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1. Introduction

A complex manifold Y enjoys the *Oka property* if for any Stein manifold X (a closed complex submanifold of some Euclidean space $\mathbb{C}^N$) and any continuous map $f\colon X \to Y$ which is holomorphic in a neighborhood of a compact $\mathcal{O}(X)$-convex subset $K \subset X$ there is a homotopy of continuous maps $f_t\colon X \to Y$, with $f_0 = f$ and f_1 holomorphic on X, such that for every $t \in [0,1]$ the map f_t is holomorphic in a neighborhood of K and close to $f = f_0$ uniformly on K. By fundamental theorems of Oka [O] and Grauert [G1], [G2], [G3] (see also [Ca], [HL]) every complex homogeneous manifold satisfies the Oka property. In 1989 Gromov [Gr2] showed that the same holds under the weaker condition that Y admits a dominating spray; for detailed exposition and further extensions see [FP1], [FP2], [FP3], [F2] and the surveys [Le], [F4].

In this paper we show that, somewhat surprisingly, the Oka property of a complex manifold Y is equivalent to a *Runge approximation property* for holomorphic maps from compact convex sets in Euclidean spaces to Y, thereby answering a question of Gromov [Gr2, p. 881, 3.4.(D)].

THE MAIN THEOREM. *If Y is a complex manifold such that any holomorphic map $f\colon K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ ($n \in \mathbb{N}$) can be approximated by holomorphic maps $\mathbb{C}^n \to Y$ then Y satisfies the Oka property.*

We shall say that a complex manifold Y satisfying the hypothesis in the Main Theorem enjoys the *convex approximation property*, abbreviated CAP. The converse implication is immediate and hence

$$\text{CAP} \Longleftrightarrow \text{the Oka property.}$$

The reader may observe that CAP is just the Oka property applied with the simplest type of pairs $K = \widehat{K}_{\mathcal{O}(X)} \subset X = \text{Stein}$, namely with X a Euclidean space $\mathbb{C}^n$ and K a compact convex set in $\mathbb{C}^n$. This provides yet another example of the importance of convexity in complex analysis (see Hörmander [H]).

The main theorem easily implies that the Oka property passes up and down in certain types of holomorphic fibrations, in particular in holomorphic covering projections (Theorem 1.2 and Corollary 1.3). In addition it greatly simplifies the proof of the Oka property in certain examples, in particular those involving complements of thin (of codimension at least two) algebraic subvarieties in affine and projective spaces, by reducing it to a problem on Euclidean spaces where one can effectively use holomorphic automorphisms. CAP can be seen as a precise opposite property to Kobayashi–Eisenman hyperbolicity; more on this below.

We now present a more effective result by narrowing down the class of compact convex sets that is needed in the proof and by specifying the dimension requirement in the convex approximation property. Let $z = (z_1, \ldots, z_n)$ be the coordinates on $\mathbb{C}^n$, with $z_j = x_j + iy_j$. Set

$$P = \{z \in \mathbb{C}^n \colon |x_j| \le 1,\ |y_j| \le 1,\ j = 1, \ldots, n\}. \tag{1.1}$$

A *special convex set* in $\mathbb{C}^n$ is a compact convex subset of the form

$$Q = \{z \in P \colon y_n \le h(z_1, \ldots, z_{n-1}, x_n)\}, \tag{1.2}$$

where h is a smooth (weakly) concave function with values in $(-1, 1)$.

DEFINITION 1. *A complex manifold Y satisfies the n-dimensional convex approximation property* (CAP_n) *if any holomorphic map $f \colon Q \to Y$ from a set of the form (1.2) can be approximated uniformly on Q by holomorphic maps $P \to Y$. Y satisfies* CAP *if it satisfies* CAP_n *for every $n \in \mathbb{N}$.*

Obviously $\mathrm{CAP}_n \Longrightarrow \mathrm{CAP}_k$ when $n > k$, but the converse fails in general for $n \le \dim Y$ as we shall see below. Applying CAP_n inductively to an increasing sequence of cubes exhausting $\mathbb{C}^n$ we see that CAP_n is equivalent to the Runge approximation of holomorphic maps $Q \to Y$ on special convex sets (1.2) by entire maps $\mathbb{C}^n \to Y$.

THEOREM 1.1. *A p-dimensional complex manifold Y satisfying* CAP_{n+p} *enjoys the Oka property for Stein manifolds of dimension at most n; furthermore, sections of any holomorphic fiber bundle with fiber Y over a Stein manifold of dimension at most n satisfy the Oka-Grauert principle.*

The *Oka-Grauert principle* means the conclusion in the definition of the Oka property for a fixed base (Stein) manifold X. We do not know whether the Runge property in CAP for a more special class of convex sets (such as balls) implies the Oka property; a priori this does not seem impossible but would require a refinement of the underlying geometry in the proof.

