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Preprint series, Vol. 42 (2004), 923

HOLOMORPHIC FLEXIBILITY OF COMPLEX MANIFOLDS

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ISSN 1318-4865

June 17, 2004

Ljubljana, June 17, 2004

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1. Introduction

One of the most interesting phenomena in complex geometry is the dichotomy *rigidity versus flexibility* for holomorphic mappings. The prototype of all rigidity theorems is undoubtedly the *Picard theorem* to the effect that every holomorphic map $\mathbb{C} \to \mathbb{P}^1 \backslash \{0, 1, \infty\}$ is constant; this leads to the notions of Kobayashi–Eisenman and Brody hyperbolicity [Ko1], [Ko2], [E], [Br], [Za] (see the surveys [PS], [D2], [D3], [Si2]).

Since the rigidity properties of a complex manifold Y are expressed in terms of the nonexistence of holomorphic maps $\mathbb{C}^n \to Y$ of certain rank, the opposite *holomorphic flexibility* of Y should signify the existence of many such maps. As the prototype we propose the *Oka property* which requires that any continuous map $f_0 \colon X \to Y$ from a Stein manifold X (a closed complex submanifold of a Euclidean space) is homotopic to a holomorphic map $f_1 \colon X \to Y$, and if f_0 is holomorphic on a compact holomorphically convex subset $K \subset X$ then the homotopy can be chosen holomorphic and uniformly close to f_0 on K. By the classical results of Oka [O] and Grauert [G1], [G2] this holds if Y is a complex homogeneous manifold; generalizations will be mentioned below. We study relations between the Oka property and other holomorphic flexibility properties:

- the *convex approximation property* concerning the Runge approximation of holomorphic maps $K \to Y$ on compact convex subsets $K \subset \mathbb{C}^n$ by entire maps $\mathbb{C}^n \to Y$,
- the jet transversality theorem for holomorphic maps $X \to Y$ from Stein manifolds to Y,
- the existence of dominating holomorphic maps $\mathbb{C}^n \to Y$,
- the homotopy principle for holomorphic submersions from Stein manifolds to Y.

A property opposite to Kobayashi-Eisenman hyperbolicity [E] is the existence of a *dominating* holomorphic map $\mathbb{C}^p \to Y$ with rank $p = \dim Y$ at $0 \in \mathbb{C}^p$. A stronger condition is the existence of a family of dominating holomorphic maps $f_y: \mathbb{C}^p \to Y$ with $f_y(0) = y$ for $y \in Y$; if f_y depends holomorphically on $y \in Y$ then the family is called a *dominating spray* [Gr3], [FP1]. All flexibility conditions which we consider lie between these two.

We begin by discussing *algebraic approximations.* Unless otherwise stated, an algebraic map in this paper is without singularities (a morphism). An *affine algebraic manifold* is a closed algebraic submanifold of some $\mathbb{C}^N$. A *spray* on a complex manifold Y is a holomorphic map $s: E \to Y$ from the total space of a holomorphic vector bundle $p: E \to Y$, satisfying $s(0_y) = y$ for all $y \in Y$ [Gr3, §1.1.B]. The spray is *algebraic* if $p: E \to Y$ is an algebraic vector bundle and the spray map $s: E \to Y$ is algebraic. A complex (algebraic) manifold Y is (algebraically) *subelliptic* if it admits finitely many (algebraic) sprays $s_j: E_j \to Y$ such that for every $y \in Y$ the vector subspaces $(ds_j)_{0_y}(E_{j,y}) \subset T_yY$ together span T_yY (Definition 1 in §2; compare with [F1]); Y is *elliptic* if there is a *dominating spray* $s: E \to Y$ with $ds_{0_y}(E_y) = T_yY$ for all $y \in Y$. Algebraic subellipticity is a Zariski local condition. For examples see §6.

THEOREM 1.1. *If X is an affine algebraic manifold and Y is an algebraically subelliptic manifold then a holomorphic map $X \to Y$ which is homotopic to an algebraic map can be approximated by algebraic maps uniformly on compacts in X. In particular, every null-homotopic holomorphic map $X \to Y$ is a limit of algebraic maps.*

A more precise result is Theorem 3.1 in §3. The following corollary will be of special importance to us.

