at least one is precisely of length s_j). If $n_j > s_j$, then a variety of pairwise disjoint minimal cyclic subspaces exist in $\operatorname{Ker} f_j(A)^{s_j}$, and a unique one if $n_j = s_j$. In particular, if $n_j = s_j = 1$, then $\operatorname{Ker} f_j(A)$ itself is the only A-invariant subspace contained in $\operatorname{Ker} f_j(A)$ and hence minimal. Consequently, if $\kappa_A(x) = m_A(x)$ with all $n_j = s_j = 1$, then $\operatorname{Ker} f_j(A)$, $j = 1, \ldots, k$, are the only minimal A-invariant subspaces, and all others are direct sums of these.

As for finding invariant subspaces of (finite) group representations we recall Masche's theorem which states that if the characteristic $\operatorname{Char}\mathbb{F}$ of the field does not divide the order of the group, then the representation is completely reducible. In this case one essentially needs to find just the minimal common invariant subspaces of the generators. (This may still involve knowing all invariant subspaces of the generators, in view of the fact that a minimal invariant subspace for the whole group need not be minimal for neither of the individual generators – however, invariant subspaces of a generator are direct sums of the minimal ones for that generator). The remaining cases where $\operatorname{Char}\mathbb{F}$ divides the order of the group could be, technically, more difficult to analyse. In contrast with inherently infinite general problem, there are only finitely many such exceptional field characteristics.

The last important issue that we need to recall is factorisation of polynomials into irreducible factors. In particular, let $A \in \mathbb{Z}_p^{n,n}$ be an invertible matrix of order m, where m is coprime with p. Then the irreducible factors $f_j(x)$ of the minimal polynomial $m_A(x)$ appear with exponent $s_j = 1$, and since $m_A(x)$ is a divisor of $x^m - 1$ we first need to find the irreducible factors of $x^m - 1$ over the prime field $\mathbb{Z}_p$. In its splitting field, $x^m - 1$ factorizes as $x^m - 1 = \Pi_{j=1}^m (x - \xi^j)$, where ξ is a primitive m-th root of unity. The additive group $\mathbb{Z}_m = \oplus_{d|m}\{j \mid \operatorname{ord}(j) = d\}$ is a disjoint union (taken over all divisors of m) of subsets consisting of all elements of order d in $\mathbb{Z}_m$. Hence $x^m - 1 = \Pi_{d|m} C_d(x)$ where $C_d(x) = \Pi_{\operatorname{ord}(j)=d}(x - \xi^j)$ is the d-th cyclotomic polynomial; in particular $C_1(x) = x - 1$. Observe that each ξ^j, $\operatorname{ord}(j) = d$, is a primitive d-th root of unity; hence $C_d(x)$ can be written in the form $C_d(x) = \Pi_{k \in \mathbb{Z}_d^*}(x - \eta^k)$, where η is a primitive d-th root of unity. The above cyclotomic polynomials are pairwise coprime and belong to $\mathbb{Z}_p[x]$, see [10, Theorem 2.45]. Finally, to find the factorization of $C_d(x)$ into irreducible factors over $\mathbb{Z}_p$, let $p \equiv \bar{p} \pmod{d}$. Clearly, $\bar{p} \in \mathbb{Z}_d^*$. Denote by $P = \langle \bar{p} \rangle$ the subgroup of $\mathbb{Z}_d^*$ generated by $\bar{p}$, and let r be its order. The action of P on $\mathbb{Z}_d^*$ by multiplication has the cosets of P as its orbits. For $k \in \mathbb{Z}_d^*$, let

$$q_k(x) = (x - \eta^{k\bar{p}})(x - \eta^{k\bar{p}^2}) \ldots (x - \eta^{k\bar{p}^{r-1}}).$$

By letting k run through a fixed set of coset representatives we obtain $\phi(d)/r$ such polynomials of degree r. Their product is obviously $C_d(x)$, they all belong to $\mathbb{Z}_p[x]$, and are moreover irreducible over $\mathbb{Z}_p$, see [10, Theorem 2.47].