Since any holomorphic map $Q \to Y$ on a compact convex set $Q \subset \mathbb{C}^n$ is homotopic to a constant map $Q \to y_0 \in Y$ through a homotopy of holomorphic maps $Q \to Y$, CAP follows from the Runge-type theorems proved in [G1], [G2], [Gr2], [FP1], [FP3], [F2]. This holds in particular if Y is *subelliptic* in the sense that it admits a finite dominating family of holomorphic sprays (Definition 2 and Theorem 3.1 in [F2]).

Recall that a continuous map is a *Serre fibration* if it satisfies the homotopy lifting property. For the definition of a *subelliptic submersion* see [F2, p. 529, Definition 2]. Using our Main Theorem we prove the following (§4).

THEOREM 1.2. *If $\pi: Y \to Y_0$ is a subelliptic submersion which is also a Serre fibration then Y satisfies the Oka property if and only if Y_0 does.*

COROLLARY 1.3. *If one of the manifolds Y, Y_0 in Theorem 1.2 is subelliptic then both of them satisfy the Oka property.*

A map π as in Theorem 1.2 will be called a *subelliptic Serre fibration.* The conclusion of Theorem 1.2 is related to a conjecture of F. Lárusson [La, p. 19]. Theorem 1.2 applies in particular when $\pi: Y \to Y_0$ is a holomorphic fiber bundle with a subelliptic fiber; examples include unramified holomorphic coverings, principal holomorphic fiber bundles, unramified elliptic fibrations, fiber bundles with projective fibers, etc.; for further examples see §6 and [F7, §6]. The Main Theorem reduces Theorem 1.2 to the equivalence of CAP for Y and Y_0; the latter property involves maps from contractible sets and these can be lifted in a Serre fibration. (Compare with Corollary 3.3.C' in [Gr2, p. 881].)

By Theorem 1.4 in [F7] the Oka property of Y implies the jet transversality theorem for holomorphic maps from any Stein manifold to Y which gives

COROLLARY 1.4. *If a complex manifold Y satisfies CAP then holomorphic maps from any Stein manifold to Y satisfy the jet transversality theorem.*

In applications one often needs the stronger *parametric Oka principle* which reduces to the *parametric convex approximation property*; see §5.

Note that CAP_1 is a precise opposite property to Kobayashi hyperbolicity which prohibits nonconstant entire maps $\mathbb{C} \to Y$. More generally, CAP_n for $n \le \dim Y$ is an opposite to n-dimensional Kobayashi-Eisenman hyperbolicity [Kb], [E]. If $n \ge p = \dim Y$ then CAP_n of Y implies the existence of *dominating holomorphic maps $f: \mathbb{C}^p \to Y$* with rank p at a generic point of $\mathbb{C}^p$. Dominability imposes strong restrictions; if such a manifold is compact then it cannot be of general Kodaira type [Kd], [CG], [KO]. For a comprehensive discussion of dominability of complex surfaces we recommend the paper [BL] by Buzzard and Lu; one can see that for complex surfaces the gap between the Oka property and dominability became rather narrow (see [F7]).

EXAMPLE. For every $1 < k \le p$ there exists a p-dimensional complex manifold which satisfies CAP_{k-1} but not CAP_k. For $k = p$ we can take $Y = \mathbb{C}^p \backslash A$ where A is a discrete subset of $\mathbb{C}^p$ which is *rigid* in the sense of Rosay and Rudin [RR, p. 60]; Y satisfies CAP_{p-1} since most holomorphic maps $\mathbb{C}^{p-1} \to \mathbb{C}^p$ avoid A, but CAP_p fails by the definition of a rigid set. For $1 < k <$

p we take $Y = \mathbb{C}^p \backslash \phi(\mathbb{C}^{p-k})$ where $\phi\colon \mathbb{C}^{p-k} \to \mathbb{C}^p$ is a proper holomorphic embedding whose complement is k-hyperbolic, i.e., every entire map $\mathbb{C}^k \to \mathbb{C}^p$ whose range omits $\phi(\mathbb{C}^{p-k})$ has rank $< k$ at every point (see [F1] for the construction of such maps). Thus CAP_k fails but CAP_{k-1} holds since most holomorphic maps $\mathbb{C}^{k-1} \to \mathbb{C}^p$ avoid the submanifold $\phi(\mathbb{C}^{p-k})$ by dimension reasons. Alternatively one may take $Y = (\mathbb{C}^k \backslash A) \times \mathbb{C}^{p-k}$ where A is a rigid discrete set in $\mathbb{C}^k$.

QUESTION. Is there an integer p depending on Y (or on $\dim Y$) such that $\mathrm{CAP}_p \implies \mathrm{CAP}_n$ for all $n \geq p$? Does this hold for $p = \dim Y$?

Our proof of Theorem 1.1 (§3) is a synthesis of several recent developments, in particular those in [F5] and [F6]. A main ingredient is Lemma 2.1 (§2) generalizing the classical Cartan lemma. Another essential tool developed in [F5] and [F6] allows us to extend a solution across a critical level of a strongly plurisubharmonic exhaustion function by reducing the problem to the noncritical case for a different strongly plurisubharmonic function.