COROLLARY 1.2. *If Y is an algebraically subelliptic manifold then every holomorphic map $K \to Y$ from a compact convex subset $K \subset \mathbb{C}^n$ can be approximated uniformly on K by algebraic maps $\mathbb{C}^n \to Y$.*

QUESTIONS. 1. Let X and Y be as in Theorem 1.1. Is every holomorphic map $X \to Y$ homotopic to an algebraic map ? (By [F1] every continuous map $X \to Y$ is homotopic to a holomorphic map.)

2. Assuming that an algebraic manifold Y satisfies the conclusion of Corollary 1.2, does it also satisfy the conclusion of Theorem 1.1 ?

The problem of approximating holomorphic maps by algebraic maps is of central importance in analytic geometry. Algebraic approximations in general do not exist even for maps between affine algebraic manifolds (for example, there are no nonconstant algebraic morphisms $\mathbb{C} \to \mathbb{C}^*$). On the other hand, according to Demailly, Lempert and Shiffman [DL], [Lm], a holomorphic map from a Runge domain in an affine algebraic variety to a semiprojective algebraic variety with isolated singularities can be approximated uniformly on compacts by *Nash algebraic* maps. (A map $U \to Y$ from an open set U in an algebraic variety X is Nash algebraic if its graph is contained in an algebraic subvariety

$\Gamma \subset X \times Y$ with $\dim \Gamma = \dim X$, [Na].) Nash algebraic approximations do not suffice for our application in Theorem 5.1 and would even lead to false statements. In the proof of that theorem we need to approximate a holomorphic map $f: K \to Y$ on a compact convex set $K \subset \mathbb{C}^n$ by a holomorphic map $\mathbb{C}^n \to Y$ whose ramification locus is a thin *algebraic* subvariety of $\mathbb{C}^n$. Global Nash algebraic maps would suffice for this purpose, but these are morphisms by the GAGA principle [Se2, p. 13, Proposition 8].

We now turn to a discussion of transversality theorems. If X and Y are smooth manifolds, $k \in \{0, 1, 2, \ldots\}$ and Z is a smooth closed submanifold of $J^k(X, Y)$ (the manifold of k-jets of smooth maps of X to Y) then for a generic smooth map $f: X \to Y$ its k-jet extension $j^k f: X \to J^k(X, Y)$ is transverse to Z (Thom [T1], [T2]; for extensions see [Ab], [Wh], [Tr], [Go], [GM], [MT]).

Transversality theorems only rarely hold for global holomorphic maps due to holomorphic rigidity. For example, the transversality theorem for one-jets of holomorphic maps $\mathbb{C}^n \to Y$ ($n \le \dim Y$) implies the existence of a holomorphic map with rank n at a generic point of $\mathbb{C}^n$, and the infinitesimal n-dimensional Kobayashi-Eisenman metric of Y [E] vanishes at most points in the range of such a map. If this holds for $n = \dim Y$ then Y is dominable by $\mathbb{C}^n$; if such Y is compact and connected then by [Kd], [CG], [KO] its Kodaira dimension $\kappa = \mathrm{kod}\, Y$ [BH, p. 29] satisfies $\kappa < \dim Y$, i.e., *Y is not of Kodaira general type.* It is believed that a compact projective manifold Y of general type has a proper subvariety $Y_0 \subset Y$ which contains the image of any nonconstant holomorphic map $\mathbb{C} \to Y$ (the *Green-Griffiths conjecture*, [GG]).

The basic transversality theorem (for 0-jets) holds for holomorphic maps to any manifold with a *submersive family* of holomorphic self-maps (Abraham [Ab]); a classical case is *Bertini's theorem* to the effect that almost any projective hyperplane in $\mathbb{P}^n = \mathbb{P}^n(\mathbb{C})$ intersects a given complex submanifold $Z \subset \mathbb{P}^n$ transversally [Be], [GM, p. 150], [Ha, p. 179]. The jet transversality theorem holds for holomorphic maps of Stein manifolds to Euclidean spaces, [Fo]. Furthermore, a holomorphic map from a Stein manifold X to any complex manifold Y can be approximated uniformly on any compact subset $K \subset X$ by holomorphic maps defined in an open neighborhood of K and satisfying the given (jet) transversality condition, [KZ] (see Theorem 4.5 below).

