University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1111 Ljubljana, Slovenia

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Andrej Vodopivec

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On embeddings of snarks in the torus

Andrej Vodopivec

Department of Mathematics
IMFM
1000 Ljubljana, Slovenia
andrej.vodopivec@fmf.uni-lj.si

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Abstract

A condition on cubic graphs G_1 and G_2 is presented, which implies that a dot product $G_1 \cdot G_2$ exists, which has an embedding in the torus. Using this condition it is proved that for every positive integer n a dot product of n copies of the Petersen graph exists, which can be embedded in the torus. This disproves a conjecture of Watkins and Tinsley and answers a question by Mohar.

1 Introduction

A *snark* is a cyclically 4-edge connected cubic graph of girth at least 5 with no 3-edge coloring. Let G_1 and G_2 be cubic graphs and let $e_1 = x_1x_2, e_2 = x_3x_4$ be two nonadjacent edges in G_1 and $f = uv$ be an edge in G_2. Let the neighbours of u, distinct from v, be y_1 and y_2 and the neighbours of v, distinct from u, be y_3 and y_4. Construct a graph G by deleting the edges e_1 and e_2 from G_1 and vertices u and v from G_2 and then adding edges x_iy_i, $i = 1, 2, 3, 4$. The graph G is called a *dot product* of graphs G_1 and G_2. It is well known that if G_1 and G_2 are snarks, then G is also a snark.

Let P^n denote a dot product of n copies of the Petersen graph. In [3] authors proposed a conjecture, that a graph P^n has genus precisely $n - 1$. This conjecture was disproved in [2], where it was shown that one of the two possible dot products P^2 has genus 2, so that the genus can be bigger than conjectured. In this paper we show, that for every positive integer n a dot product of n Petersen graphs exists, which can be embedded in the torus and has therefore genus 1, so the value of the genus can also be (much) smaller than the conjectured value. Mohar asked [1] what is the smallest integer g such that there are infinitely many snarks of genus g. We answer this question by proving that $g = 1$.

2 Main result

Let G_1 and G_2 be cubic graphs embedded in the torus. Let $e_1 = x_1x_2$ and $e_2 = x_3x_4$ be two edges of G_1 such that in the embedding of G_1, there are two facial cycles $C_1 = x_1x_2P_1x_3x_4P_2x_1$ and $C_2 = x_2x_1P_4x_4x_3P_3x_2$. Then we say that the edges e_1 and e_2 satisfy property $\mathcal{P}$. Let $f = uv$ be an edge in G_2 such that the neighbours of u, distinct from v, are y_1, y_2 and the neighbours of v, distinct from u are y_3, y_4 and in the embedding of G_2 there are distinct facial cycles $D_1 = y_1uvy_4R_4y_1$, $D_2 = y_3vuy_2R_3y_3$ and $D_3 = y_2uy_1R_2y_4vy_3R_1y_2$.

Lemma 2.1 *Let G_1 and G_2 be as above. Then a dot product $G = G_1 \cdot G_2$ exists which has an embedding in the torus. Furthermore, two of the edges $e'_i = x_iy_i$ and $e'_j = x_jy_j$ in G have property $\mathcal{P}$.*

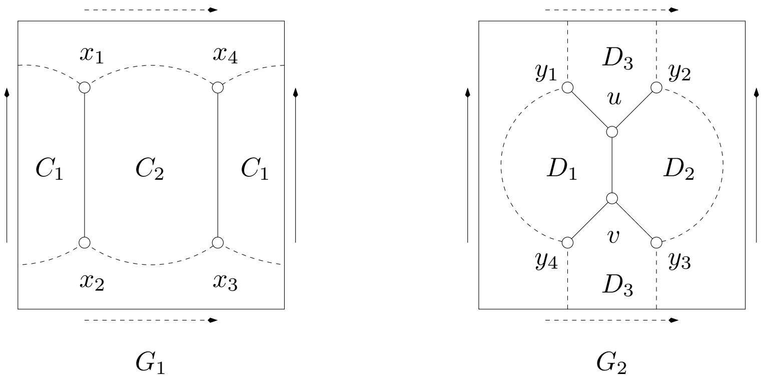

Figure 1: The configuration of faces in G_1 and G_2.

Proof. Let G_1 and G_2 be embedded as in lemma. Let G be the dot product as described in the Introduction. We define the embedding of G by specifying local rotations at its vertices. Denote with X the set $\{x_i, y_i \mid i = 0, 1, 2, 3, 4\}$. The rotations at vertices in $V(G) \setminus X$ are the same as the rotations in the embeddings of G_1 and G_2. The rotations at vertices in X are the same as the rotations in the embeddings of G_1 and G_2 where we naturally replace the deleted edges with the added ones. This is clearly an embedding in an orientable surface. To prove that this surface is the torus, we count the facial cycles. The facial cycles, which do not contain any of the vertices from X are facial cycles in the embedding of G_1 or G_2. The facial cycles, which contain vertices from X are $F_1 = x_2P_1x_3y_3R_1y_2x_2$, $F_2 = x_1y_1R_2y_4x_4P_2x_1$, $F_3 = x_2P_1x_3y_3R_1y_2x_2$ and $F_4 = x_1P_4x_4y_4R_4y_1x_1$. So we have replaced five facial cycles C_1, C_2, D_1, D_2, D_3 with four facial cycles F_1, F_2, F_3, F_4. We have $|V(G)| = |V(G_1)| + |V(G_2)| - 2$, $|E(G)| = |E(G_1)| + |E(G_2)| - 3$ and $|F(G)| = |F(G_1)| + |F(G_2)| - 1$. So $|V(G)| - |E(G)| + |V(G)| = 0$ and this is an embedding in the torus. It is also easy to see that edges x_1y_1 and y_4x_4 satisfy the property $\mathcal{P}$. □

Corollary 2.2 *For every positive integer n there exists a dot product of n copies of the Petersen graph, that can be embedded in the torus.*

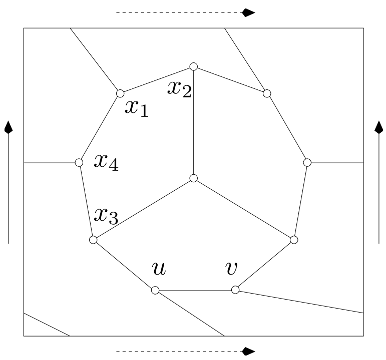

Figure 2: The Petersen graph in the torus.

Proof. An embedding of the Petersen graph in the torus is shown in Figure 2. It is easy to check that if we take the edges x_1x_2 and x_3x_4 in one copy and the edge uv in the other, the conditions of Lemma 2.1 are satisfied for both copies. The corollary follows. □

Corollary 2.3 *There exist arbitrary large snarks, that can be embedded in the torus.*

References

[1] B. Mohar, private communication.

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[3] F. C. Tinsley, J. J. Watkins, A study of snark embeddings, Graphs and applications (Boulder, Colo., 1982), 317–332, Wiley-Intersci. Publ., 1985.