2. A Cartan type splitting lemma.

Let X be a complex manifold and A, B compact subsets of X satisfying

(i) $A \cup B$ admits a basis of Stein neighborhoods in X, and

(ii) $\overline{A\backslash B} \cap \overline{B\backslash A} = \emptyset$ (the separation property).

Such (A, B) will be called a *Cartan pair* in X. Set $C = A \cap B$. Let D be a compact subset with a basis of Stein neighborhoods in a complex manifold T.

LEMMA 2.1. (Assumptions as above.) *If $\gamma(x,t) = (x, c(x,t)) \in X \times T$ ($x \in X$, $t \in T$) is an injective holomorphic map in an open neighborhood $\Omega_C \subset X \times T$ of $C \times D$ which is sufficiently uniformly close to the identity map then there exist open neighborhoods $\Omega_A, \Omega_B \subset X \times T$ of $A \times D$ respectively $B \times D$ and injective holomorphic maps $\alpha\colon \Omega_A \to X \times T$, $\beta\colon \Omega_B \to X \times T$ of the form $\alpha(x,t) = (x, a(x,t))$, $\beta(x,t) = (x, b(x,t))$, close to the identity map on their respective domains and satisfying*

$$\gamma = \beta \circ \alpha^{-1} \tag{2.1}$$

in a neighborhood of $C \times D$ in $X \times T$.

We shall use Lemma 2.1 with D a closed ball in $T = \mathbb{C}^p$ for various values of $p \in \mathbb{N}$. Lemma 2.1 generalizes the classical Cartan lemma [GR, p. 199] in which A, B and $C = A \cap B$ are cubes in $\mathbb{C}^n$ and a, b, c are invertible linear functions of $t \in \mathbb{C}^p$ depending holomorphically on the base variable x.

Lemma 2.1 follows from Theorem 4.1 in [F5] by applying the latter result with the Cartan pair $(A \times D, B \times D)$ in the manifold $X \times T$ and with the 'vertical' foliation $\mathcal{F}$ with leaves $\{x\} \times T$, $x \in X$. The map γ in Lemma 2.1 is an $\mathcal{F}$-map in the terminology of [F5], meaning that it preserves the connected components of the leaves of $\mathcal{F}$ ('plaques') in distinguished local charts. If γ is sufficiently close to the identity in a neighborhood of $C \times D$ then Theorem 4.1 in [F5] furnishes a required decomposition where α and β are $\mathcal{F}$-maps which are close to the identity in a neighborhood of $A \times D$ respectively $B \times D$. The proof of Theorem 4.1 in [F5] uses a Nash-Moser iteration in which the linearized problem is solved by the $\overline{\partial}$-technique.

Certain generalizations are possible. For example, if Σ is a closed complex subvariety of $X \times T$ which does not intersect $C \times D$ then one can choose α and β in the decomposition (2.1) such that they are tangent to the identity map to a given finite order along Σ (Theorem 4.1 in [F5] is proved in this form).

3. Proof of the Main Theorem.

We use Grauert's bumping method as in Henkin and Leiterer [HL] or [Gr2], [FP1], with the additions introduced in [F5] and [F6]. Assume that Y is a complex manifold satisfying CAP. Let X be a Stein manifold, $K \subset X$ a compact holomorphically convex subset of X and $f: X \to Y$ a continuous map which is holomorphic in an open set $U \subset X$ containing K. We shall modify f in a countable sequence of steps to obtain a holomorphic map $X \to Y$ homotopic to f and approximating it uniformly on K. The goal of every step is to enlarge the domain of holomorphicity and thus obtain a sequence of maps $X \to Y$ which converges uniformly on compacts in X to a holomorphic map.

Choose a smooth strongly plurisubharmonic Morse exhaustion function $\rho: X \to \mathbb{R}$ such that $\rho|_K < 0$ and $\{\rho \le 0\} \subset U$. Let $X_t = \{\rho \le t\}$. It suffices to prove that for any $0 \le c_0 < c_1$ such that c_0 and c_1 are regular values of ρ, a continuous map $f: X \to Y$ which is holomorphic on X_{c_0} can be deformed by a homotopy of maps $f_t: X \to Y$ ($t \in [0,1]$) to a map f_1 which is holomorphic on X_{c_1}, where f_t is holomorphic and arbitrarily uniformly close to $f = f_0$ on X_{c_0} for every $t \in [0,1]$. There are two main cases to consider:

The noncritical case: $d\rho \ne 0$ on the set $\{x \in X: c_0 \le \rho(x) \le c_1\}$.

The critical case: there is a point $p \in X$ with $c_0 < \rho(p) < c_1$ such that $d\rho_p = 0$. (We may assume that there is a unique such p.)