THEOREM 1.3. *Holomorphic maps from a Stein manifold to a subelliptic manifold satisfy the jet transversality theorem. Furthermore, algebraic maps from an affine algebraic manifold to an algebraically subelliptic manifold satisfy the jet transversality theorem on compact subsets.*

The last statement in Theorem 1.3 means that the map is global but satisfies the transversality condition only on a compact subset. For precise statements and proofs see §4.

We say that maps from a Stein manifold X to a complex manifold Y satisfy the *Oka principle* if for any compact holomorphically convex subset $K \subset X$ and any continuous map $f: X \to Y$ which is holomorphic on (a neighborhood of) K

there is a homotopy $f_t\colon X \to Y$ ($t \in [0,1]$) such that $f_0 = f$, f_t is holomorphic on K and $f_t|_K$ is arbitrarily uniformly close to $f|_K$ for every $t \in [0,1]$, and f_1 is holomorphic on X. If this holds for every such pair $K \subset X$ then Y is said to enjoy the *Oka property*. A subelliptic manifold enjoys the Oka property [F1].

THEOREM 1.4. *If Y enjoys the Oka property then holomorphic maps from any Stein manifold to Y satisfy the jet transversality theorem.*

Theorem 1.3 is a special case of Theorem 1.4 since every subelliptic manifold enjoys the Oka property [F1]. However, our proof of Theorem 1.3 (§4) is much more elementary than the proof of the Oka property and it also applies in the algebraic category; for these reasons we separate the two results.

The Oka principle has a long history beginning with Oka [O] and Grauert [G1], [G2], [G3], [Ca] who proved it for all complex homogeneous manifolds; for extensions see [FR], [Gr3], [HK], [HK], [Wi], [FP1], [FP2], [FP3], [F1], [L1], [L2] and the surveys [Le], [F5]. The following flexibility condition, called the *convex approximation property*, was introduced recently in [F6]:

CAP: *Any holomorphic map $f\colon K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ ($n \in \mathbb{N}$) can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \to Y$.*

CAP can be seen as a precise opposite property to Kobayashi-Eisenman hyperbolicity [E]. Observe that Corollary 1.2 establishes the property analogous to CAP for algebraic maps to subelliptic manifolds. It turns out [F6] that

$$\text{CAP} \Longleftrightarrow \text{the Oka property.}$$

That is, maps from convex sets in Euclidean spaces (which are model Stein manifolds) to a given manifold Y determine the answer to the basic dichotomy *flexibility versus rigidity* for maps from all Stein manifolds to Y. This equivalence implies that the Oka property both lifts and descends in a *subelliptic submersive Serre fibration* (SSSF) $\pi\colon Y \to Y_0$, in the sense that Y and Y_0 satisfy the Oka property at the same time (Theorem 1.2 in [F6]; this is related to a conjecture of F. Lárusson [L2, p. 19]). The definition of a *subelliptic submersion* is given in §2 below; a *Serre fibration* is a map satisfying the homotopy lifting property. Examples of SSSFs include holomorphic fiber bundles with subelliptic fibers and, more generally, subelliptic submersions which are topological fiber bundles (e.g., the unramified elliptic fibrations).

COROLLARY 1.5. *Let $Y = Y_m \to Y_{m-1} \to \cdots \to Y_0$ where every map $Y_j \to Y_{j-1}$ ($j = 1, 2, \ldots, m$) is a subelliptic submersive Serre fibration. If one of the manifolds Y_j enjoys the Oka property (and hence the jet transversality theorem for holomorphic maps from Stein manifolds to Y) then so do all the others. This holds in particular if $Y_0 =$ point; such Y is called semisubelliptic.*

In the proof of Corollary 1.5 one only needs the lifting property for maps from compact contractible (convex) sets in Euclidean spaces. A special case worth mentioning are the unramified holomorphic covering projections $Y \to Y_0$.

A Riemann surface enjoys the Oka property if and only if it is non-hyperbolic, i.e., one of the surfaces $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^*$ or a torus; each of these admits a dominating spray (Proposition 6.5 in §6). Hence Corollary 1.5 implies

COROLLARY 1.6. *Let Y be a complex surface with the structure of a holomorphic fiber bundle $\pi: Y \to C$ over a curve C. If C and the fiber $\pi^{-1}(x)$ ($x \in C$) are non-hyperbolic then Y enjoys the Oka property and the jet transversality theorem for maps from Stein manifolds.*

The same holds if $\pi: Y \to C$ is an unramified elliptic fibration (without multiple fibers) over a non-hyperbolic curve C; see (6.4) in §6. Conversely, the Oka property of Y in Corollary 1.6 clearly implies non-hyperbolicity of the base C, but does it also imply non-hyperbolicity of the fiber $\pi^{-1}(x)$? Clearly the answer is affirmative when the bundle is trivial.