4.2 R-invariant subspaces

The characteristic polynomial of R is $\kappa_R(x) = (x - 1)(x^5 - 1)$ and its minimal polynomial is

$$m_R(x) = x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1).$$

Let $K_0 = \operatorname{Ker}(R - I)$ and $K = \operatorname{Ker}(R^4 + R^3 + R^2 + R + I)$. By straightforward computation we find that $K_0 = \langle v_1, v_2\rangle$ and $K = \langle b_1, b_2, b_3, b_4\rangle$, where

$$\begin{array}{rclcrcl} v_1 & = & (1,0,1,0,1,1)^t, & \quad & b_1 & = & (0,1,0,0,0,-1)^t, \\ v_2 & = & (0,1,1,1,1,1)^t, & & b_2 & = & (0,0,1,0,0,-1)^t, \\ & & & & b_3 & = & (0,0,0,1,0,-1)^t, \\ & & & & b_4 & = & (0,0,0,0,1,-1)^t. \end{array}$$

Clearly, K_0 and K are R-invariant, and each minimal R-invariant subspace is contained either in K_0 or in K. Since K_0 is an eigenspace, those contained in K_0 are exactly its 1-dimensional subspaces; these can be conveniently parameterized as

$$\begin{array}{rclcl} U(\infty) & = & \langle v_1 \rangle, & & \\ U(s) & = & \langle s v_1 + v_2 \rangle & = & \langle (s,1,1+s,1,1+s,1+s)^t \rangle, \quad s \in \mathbb{Z}_p. \end{array}$$

As for finding those minimal R-invariant subspaces contained in K we have to distinguish two cases according to whether $p = 5$ or not. Namely, if $p = 5$, then $m_R(x) = (x-1)^5$ and hence $K_0 \subseteq K$. Therefore, all minimal R-invariant subspaces have already been found above. However, this is not sufficient to determine the non-minimal ones since the group $\langle R \rangle$ is not completely reducible. On the other hand, if $p \neq 5$, then $K \cap K_0$ is trivial, implying that R might have non-trivial invariant subspaces other than those contained in K_0. But the group $\langle R \rangle$ is now completely reducible, and all remaining R-invariant subspaces are just direct sums of the minimal ones contained in K_0 or K.

Case $p \neq 5$.

To find the minimal R-invariant subspaces contained in K it proves useful to consider R as a matrix over the splitting field $\mathbb{F} = \mathbb{Z}_p(\xi)$, where ξ is a primitive 5-th root of unity. Then $m_R(x)$ splits over $\mathbb{F}$ into five distinct linear factors, namely

$$m_R(x) = (x-1)(x-\xi)(x-\xi^2)(x-\xi^3)(x-\xi^4).$$

Let $K_j = \mathrm{Ker}(R - \xi^j I)$, $j \in \{1,\ldots,4\}$, be the respective eigenspaces. From the structure of the characteristic and the minimal polynomial we infer that K_1, K_2, K_3 and K_4 are 1-dimensional. Therefore, the minimal R-invariant subspaces over $\mathbb{F}$ contained in K are exactly K_1, K_2, K_3 and K_4. By computation we obtain that $K_j = \langle z_j \rangle$, where

$$z_j \quad = \quad (0, \xi^j, \xi^{2j}, \xi^{3j}, \xi^{4j}, 1)^t, \quad j \in \{1,\ldots,4\}.$$