A reduction of the critical case to the noncritical one, based on a technique developed in [F5, §6], has been explained in [F6, §6]. It is accomplished in three steps, the first two of which do not require any special properties of Y:

Step 1: Let $f: X \to Y$ be a continuous map which is holomorphic in a neighborhood of X_c for some $c < \rho(p)$ close to $\rho(p)$. By a small modification we make f smooth on a totally real handle E attached to X_c and passing through the critical point p. (In suitable local coordinates near p this handle is the stable manifold of p for the gradient flow of ρ).

Step 2: We approximate f on the handlebody $X_c \cup E$ by a map which is holomorphic in an open neighborhood (Theorem 3.2 in [F6]).

Step 3: We approximately extend the map from the second step across the critical level $\{\rho = \rho(p)\}$ by applying the noncritical case for a different strongly plurisubharmonic function on X constructed for this purpose. After reaching $X_{c'}$ for some $c' > \rho(p)$ we revert back to ρ and continue (by the noncritical case) to the next critical level. For the details see [F5, §6] and [F6].

It remains to explain the noncritical case. Let $z = (z_1, \ldots, z_n)$, $z_j = u_j + iv_j$, denote the coordinates on $\mathbb{C}^n$, $n = \dim X$. Set

$$P = \{z \in \mathbb{C}^n : |u_j| < 1,\ |v_j| < 1,\ j = 1, \ldots, n\} \tag{3.1}$$

and $P' = \{z \in P : v_n = 0\}$. Let $A \subset X$ be a compact strongly pseudoconvex domain in X. We say that a compact subset $B \subset X$ is a *convex bump* on A if there exist an open neighborhood $V \subset X$ of B, a biholomorphic map $\phi: V \to P$ onto the set (3.1) and smooth strongly concave functions $h, \widetilde{h}: P' \to [-s, s]$ for some $0 < s < 1$ such that $h \le \widetilde{h}$, $h = \widetilde{h}$ near the boundary of P' and

$$\phi(A \cap V) = \{z \in P : v_n \le h(z_1, \ldots, z_{n-1}, u_n)\},$$
$$\phi((A \cup B) \cap V) = \{z \in P : v_n \le \widetilde{h}(z_1, \ldots, z_{n-1}, u_n)\}.$$

We also require that (i) $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$ and (ii) $C = A \cap B$ is Runge in A (i.e., every function holomorphic in a neighborhood of C can be approximated on C by functions holomorphic in a neighborhood of A).

PROPOSITION 3.1. *Assume that $A, B \subset X$ are as above. Let Y be a p-dimensional complex manifold satisfying* CAP_{n+p}. *Choose a distance function d on Y induced by a Riemannian metric. For any holomorphic map $f_0: A \to Y$ and $\epsilon > 0$ there is a holomorphic map $f_1: A \cup B \to Y$ satisfying $\sup_{x \in A} d(f_0(x), f_1(x)) < \epsilon$. The analogous result holds for sections of a holomorphic fiber bundle with fiber Y over X provided that the bundle is trivial over the neighborhood $V \subset X$ of B.*

If f_0 and f_1 are sufficiently close on A then there clearly exists a holomorphic homotopy from f_0 to f_1. As we shall see in the proof, if Y satisfies CAP_N with $N = p + [\frac{1}{2}(3n+1)]$ then we may omit the hypothesis that C is Runge in A. We don't know whether CAP_p would suffice.

Assuming Proposition 3.1 we can complete the proof of the noncritical case (and hence of Theorem 1.1) as follows. Using Narasimhan's lemma on local convexification of strongly pseudoconvex domains one can obtain a finite sequence $X_{c_0} = A_0 \subset A_1 \subset \ldots \subset A_{k_0} = X_{c_1}$ of compact strongly pseudoconvex domains in X such that for every $k = 0, 1, \ldots, k_0 - 1$ we have $A_{k+1} = A_k \cup B_k$ where B_k is a convex bump on A_k (Lemma 12.3 in [HL2]). Furthermore, the sets B_k may be chosen small (contained in elements of a prescribed open covering of X). The separation condition (i) is trivial to satisfy while (ii) is only a small addition (for example, one can use local convexification of a strongly pseudoconvex domain A given by holomorphic functions defined in a neighborhood of A; see [Fn, p. 530, Proposition 1] or [HC]). It remains to apply Proposition 3.1 inductively to every pair (A_k, B_k), $k = 0, 1, \ldots, k_0 - 1$.

Proof of Proposition 3.1. Let P be the cube (3.1). Choose $r' < r < 1$ such that $\phi(B) \subset r'\overline{P}$. The set

$$Q = \phi(A \cap V) \cap r\overline{P} = \{z \in r\overline{P}: v_n \le h(z', u_n)\} \tag{3.2}$$

is special compact convex in $\mathbb{C}^n$ of the form (1.2) (with respect to the closed cube $r\overline{P}$), and $C = A \cap B$ is contained in $Q_0 = \phi^{-1}(Q) \subset X$.

By the hypothesis f_0 is holomorphic in an open neighborhood $U \subset X$ of A. Set $F_0(x) = (x, f_0(x)) \in X \times Y$ for $x \in U$.