The jet transversality theorem implies that 'singularities' of a generic holomorphic map $X \to Y$ satisfy the codimension conditions given by Proposition 2 in [Fo]. This implies the following.

COROLLARY 1.7. *Let X be a Stein manifold. If Y enjoys the Oka property (in particular, if Y is semisubelliptic) and $\dim Y \geq 2\dim X$ then a generic holomorphic map $X \to Y$ is an immersion, and if $\dim Y \geq 2\dim X + 1$ then a generic map is an injective immersion.*

For the last statement concerning injective immersions one needs a multi-jet transversality theorem which is an easy extension; see [Fo, §1.3] for $Y = \mathbb{C}^N$.

In summary, the following hold for any p-dimensional complex manifold:

$$\text{elliptic} \Longrightarrow \text{subelliptic} \Longrightarrow \text{semisubelliptic} \Longrightarrow \text{the Oka property} \Longleftrightarrow$$
$$\Longleftrightarrow \text{CAP} \Longrightarrow \text{jet transversality theorem} \Longrightarrow \text{dominability by } \mathbb{C}^p.$$

Both the strongest and the weakest of the above properties concern the existence of dominating maps $\mathbb{C}^p \to Y$. If Y is compact and connected then

$$\text{dominability by } \mathbb{C}^p \Longrightarrow \ \text{kod}\, Y < \dim Y.$$

In the algebraic category we have the implications

$$\begin{array}{ccccc} \text{elliptic} & \Longrightarrow & \text{subelliptic} & \Longrightarrow & \text{CAP} \\ & & \Downarrow & & \Downarrow \\ & & \text{jet transversality} & \Longrightarrow & \text{dominability} \end{array}$$

where CAP is now interpreted in the sense of Corollary 1.2. The first vertical arrow holds by Theorem 1.3. The algebraic analogue of the Oka property is poorly understood at this time.

We don't know whether any of the above implications can be reversed in general. If Y is Stein then the Oka property with interpolation on closed complex submanifolds in Stein manifolds implies (and hence is equivalent to) the ellipticity of Y [Gr3, 3.2.A.], [FP3, Proposition 1.2].

Although we hope that the above results might be of independent interest, we have the following specific application in mind. Given a continuous map $f: X \to Y$ from a Stein manifold X to a complex manifold Y for which there exists a surjective complex vector bundle map $TX \to f^*TY$, is f homotopic to a holomorphic submersion $X \to Y$? In the $\mathcal{C}^\infty$ category the answer to the analogous question is affirmative if X is an open manifold [P], [Gr1], but in the holomorphic category rigidity of Y is a powerful obstruction. If the answer is affirmative we say that holomorphic submersions $X \to Y$ satisfy the *homotopy principle*. By Theorem II in [F3] this holds if $Y = \mathbb{C}^p$ and $\dim X > p$. By [F4] holomorphic submersions from n-dimensional Stein manifolds to Y satisfy the homotopy principle with approximation on compact holomorphically convex subsets if and only if Y satisfies the following approximation property:

PROPERTY S_n: *Every holomorhic submersion $K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of entire submersions $\mathbb{C}^n \to Y$.*

Analysis of this property led to the algebraic approximation and transversality theorems presented in the paper. Our main result, Theorem 5.1 in §5, asserts that the homotopy principle holds for submersions of Stein manifolds to lower dimensional algebraic manifolds which are Zariski locally equivalent to complements of thin algebraic subvarieties in a Euclidean space, and to their unramified holomorphic quotients. As pointed out in [F6], the analogy between CAP_n (which is defined as CAP for a fixed $n \in \mathbb{N}$) and Property S_n is not merely coincidental but is also reflected in the proofs.

How big is the gap between (semi-) subellipticity, the Oka property and dominability ? For Riemann surfaces all three conditions coincide (Subsection 5 in §6). A more realistic test case are the complex surfaces. If a connected compact complex surface Y is dominable by $\mathbb{C}^2$ then $\mathrm{kod}\, Y \in \{-\infty, 0, 1\}$ and hence one can use the Enriques-Kodaira classification [BH]. A fairly complete list of compact (and compactifiable) dominable surfaces was obtained by Buzzard and Lu [BL]. By inspecting their list one sees that the existing results are rather good in the class of minimal surfaces (not containing any -1 rational curves) with the exception of the K3 surfaces that are not elliptic fibrations (Subsection 6 in §6).