Which direct sums of these minimal subspaces over $\mathbb{F}$ can be represented over $\mathbb{Z}_p$ (and hence giving rise to minimal R-invariant subspaces over $\mathbb{Z}_p$) depends on the prime factorization of $f(x) = x^4 + x^3 + x^2 + x + 1$ over $\mathbb{Z}_p$. Now, over the prime field, $f(x)$ factorizes into $\phi(5)/r$ irreducible polynomials, where $r \neq 0$ is the order of $\bar{p}$ in $\mathbb{Z}_5^*$. This order can attain values 1, 2 or 4, depending on whether p is congruent to 1, -1 or ± 2 modulo 5, respectively. Therefore,

$$f(x) = \begin{cases} (x-\xi)(x-\xi^2)(x-\xi^3)(x-\xi^4) & p \equiv 1 \pmod 5 \\ (x^2 - \nu_1 x + 1)(x^2 - \nu_2 x + 1) & p \equiv -1 \pmod 5, \ \ \nu_1 = \xi + \xi^4, \ \nu_2 = \xi^2 + \xi^3 \\ (x^4 + x^3 + x^2 + x + 1) & p \equiv \pm 2 \pmod 5. \end{cases}$$

Note that when $p \equiv \pm 1 \pmod 5$, the two elements $\nu_1 = \xi + \xi^4$ and $\nu_2 = \xi^2 + \xi^3$ (which belong to $\mathbb{Z}_p$) satisfy the equation $\nu^2 + \nu - 1$, and can be expressed in the form $\nu_{1,2} = (-1 \pm \sqrt{3})/2$. In particular, $\nu_2 = -(\nu_1 + 1) = -1/\nu_1$.

SUBCASE $p \equiv 1 \pmod 5$.

Here $\mathbb{F} = \mathbb{Z}_p$. Therefore, $p+5$ minimal R-invariant subspaces exist, and all are 1-dimensional. These are: K_j, $j \in \{1, \ldots, 4\}$, and $U(s)$, $s \in \mathbb{Z}_p \cup \{\infty\}$.

SUBCASE $p \equiv -1 \pmod 5$.

Here $\mathbb{F}$ is a degree-2 extension of $\mathbb{Z}_p$. In addition to the 1-dimensional subspaces $U(s)$ contained in K_0, there are two minimal R-invariant subspaces contained in K. Namely, the 2-dimensional subspaces $L_i = \operatorname{Ker}(R^2 - \nu_i R + I)$, $i \in \{1, 2\}$. Their bases can be computed either directly or, via computation in the splitting field $\mathbb{F}$, by finding appropriate bases of $L_1 = K_1 \oplus K_4$ and $L_2 = K_2 \oplus K_3$ having coefficients in $\mathbb{Z}_p$. In either way we easily find that $L_i = \langle w_{i,1}, w_{i,2} \rangle$, $i \in \{1, 2\}$, where

$$\begin{aligned} w_{i,1} &= (0, 1, 0, -1, -\nu_i, \nu_i)^t, \\ w_{i,2} &= (0, 0, 1, \nu_i, -\nu_i, -1)^t. \end{aligned}$$

SUBCASE $p \equiv \pm 2 \pmod 5$.

Here $\mathbb{F}$ is a degree-4 extension of $\mathbb{Z}_p$. Besides the 1-dimensional subspaces $U(s)$ there is a single remaining minimal R-invariant subspace, namely, K itself.

This completes the analysis of minimal R-invariant subspaces when $p \neq 5$. By Masche's theorem, all other R-invariant subspaces are direct sums of the minimal ones.

CASE $p = 5$.

From the characteristic and the minimal polynomials $\kappa_R(x) = (x-1)^6$ and $m_R(x) = (x-1)^5$ we infer that the Jordan normal form of R consists of two elementary Jordan matrices, one of size 1 and one of size 5. This implies that there exists a strictly increasing nested sequence of length 5 of R-invariant subspaces