LEMMA 3.2. *There are a neighborhood $U_1 \subset U$ of A in X, a neighborhood $W \subset \mathbb{C}^p$ of $0 \in \mathbb{C}^p$ and a holomorphic map $F(x,t) = (x, f(x,t)) \in X \times Y$, defined for $x \in U_1$, $t \in W$, such that $f(\cdot, 0) = f_0$ and $f(x, \cdot): W \to Y$ is injective holomorphic for every x in a neighborhood of $C = A \cap B$.*

Proof. $F_0(U)$, being a closed Stein submanifold of the complex manifold $U \times Y$, admits an open Stein neighborhood in $U \times Y$ by [S]. Let $\pi_X: X \times Y \to X$ denote the projection $(x,y) \to x$. The set $E = \ker d\pi_X \subset T(X \times Y)$ is a holomorphic vector subbundle of rank $p = \dim Y$ consisting of vectors $\xi \in T(X \times Y)$ are tangent to the fibers of π_X. Note that E is trivial over a neighborhood of $F_0(Q_0)$ in $X \times Y$ (due to contractibility of Q_0) and hence is generated there by p holomorphic sections, i.e., vector fields tangent to the fibers of π_X. Since C is Runge in A, these sections can be approximated uniformly on $F_0(C)$ by holomorphic sections $\xi_1, \ldots, \xi_p$ of E, defined in a neighborhood of $F_0(A)$ in $X \times Y$, which still generate E over a neighborhood of $F_0(C)$. The flow θ^j_t of ξ_j is well defined for sufficiently small $t \in \mathbb{C}$. The map

$$F(x, t_1, \ldots, t_p) = \theta^1_{t_1} \circ \cdots \circ \theta^p_{t_p} \circ F_0(x) \in X \times Y,$$

defined and holomorphic for x in a neighborhood of A and for $t = (t_1, \ldots, t_p)$ in a neighborhood of the origin in $\mathbb{C}^p$, satisfies Lemma 3.2.

We continue with the proof of Proposition 3.1. Let F and $W \subset \mathbb{C}^p$ be as in Lemma 3.2. Choose a closed cube $D \subset W$ centered at 0. Then $\widetilde{Q} := Q \times D \subset \mathbb{C}^{n+p}$ is a special compact convex set of the form (1.2) with respect to the closed cube $\widetilde{P} := r\overline{P} \times D \subset \mathbb{C}^{n+p}$. Since Y is assumed to satisfy CAP_{n+p}, we can approximate the map $\widetilde{f}(z,t) := f(\phi^{-1}(z), t) \in Y$ uniformly on a compact convex neighborhood of $\widetilde{Q}$ by an entire map $\widetilde{g}: \mathbb{C}^{n+p} \to Y$. (This is the only place in the proof where CAP is used.) Setting $g(x,t) = \widetilde{g}(\phi(x), t)$ for $x \in V$, $t \in \mathbb{C}^p$ gives a holomorphic map $g: V \times \mathbb{C}^p \to Y$ which approximates f uniformly in a neighborhood of $Q_0 \times D$ in $X \times \mathbb{C}^p$. It follows that $g(x, \cdot): \mathbb{C}^p \to Y$ is uniformly close to $f(x, \cdot)$ in a neighborhood of D (and hence injective holomorphic) for every fixed x in a neighborhood of C.

If the approximation is sufficiently close, there is a (unique) injective holomorphic map $\gamma(x,t) = (x, c(x,t)) \in X \times \mathbb{C}^p$, defined and uniformly close to the identity in an open neighborhood $\Omega \subset X \times \mathbb{C}^p$ of $C \times D$, satisfying

$$f(x,t) = (g \circ \gamma)(x,t) = g(x, c(x,t)), \quad (x,t) \in \Omega. \tag{3.3}$$

Lemma 2.1 gives a decomposition $\gamma = \beta \circ \alpha^{-1}$ with $\alpha(x,t) = (x, a(x,t))$, $\beta(x,t) = (x, b(x,t))$ close to the identity maps in their respective domains $\Omega_A \supset A \times D$, $\Omega_B \supset B \times D$. From (3.3) we obtain

$$f(x, a(x,t)) = g(x, b(x,t)), \quad (x,t) \in C \times D.$$

Setting $t = 0$ the two sides define a holomorphic map $f_1 \colon A \cup B \to Y$ which approximates $f_0 = f(\cdot\,, 0)$ uniformly on A (since $a(x,0) \approx 0$ for $x \in A$).

This completes the proof of Proposition 3.1 (and hence of Theorem 1.1) for maps $X \to Y$. The same proof applies to sections of a holomorphic fiber bundle $Z \to X$ with fiber Y since we may choose our bumps B small enough to insure that the bundle is trivial over a suitable neighborhood of B in X.

REMARKS. 1. The restriction of the rank p vector bundle E to the n-dimensional Stein manifold $F_0(U) \subset U \times Y$ is generated by $p + [\frac{1}{2}(n+1)]$ sections (Lemma 5 in [Fo, p. 178]). Without assuming that C is Runge in A this enables us to complete the proof if Y satisfies CAP_N, $N = p + [\frac{1}{2}(3n+1)]$.