After preliminaries (§2) we discuss algebraic approximations in §3, transversality theorems in §4, holomorphic submersions in §5 and examples and open problems in §6.

2. Preliminaries.

We denote by $\mathcal{H}_X$ the sheaf of germs of holomorphic functions on a complex manifold X and by $\mathcal{H}(X)$ the algebra of all global holomorphic functions on X. Given a pair of complex manifolds X, Y we denote by $\mathcal{H}(X, Y)$ the set of all holomorphic maps $X \to Y$ (so $\mathcal{H}(X) = \mathcal{H}(X, \mathbb{C})$). The compact-open topology on $\mathcal{H}(X, Y)$ is metrizable by a complete metric and hence $\mathcal{H}(X, Y)$ is a *Baire space*. A property of $f \in \mathcal{H}(X, Y)$ is said to be *generic* if it holds for all f in a *residual subset* (a countable intersection of open dense subsets).

A function (or map) is said to be *holomorphic on a compact set* K in a complex manifold X if it is holomorphic in an open set $U \subset X$ containing K. A homotopy $\{f_t\}$ is holomorphic on K if there is a neighborhood U of K such that every f_t is holomorphic on U.

An *affine manifold* is a closed complex or algebraic submanifold of a Euclidean space $\mathbb{C}^N$. The class of affine complex manifolds coincides with the class of *Stein manifolds* [St], [GR] according to the embedding theorems [R], [Bi], [Ns], [EG], [Sch]. A compact subset K in a closed affine submanifold $X \subset \mathbb{C}^N$ is said to be $\mathcal{H}(X)$-convex (*holomorphically convex* in X) if for any $p \in X \backslash K$ there exists $f \in \mathcal{H}(X)$ with $|f(p)| > \sup_K |f|$. Since any $f \in \mathcal{H}(X)$ extends to a function $\widetilde{f} \in \mathcal{H}(\mathbb{C}^N)$, a set $K \subset X$ is $\mathcal{H}(X)$-convex if and only if it is $\mathcal{H}(\mathbb{C}^N)$-convex, i.e., *polynomially convex*.

We denote by $\mathcal{O}_X$ the structure sheaf of an algebraic manifold X and by $\mathcal{O}(X)$ the algebra of all regular algebraic functions on X. Similarly $\mathcal{O}(X, Y)$ denotes the set of all regular algebraic maps (morphisms) between algebraic manifolds X, Y. Clearly $\mathcal{O}(X, Y) \subset \mathcal{H}(X, Y)$ but $\mathcal{O}(X, Y)$ need not be closed in $\mathcal{H}(X, Y)$. If X and Y are compact then $\mathcal{O}(X, Y) = \mathcal{H}(X, Y)$ by Serre's GAGA principle [Se2]. In our case the source manifold X will be affine but Y will often be compact. For basic results on algebraic manifolds and maps we refer to Serre [Se1], [Se2] and Hartshorne [Ha].

The notion of *spray* has already been defined in §1. More generally, a *fiber-spray* for a holomorphic submersion $h: Y \to Y'$ is a triple (E, p, s) consisting of a holomorphic vector bundle $p: E \to Y$ and a holomorphic map $s: E \to Y$ such that for each $y \in Y$ we have $s(0_y) = y$ and $s(E_y) \subset Y_{h(y)} = h^{-1}(h(y))$. (See [Gr3, §1.1.B] and [FP1].) Let $VT_yZ = \ker dh_z \subset T_yY$ for $y \in Y$.

DEFINITION 1. *A holomorphic submersion $h: Y \to Y'$ is subelliptic if each point in Y' has an open neighborhood $U \subset Y'$ such that $h: Y|_U \to U$ admits finitely many fiber-sprays (E_j, p_j, s_j) $(j = 1, \ldots, k)$ satisfying*

$$(ds_1)_{0_y}(E_{1,y}) + (ds_2)_{0_y}(E_{2,y}) \cdots + (ds_k)_{0_y}(E_{k,y}) = VT_yY \tag{2.1}$$

for each $y \in Y|_U$. A collection of sprays satisfying (2.1) is said to be dominating at y. The submersion is elliptic if the above holds with $k = 1$. A complex manifold Y is (sub-)elliptic if the trivial submersion $Y \to$ point is such, and is algebraically (sub-)elliptic if there exist algebraic sprays on Y satisfying the above domination property.