$$K_0 = W_0 \leq W_1 \leq W_2 \leq W_3 \leq W_4 = \mathbb{Z}_5^6,$$

where $W_j = \operatorname{Ker}(R - I)^{j+1}$. By choosing a Jordan basis, say

$$\begin{aligned} t_0' &= (1, 3, 4, 3, 4, 4)^t, \\ t_0 &= (0, 1, 1, 1, 1, 1)^t, \\ t_1 &= (0, 0, 1, 2, 3, 4)^t, \\ t_2 &= (0, 0, 0, 1, 3, 1)^t, \\ t_3 &= (0, 0, 0, 0, 1, 4)^t, \\ t_4 &= (0, 0, 0, 0, 0, 1)^t, \end{aligned}$$

we have that $W_j = \langle t_0', t_0, \ldots, t_j \rangle$. Further, for $j, s \in \mathbb{Z}_5$ let

$$W_j(s) = \langle t_0, \ldots, t_{j-1}, s t_0' + t_j \rangle \quad \text{and} \quad W_j(\infty) = \langle t_0, \ldots, t_{j-1}, t_0' \rangle.$$

Note that $W_j(\infty) = W_{j-1}$ for $j \neq 0$, and that $W_0(\infty) = \langle t_0' \rangle$. Moreover, $\dim(W_j(s)) = j+1$. The following lemma resolves the question of R-invariant subspaces in case $p = 5$.

Lemma 4.1 *A non-trivial subspace $W \leq \mathbb{Z}_5^6$ is R-invariant if and only if $W = W_j(s)$ for some $j \in \mathbb{Z}_5$ and $s \in \mathbb{Z}_5 \cup \{\infty\}$. Moreover, $W_j(s) = W_{j'}(s')$ if and only if $j = j'$ and $s = s'$.*

PROOF. The fact that the subspaces $W_j(s)$ are indeed R-invariant and pairwise distinct is obvious. The proof that every R-invariant subspace is one of $W_j(s)$ is by induction on the dimension.

If W is a 1-dimensional R-invariant subspace, then W is contained in $K_0 = \langle t_0', t_0\rangle$, and is obviously one of the spaces $W_0(s)$, $s \in \mathbb{Z}_5 \cup \{\infty\}$. Now, suppose the claim holds for all k-dimensional subspaces, and let W be a $(k+1)$-dimensional R-invariant subspace. If there were an element $w \in W \setminus W_k$, then the $k+2$ vectors $w, (R-I)w, \ldots, (R-I)^{k+1}w$ in W would be linearly independent and $\dim(W) \geq k+2$, a contradiction. Hence W is a codimension-1 subspace of W_k. If $W = W_k(0)$, then the claim holds. On the other hand, if $W \neq W_k(0)$, then $W' = W \cap W_k(0)$ is an R-invariant subspace of dimension k, and by the induction hypothesis, W' is one of the spaces $W_{k-1}(s)$, $s \in \mathbb{Z}_5 \cup \{\infty\}$. However, since $W_{k-1}(s)$ contains t_0' whenever $s \neq 0$, and since W' is contained in $W_k(0)$, we coclude that $W' = W_{k-1}(0) = \langle t_0, \ldots, t_{k-1}\rangle$. Consequently, $W = \langle t_0, \ldots, t_{k-1}, w\rangle$ where $w = \lambda_0' t_0' + \lambda_0 t_0 + \ldots + \lambda_{k-1} t_{k-1} + \lambda_k t_k$ is an arbitrary element of $W_k \setminus W$. Therefore, $W = \langle t_0, \ldots, t_{k-1}, \lambda_0' t_0' + \lambda_k t_k\rangle$ Finally, if $\lambda_k = 0$, then $W = W_k(\infty)$, and if $\lambda_k \neq 0$, then $W = W_k(\lambda_0'/\lambda_k)$. ∎