2. From (3.3) we can also complete the proof as follows. Since $c(x,t) \approx t$ uniformly on a neighborhood of $C \times D$, Proposition 5.2 in [FP1] furnishes a pair of holomorphic maps $t = \alpha(x)$, $t = \beta(x)$, defined and close to the zero map in an open neighborhood of A respectively of B in X, satisfying $c(x, \alpha(x)) = \beta(x)$ for $x \in C$. From (3.3) we obtain for $x \in C$

$$f(x, \alpha(x)) = g(x, c(x, \alpha(x))) = g(x, \beta(x))$$

which concludes the proof as before. We find Lemma 2.1 more appealing and closer in spirit to the constructions in [F5] and [F6].

3. The reader may observe that CAP_n is reminiscent of the Property S_n introduced in [F6] which requires that any holomorphic submersion $f \colon K \to Y$ from a special compact convex set $K \subset \mathbb{C}^n$, $n \geq \dim Y$, is approximable by entire submersions $\mathbb{C}^n \to Y$. By Theorem 2.1 in [F6] Property S_n of Y implies that holomorphic submersions from any n-dimensional Stein manifold to Y satisfy the homotopy principle analogous to the one for smooth submersions [P], [Gr1]. The similarity of the two conditions is not merely apparent: Our proof unifies the construction of holomorphic maps from Stein manifolds with that of noncritical holomorphic functions and submersions in [F5] and [F6].

4. Proof of Theorem 1.2.

Let $\pi \colon Y \to Y_0$ satisfy the hypotheses of Theorem 1.2. Assume first that Y_0 satisfies the Oka property. Let $f \colon U \to Y$ be a holomorphic map from an open convex set $U \subset \mathbb{C}^n$ and let $K \subset L \subset U$ be compact convex sets with $K \subset \operatorname{Int} L$. Set $g = \pi \circ f \colon U \to Y_0$. Since Y_0 satisfies CAP, there is an entire map $g_1 \colon \mathbb{C}^n \to Y$ which approximates g uniformly on L. By Lemma 3.4 in [F6] there exists a holomorphic retraction ρ_x of an open neighborhood of the fiber $R_x = \pi^{-1}(g_1(x)) \subset Y$ in the manifold Y onto R_x which depends holomorphically on $x \in U$ (Lemma 3.4 in [F6]). If g and g_1 are sufficiently uniformly close on L then $f(x)$ belongs to the domain of ρ_x for every $x \in L$ and we define $f_1(x) = \rho_x(f(x))$ for $x \in L$. Then f_1 is holomorphic and approximates f uniformly on L, and $\pi \circ f_1 = g_1$. Since π is a Serre fibration, f_1 extends to a continuous map $f_1 \colon \mathbb{C}^n \to Y$ satisfying $\pi \circ f_1 = g_1$ on $\mathbb{C}^n$. Since g_1 is holomorphic and π is a subelliptic submersion, Theorem 1.3 in [F3] furnishes an entire map $\widetilde{f} \colon \mathbb{C}^n \to Y$ such that $\pi \circ \widetilde{f} = g_1$ and $\widetilde{f}|_K$ approximates $f_1|_K$, and hence $f|_K$. (Since π is unramified, the quoted theorem is an immediate consequence of Theorem 1.5 in [FP2].) This shows that Y satisfies CAP and hence the Oka property. (Compare with Corollary 3.3.C' in [Gr2, p. 881].)

Conversely, assume that Y satisfies the Oka property. Fix a holomorphic map $g \colon K \to Y_0$ from a compact convex set $K \subset \mathbb{C}^n$. Since π is a Serre fibration and K is contractible, there is a continuous lifting $f_0 \colon K \to Y$ with $\pi \circ f_0 = g$. Since π is a subelliptic submersion, Theorem 1.3 in [F3] gives a homotopy of liftings $f_t \colon K \to Y$ $(t \in [0,1])$ with $\pi \circ f_t = g$ for every $t \in [0,1]$ such that f_1 is holomorphic. By CAP of Y we can approximate f_1 uniformly on K by an entire map $\widetilde{f} \colon \mathbb{C}^n \to Y$. Then $\widetilde{g} = \pi \circ \widetilde{f} \colon \mathbb{C}^n \to Y_0$ is entire and approximates g uniformly on K. Thus Y_0 satisfies CAP and hence (by the Main Theorem) the Oka property. This completes the proof of Theorem 1.2.

Contractibility of K was essential in the last part and we do not know how to do this without using the Main Theorem. The existence of sprays on Y does not help since these sprays do not necessarily pass down to Y_0.

5. The parametric convex approximation property.

We recall the notion of the *parametric Oka property* (POP); see Theorem 1.5 in [FP2] and Gromov's (Ell_∞) property [Gr2, §3.1].