Examples of subelliptic submersions can be found in [G] (see especially sections 0.5.B and 3.4.F), [FP1], [F1]; for further examples see §6 below.

3. Algebraic approximation.

All algebraic maps are assumed to be morphisms (without singularities).

THEOREM 3.1. *Let X be an affine algebraic manifold, K a compact $\mathcal{H}(X)$-convex subset of X and Y an algebraically subelliptic manifold (Definition 1 in §2). If $f_t: K \to Y$ ($t \in [0,1]$) is a homotopy of holomorphic maps such that f_0 is the restriction to K of an algebraic map $X \to Y$ then there exists an algebraic map $F: X \times \mathbb{C} \to Y$ such that $F(\cdot\,,0) = f_0$ and $F(\cdot\,,t)$ approximates f_t uniformly on K for every $t \in [0,1]$.*

COROLLARY 3.2. *Let $K \subset X$ and Y be as in Theorem 1.1. Every null-homotopic holomorphic map $K \to Y$ is a uniform limit of algebraic maps $X \to Y$. In particular, every holomorphic map $K \to Y$ on a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of algebraic maps $\mathbb{C}^n \to Y$.*

We do not know whether Corollary 3.2 remains valid for homotopically nontrivial holomorphic maps $K \to Y$. The result fails if Y is subelliptic but not algebraically subelliptic; for example, there are no nonconstant algebraic morphisms $\mathbb{C} \to \mathbb{C}^*$.

Theorem 3.1 is a special case of the following result.

THEOREM 3.3. *Let $h: Z \to X$ be an algebraic submersion onto a closed affine algebraic manifold X. Assume that Z admits a family of algebraic fiber-sprays (E_j, p_j, s_j) ($j = 1, \ldots, k$) satisfying the domination property (2.1). Let $K \subset X$ be a compact $\mathcal{H}(X)$-convex set and $f_t: K \to Z$ ($t \in [0,1]$) a homotopy of holomorphic sections such that f_0 extends to an algebraic section $X \to Z$. Then there is an algebraic map $F: X \times \mathbb{C} \to Z$ such that $h(F(x,t)) = x$ for all $(x,t) \in X \times \mathbb{C}$, $F(\cdot\,,0) = f_0$, and $F(\cdot\,,t)$ approximates f_t uniformly on K for every $t \in [0,1]$.*

An algebraic submersion satisfying the hypotheses of Theorem 3.3 will be called *algebraically subelliptic* (compare with Definition 1 in §2). This condition is Zariski local with respect to the base manifold X, i.e., if every point $x \in X$ admits a Zariski neighborhood $U \subset X$ such that the restriction of the submersion to U is algebraically subelliptic over U then the submersion is algebraically subelliptic over X. This is proved in the same way as Proposition 1.3 in [F1] (see also Lemmas 3.5.B. and 3.5.C. in [Gr3]).

Theorem 3.3 is an 'algebraic' analogue of Theorem 3.1 in [F1]. (See also [Gr3], as well as Theorems 4.1 and 4.2 in [FP1].) It seems that the first approximation results of this type were proved by Grauert [G1], [G2] for holomorphic maps of Stein manifolds to complex homogeneous manifolds.

Proof of Theorem 3.3. Let (E, p, s) be the (algebraic) composed fiber-spray on Z obtained from the fiber-sprays (E_j, p_j, s_j) ($j = 1, \ldots, k$) (see [Gr3, §1.3] and [FP1, Definition 3.3]). We recall the construction of E. If $k = 1$, we take $(E, p, s) = (E_1, p_1, s_1)$. If $k \geq 2$, we set $E^{(2)} = s_1^*(E_2) \to E_1$ (the pull-back to E_1 of the vector bundle $p_2: E_2 \to Z$ by the first spray map $s_1: E_1 \to Z$) and define the spray map $s^{(2)}: E^{(2)} \to Z$ by $s^{(2)}(e_1, e_2) = s_2(e_2)$ (here $e_1 \in E_1$ and $e_2 \in E_2$ are such that $s_1(e_1) = p_2(e_2)$). Continuing inductively we obtain a sequence of algebraic vector bundle projections

$$E = E^{(k)} \longrightarrow E^{(k-1)} \longrightarrow \cdots \longrightarrow E^{(2)} \longrightarrow E_1 \longrightarrow Z.$$