4.3 $\langle R, T\rangle$-invariant subspaces

First recall that $\tau\rho\tau^{-1} = \rho^2$ and hence $T^{-1}RT = R^2$. Therefore, T acts on the set of R-invariant subspaces and in particular, on the subset of minimal ones. Now, if λ is an eigenvalue of R, then the linear transformation T maps the eigenspace $\mathrm{Ker}(R - \lambda I)$ to the eigenspace $\mathrm{Ker}(R - \lambda^2 I)$. Hence $K_0 = \mathrm{Ker}(R - I)$ is T-invariant. Moreover, if $p \neq 5$, then T (viewed as a linear transformation over the splitting field $\mathbb{F}$ of R) acts on the set $\{K_1, K_2, K_3, K_4\}$ of minimal invariant subspaces of K as a cyclic permutation (K_1, K_2, K_4, K_3). Consequently, if $p \neq 5$, then each minimal $\langle R, T\rangle$-invariant subspace is either K or is contained in K_0. Thus, we need to consider the restriction $T_0 = T|_{K_0}$ of T on K_0 only. In the ordered basis $\{v_1, v_2\}$ the restriction T_0 is represented by the matrix

$$T_0 = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}.$$

The characteristic polynomial of T_0 is $\kappa_{T_0}(x) = x^2 + 1$, and hence T_0 has eigenvectors in K_0 if and only if -1 is a square in $\mathbb{Z}_p$. This depends on the congruence class of the prime p modulo 4.

CASE $p \equiv -1 \pmod 4$.

Then -1 is not a square in $\mathbb{Z}_p$, and so K_0 and K are the only proper non-trivial $\langle R, T\rangle$-invariant subspaces (see rows 5 and 7 in Table 4). As we shall see later, K_0 and K are not A-invariant (unless $p = 2$, when K_0 is A-invariant). Therefore the maximal subgroup of $\mathrm{Aut}\,\mathcal{P}$ that lifts along the two covering projections is $\langle \rho, \tau\rangle \cong \mathrm{AGL}(1, 5)$.

CASE $p = 2$.

Then 1 is a square of -1. The matrix representation of T_0 is an elementary Jordan matrix having a unique eigevector v_1. Therefore, the proper non-trivial $\langle R, T\rangle$-invariant subspaces are $\langle v_1\rangle$, K_0, K and $\langle v_1\rangle \oplus K$ (see rows 3, 5, 7, 10 of Table 4). The first two subspaces are also A-invariant while last two are not, see Subsection 4.4.

CASE $p \equiv 1 \pmod 4$.

Then there exists an element $\iota \in \mathbb{Z}_p$ such that $\iota^2 = -1$. The eigenvalues of T_0 are ι and $-\iota$, with respective eigenvectors

$$\begin{array}{rclcl} u_1 & = & (3-\iota)v_1 - 2v_2 & = & (3-\iota, -2, 1-\iota, -2, 1-\iota, 1-\iota)^t, \\ u_2 & = & (3+\iota)v_1 - 2v_2 & = & (3+\iota, -2, 1+\iota, -2, 1+\iota, 1+\iota)^t, \end{array}$$

spanning the two minimal T-invariant subspaces of K_0.

SUBCASE $p \neq 5$.

By Masche's theorem, the non-trivial proper $\langle R, T\rangle$-invariant subspaces are $\langle u_1\rangle$, $\langle u_2\rangle$, $K_0 = \langle u_1\rangle \oplus \langle u_2\rangle$, K, $\langle u_1\rangle \oplus K$, and $\langle u_2\rangle \oplus K$ (see rows 1,2, 5, 7, 8, 9 of Table 4). None of these subspaces is A-invariant.

SUBCASE $p = 5$.