Let P be a compact Hausdorff space (the parameter space) and $P_0 \subset P$ a closed subset of P which is a strong deformation retract of some neighborhood in P. (In applications P is a polyhedron and P_0 a subpolyhedron.) Given a Stein manifold X and a compact holomorphically convex subset K in X we consider continuous maps $f \colon X \times P \to Y$ such that for every $p \in P$ the map $f^p = f(\cdot\,, p) \colon X \to Y$ is holomorphic in an open neighborhood of K in X (independent of $p \in P$), and for every $p \in P_0$ the map f^p is holomorphic on X. We say that Y satisfies the *parametric Oka property* (POP) if for every such data (X, K, P, P_0, f) there is a homotopy $f_t \colon X \times P \to Y$ $(t \in [0,1])$ consisting of maps satisfying the same properties as $f_0 = f$ such that

(i) the homotopy is fixed on P_0 (i.e., $f_t^p = f^p$ when $p \in P_0$ and $t \in [0,1]$),

(ii) f_t approximates f uniformly on $K \times P$ for all $t \in [0,1]$,

(iii) $f_1^p\colon X \to Y$ is holomorphic for every $p \in P$.

The *parametric convex approximation property* (PCAP) of a manifold Y means by definition that the above holds for any special compact convex set $K \subset \mathbb{C}^n$ of the form (1.2) for any $n \in \mathbb{N}$.

THEOREM 5.1. *If a complex manifold Y satisfies PCAP then it also satisfies the parametric Oka property (and hence* PCAP $\Longleftrightarrow$ POP*).*

To prove Theorem 5.1 it suffices to follow the proof of Theorem 1.1 (§3), using the requisite tools with continuous dependence on the parameter $p \in P$. Precise arguments of this kind can be found in [FP1], [FP2] and we leave out the details.

An analogue of Theorem 1.2 holds for lifting/descending of the parametric Oka property (POP) in a subelliptic Serre fibration $\pi\colon Y \to Y_0$. The implication

$$\text{POP of } Y_0 \Longrightarrow \text{POP of } Y$$

holds for any compact Hausdorff parameter space P and is proved as before by using the parametric version of all tools. However, we can prove the converse implication only for a *contractible* parameter space P since we must lift a map $K \times P \to Y_0$ (with K a compact convex set in $\mathbb{C}^n$) to a map $K \times P \to Y$. One may try to avoid this difficulty by subdividing a sufficiently 'nice' P (e.g. a polyhedron) into a union of contractible subsets and applying the approximation argument in the proof of Theorem 1.2 to each piece separately. However, we don't know how to patch together the individual maps obtained in this way.

One can obtain a similar 'convex approximation' characterization of the Oka property for sections of a holomorphic submersion $h\colon Z \to X$ over a Stein base manifold X (Theorem 1.5 in [FP2]). In this case we must localize the approximation condition to pairs $K_0 \subset K_1$ of small compact convex sets in X ('small' in the sense that the submersion h admits a dominating family of fiber-sprays over an open neighborhood of K_1 in X, and 'convex' in suitable local holomorphic coordinates on X). As in the proof of Proposition 3.1 we initially 'thicken' the section in the fiber directions by introducing additional variables, approximate the thickened section and patch them by Lemma 2.1.

6. Examples and open problems.

We limit ourselves to a few examples which are either new or else we can give a new and simpler proof of the Oka property. A fairly comprehensive list of known examples can be found in §6 of [F7] (see also §3 in [Gr2]).

EXAMPLE 1. Let Y_0 be one of the manifolds $\mathbb{C}^p$, $\mathbb{P}_p$ or a complex Grassmanian of dimension $p \geq 2$. *If $A \subset Y_0$ is a closed algebraic subvariety of complex codimension at least two then $Y = Y_0 \backslash A$ satisfies the Oka property.*

This was proved in [F2] by showing that such Y is subelliptic (i.e., it admits a finite dominating family of sprays) and hence the main result of [F2] applies. Here we give a simpler argument to the effect that Y enjoys CAP.

Let $f: Q \to Y$ be a holomorphic map from a special compact convex set $Q \subset P \subset \mathbb{C}^n$ (1.2). An elementary argument shows that f can be approximated uniformly on Q by an algebraic map $\widetilde{f}: \mathbb{C}^n \to Y$ such that $\Sigma = \widetilde{f}^{-1}(A)$ is an algebraic subvariety of codimension at least two which does not intersect Q. (If $Y_0 = \mathbb{C}^p$ we may take a suitable polynomial approximation of f, and the other cases easily reduce to this one; see e.g. [F6].) By Lemma 3.4 in [F5] there is a holomorphic automorphism ψ of $\mathbb{C}^n$ which approximates the identity map in a neighborhood of Q and satisfies $\psi(P) \cap \Sigma = \emptyset$. Then $f_1 = \widetilde{f} \circ \psi$ maps P holomorphically to $Y = Y_0 \backslash A$ and approximates f uniformly on Q. This proves that Y enjoys CAP.