By Lemma 4.1, the set of non-trivial R-invariant subspaces is $\mathcal{W} = \{W_j(s) \mid j \in \mathbb{Z}_5, s \in \mathbb{Z}_5 \cup \{\infty\}\}$. Since T normalizes $\langle R\rangle$, it acts on $\mathcal{W}$. Moreover, since T normalizes $\langle (R-I)^{j+1}\rangle$ for every $j \in \mathbb{Z}_5$, it follows that the subspaces $W_j(\infty) = \mathrm{Ker}(R-I)^{j+1}$ are T-invariant. To find the action of T on the remaining subspaces in $\mathcal{W}$ we first compute the matrix representation $T_{\mathcal{J}}$ of T in the Jordan basis $\{t_0', t_0, \ldots, t_4\}$ of R:

$$T_{\mathcal{J}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 4 \\ 0 & 3 & 2 & 4 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

It is now a matter of a straightforward computation to verify that T fixes the subspaces

$$W_0(0),\ W_1(0),\ W_2(0),\ W_2(1),\ W_2(2),\ W_2(3),\ W_2(4),\ W_3(0),\ W_4(4), \tag{3}$$

swaps the pairs $(W_0(1), W_0(4))$, $(W_0(2), W_0(3))$, $(W_4(0), W_4(3))$, $(W_4(1), W_4(2))$, and acts on the remaining eight subspaces as a product of two 4-cycles $(W_1(1), W_1(2), W_1(4), W_1(3))$ and $(W_3(1), W_3(3), W_3(4), W_3(2))$.

Hence, among the thirty non-trivial proper R-invariant subspaces, exactly fourteen are T-invariant. Namely, the five subspaces $W_j(\infty)$, $j \in \mathbb{Z}_p$, and the nine subspaces in (3). Together with the full space $\mathbb{Z}_5^6$, they give rise to the fifteen covering projections in Table 5. Exactly two of them are also A-invariant, namely, $W_2(4)$ and $\mathbb{Z}_5^6$.

4.4 $\langle R, A\rangle$-invariant subspaces

Since the group $\langle R\rangle$ is not normalised by A, the set of invariant R-subspaces is not preserved by the action of A (as it by the action of T). Yet, if $p \neq 2, 3, 5$, then each $\langle R, A\rangle$-invariant subspace is a direct sum minimal ones. Moreover, if $p \neq 5$, then each $\langle R, A\rangle$-invariant subspace is a direct sum of minimal R-invariant subspaces.

CASE $p \neq 5$.

Suppose first that a non-trivial $\langle R, A\rangle$-invariant subspace W contains no minimal R-invariant subspaces contained in K. Then W is either K_0 or one of its 1-dimensional subspaces $U(s)$, $s \in \mathbb{Z}_p \cup \{\infty\}$. If $p \neq 2$, then one easily checks that $AK_0 \cap K_0$ is trivial, a contradiction. If $p = 2$, then A acts on K_0 as the identity, and hence all three 1-dimensional subspaces of K_0 are minimal

$\langle R, A\rangle$-invariant. We already know that the subspace spanned by $v_1 = (1,0,1,0,1,1)$ is also T-invariant (see row 3 of Table 4). On the other hand, T swaps the remaining two 1-dimensional subspaces of K_0. Hence the corresponding covering projections are isomorphic (see row 4 of Table 4).

Suppose now that W intersects K non-trivially. As in Subsection 4.2, we first find $\langle R, A\rangle$-invariant subspaces over the splitting field $\mathbb{F}$ of R. Recall that the minimal R-invariant subspaces contained in K are the 1-dimensional subspaces $K_j = \mathrm{Ker}(R-I)^j$, $j \in \{1,\dots,4\}$. Since W contains at least one K_j, it contains the subspace

$$K_j{}^A = \langle K_j, AK_j, A^2K_j\rangle.$$

By computation we find that

$$K_1{}^A = K_4{}^A = K_1+K_4+U(\nu_1-2) = \langle (1,0,0,\nu_2,0,1)^t, (0,1,0,-1,-\nu_1,\nu_1)^t, (0,0,1,\nu_1,-\nu_1,-1)^t\rangle,$$

$$K_2{}^A = K_3{}^A = K_2+K_3+U(\nu_2-2) = \langle (1,0,0,\nu_1,0,1)^t, (0,1,0,-1,-\nu_2,\nu_2)^t, (0,0,1,\nu_2,-\nu_2,-1)^t\rangle.$$

These two subspaces are clearly minimal $\langle R, A\rangle$-invariant. Moreover, since their sum is $\mathbb{Z}_p^6$, these two are the only non-trivial proper $\langle R, A\rangle$-invariant subspaces. Observe that $K_1{}^A$ and $K_2{}^A$ are swaped by T, implying that the corresponding covering projections are isomorphic.

Since the realizability of $K_1{}^A$ over $\mathbb{Z}_p$ depends on whether ν_1 and ν_2 belong to $\mathbb{Z}_p$, it follows that the existence of proper $\langle R, A\rangle$-invariant subspaces intersecting K non-trivially depends on the congruence class of p modulo 5.

The above discussion can now be summarized as follows.

Subcase $p \equiv \pm 1 \pmod 5$.

The proper non-trivial $\langle R, A\rangle$-invariant subspaces are $K_1{}^A$ and $K_2{}^A$, giving rise to isomorphic covering projections (see row 6 of Table 4).

Subcase $p \equiv \pm 2 \pmod 5$.

If $p \neq 2$, then there are no proper non-trivial $\langle R, A\rangle$-invariant subspaces. If $p = 2$, then all three 1-dimensional subspaces of K_0 are $\langle R, A\rangle$-invariant, giving rise to two non-isomorphic covering projections (see rows 3 and 4 of Table 4).

Case $p = 5$.

Recall that there are 30 proper non-trivial R-invariant subspaces, namely, $W_j(s)$, $j \in \mathbb{Z}_5$, $s \in \mathbb{Z}_5 \cup \{\infty\}$. Similarly as in Subsection 4.3 one can check that exactly one of them is also A-invariant, namely, the subspace $W_2(4)$ (see row 10 of Table 5).

4.5 Isomorphisms of covering projections in Tables 4 and 5

It remains to show that all covering projections in Tables 4 and 5 are pairwise non-isomorphic. If two covering projections are isomorphic, then: (a) they arise from voltage assignments valued in the same group, and (b) the maximal groups that lift are isomorphic (see Proposition 2.1). Note that as soon as two covering projections in Tables 4 and 5 are admissible for isomorphic groups, they are in fact admissible for the same group. The possiblity that the projections in Table 4 be isomorphic is thus reduced to checking the pairs in rows 1, 2 and rows 9, 10. In Table 5, however, the number of pairs that need to be checked is larger. That is, one has to check the pairs contained in the following sets of rows: $\{1,2\}$, $\{3,4\}$, $\{5,6,7,8,9\}$ and $\{11,12,13,14\}$.

By Theorem 2.3, two G-admissible derived covering projections associated with $G^{\#^*_h}$-invariant subspaces $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic if and only if there exists an automorphism $\varphi \in \operatorname{Aut}\mathcal{P}$ such that $\varphi^{\#^*_h}$ maps $\mathcal{S}'$ to $\mathcal{S}$. Clearly, if φ is such an automorphism, then every automorphism in the coset φG has the same property. Hence it suffices check whether $\mathcal{S}$ is the image of $\mathcal{S}'$ under an element from a fixed transversal of G in $\operatorname{Aut}\mathcal{P}$. For the groups $\langle\rho,\tau\rangle$ and $\langle\rho,\alpha\rangle$ we choose the transversals

$$\mathcal{T}_{\rho,\tau} = \{\mathrm{id}, \alpha, \alpha^2, \sigma, \sigma\alpha, \sigma\alpha^2\} \qquad \text{and} \qquad \mathcal{T}_{\rho,\alpha} = \{\mathrm{id}, \tau\},$$

where $\sigma = (3,4) \in S_5$. We leave to the reader to check that none of the above transversal elements gives rise to an isomorphism of the respective covering projections. This completes the proof of Theorem 3.1. ■

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