A similar argument holds if Y_0 is a compact algebraic manifold such that every point $y_0 \in Y_0$ has a Zariski open neighborhood $U \subset Y_0$ which is biregularly equivalent to $\mathbb{C}^p$; see e.g. Proposition 5.5 in [F6].

EXAMPLE 2. The *Hopf manifolds* are holomorphic quotients of $\mathbb{C}^q \backslash \{0\}$ by an infinite cyclic group of dilations of $\mathbb{C}^q$, $q \geq 2$ [BH, p. 225]. Since $\mathbb{C}^q \backslash \{0\}$ enjoys CAP (see Example 1), Corollary 1.3 implies that every Hopf manifold enjoys the Oka property. These manifolds are non-algebraic and even non-Kählerian.

EXAMPLE 3. *The complement of any finite set of points in a complex torus of dimension $q \geq 2$ enjoys the Oka property.* (In [BL] it was proved that such complement is dominated by $\mathbb{C}^q$. This is clearly false for $q = 1$ since the complement of a point in an elliptic curve is Kobayashi hyperbolic.)

To see this, let $\pi: \mathbb{C}^q \to \mathbb{C}^q/\Gamma = \mathbf{T}^q$ be the universal covering of a torus $\mathbf{T}^q$, Γ being a lattice in $\mathbb{C}^q$ of maximal rank $2q$. Choose finitely many points $t_1, \ldots, t_m \in \mathbf{T}^q$ and preimages $p_j \in \mathbb{C}^q$ with $\pi(p_j) = t_j$ for $j = 1, \ldots, m$. The discrete set $\Gamma_0 = \cup_{j=1}^m (\Gamma + p_j) \subset \mathbb{C}^q$ is tame in $\mathbb{C}^q$ according to Proposition 4.1 in [BL] (that proposition is stated for $q = 2$, but the result remains valid for $q \geq 2$). Hence the complement $Y = \mathbb{C}^q \backslash \Gamma_0$ admits a dominating spray and therefore satisfies the Oka property [Gr2], [FP1]. Since $\pi|_Y: Y \to \mathbf{T}^q \backslash \{t_1, \ldots, t_m\}$ is a holomorphic covering projection, Corollary 1.3 implies that the latter set also enjoys the Oka property.

The same argument applies to $Y = \mathbb{C}^q/\text{lattice} = \mathbf{T}^k \times (\mathbb{C}^*)^l \times \mathbb{C}^m$ where $k + l + m = q$. Any such Y is elliptic since the dominating spray $\mathbb{C}^q \times \mathbb{C}^q \to \mathbb{C}^q$, $(z, t) \to z + t$, is Γ-equivariant and hence it passes down to a dominating spray on Y. Thus Y satisfies the Oka property, and we can see as before that the same is true for the complement of any finite set in Y.

EXAMPLE 4. Assume that $\pi: Y \to B$ is a principal fiber bundle with complex Lie group G as a fiber. The group G acts on Y by translations along fibers and B is the space of orbits. G admits a dominating spray $\mathbf{g} \times G \to G$,

$(v, g) \to e^v g$, where $\mathbf{g}$ is the Lie algebra of G. Hence Corollary 1.3 implies that Y and B satisfy the Oka property at the same time. In particular, the Oka property of the total space Y implies the Oka property of the orbit space B.

In conclusion we mention some open problems.

QUESTION 1. Let B be a closed ball in $\mathbb{C}^p$ for some $p \geq 2$. Does $\mathbb{C}^p \backslash B$ satisfy CAP (and hence the Oka property) ? Does $\mathbb{C}^p \backslash B$ admit any nontrivial holomorphic sprays ?

In other words, given a compact convex set $K \subset \mathbb{C}^n$ and a holomorphic map $f \colon K \to \mathbb{C}^p$ with $f(K) \cap B = \emptyset$, is it possible to approximate f by entire maps $\mathbb{C}^n \to \mathbb{C}^p$ which avoid B ? Of course the same question makes sense for more general compact subsets $B \subset \mathbb{C}^q$, for example the convex ones.

Using proper map techniques it might be possible to prove that $\mathbb{C}^p \backslash B$ satisfies CAP_n for n small compared to p (perhaps even for all $n < p$), but the real test case is $n \geq p$. What makes this problem particularly intriguing is the absence of any obvious obstruction; indeed $\mathbb{C}^p \backslash B$ is a union of Fatou-Bieberbach domains.

QUESTION 2. Suppose that any holomorphic map from a ball $B \subset \mathbb{C}^n$ to Y (for any $n \in \mathbb{N}$) can be approximated by entire maps $\mathbb{C}^n \to Y$. Does Y enjoy the Oka property ? (This question appears in [Gr2, p. 881, 3.4.(D)].)

QUESTION 3. Let $\pi \colon Y \to Y_0$ be a holomorphic fiber bundle. Does the Oka property of Y imply the Oka property of the base Y_0 and of the fiber ? (Clearly the answer is affirmative if $Y = Y_0 \times F$.)

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