The composed bundle $E \to Z$ has a well defined zero section which can be identified with Z, and $TE|_Z \simeq TZ \oplus E_1 \oplus \cdots \oplus E_k$. Furthermore, denoting by $s: E \to Z$ the composed spray, the restriction of its differential $ds_{0_z}: T_{0_z}E \to T_z Z$ to the fiber $E_{j,z}$ of E_j over z (in the above direct sum decomposition) equals $(ds_j)_{0_z}: T_{0_z}E_{j,z} \to VT_z Z$. Hence the domination property (2.1) is equivalent to the following domination property of the composed spray:

$$(ds)_{0_z}(T_{0_z}E_z) = VT_z Z, \quad z \in Z. \tag{3.1}$$

Although E does not have a vector bundle structure over Z, it admits a (non-canonical) holomorphic vector bundle structure over any Stein subset $\Omega \subset Z$ ([Gr3, §1.3], [FP1, Corollary 3.5]).

In the rest of the proof we shall only work with the composed spray bundle (E, p, s) and will no longer need the sprays (E_j, p_j, s_j) (so k in the sequel no longer means the number of these sprays). Choose open Stein neighborhoods $V \subset\subset U \subset\subset X$ of K such that the homotopy f_t in Theorem 3.3 is defined in U. Set $V_t = f_t(V) \subset Z$ for $t \in [0,1]$.

LEMMA 3.4. *There are numbers $k \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_k = 1$ such that for every $j = 0, 1, \ldots, k-1$ there exists a homotopy of holomorphic sections ξ_t of the restricted bundle $E|_{V_{t_j}} \to V_{t_j}$ ($t \in I_j = [t_j, t_{j+1}]$) such that ξ_{t_j} is the zero section and $s(\xi_t(z)) = f_t(h(z))$ for all $t \in I_j$ and $z \in V_{t_j}$.*

Proof. Assume first that there exists a Stein open set $\Omega \subset Z$ containing $\cup_{t\in[0,1]} \overline{V}_t$. By Corollary 3.5 in [FP1] $E|_\Omega$ admits the structure of a holomorphic vector bundle over Ω and a holomorphic direct sum splitting $E|_\Omega = H \oplus H'$ where H' is the kernel of ds at the zero section of E and H is some complementary subbundle [GR, p. 256, Theorem 7]. It follows from (3.1) that for every $z \in \Omega$ the restriction $s: H_z \to Z_{h(z)} = h^{-1}(h(z))$ maps a neighborhood of 0_z in H_z biholomorphically onto a relative neighborhood of z in the fiber $Z_{h(z)}$. The size of this neighborhood (and its image in the corresponding fiber of Z) can be chosen uniform for points in any compact subset of Ω, hence on $\cup_{t\in[0,1]}\overline{V}_t$. For every $t \in [0,1]$ the local inverse of $s: H|_{V_t} \to Z$ at the zero section gives a homotopy of sections ξ_τ of $H|_{V_t}$ ($\tau \in J_t = [t, t+\epsilon] \cap [0,1]$), with ξ_t the zero section, such that $s(\xi_\tau(z)) = f_\tau(h(z))$ for $\tau \in I_t$ and $z \in V_t$. By uniformity we can choose $\epsilon > 0$ independent of $t \in [0,1]$ and the lemma follows.

For the general case observe that $f_t(U)$, being a closed Stein submanifold of $Z|_U$, has an open Stein neighborhood in Z [Si1], [D1]. By compactness there are Stein open sets $\Omega_j \subset Z$ ($j = 1, 2, \ldots, m$) and a partitition $[0,1] = \cup_{j=1}^m I_j$

into adjacent closed intervals such that $\cup_{t\in I_j}\overline{V}_t \subset \Omega_j$. Applying the above separately for each I_j proves Lemma 3.4.

Let $(E^{(k)}, p^{(k)}, s^{(k)})$ denote the k-th iterate of (E, p, s) (Definition 3.3 in [FP1], p. 132), and let (E', p', s') denote the pull-back of $(E^{(k)}, p^{(k)}, s^{(k)})$ to X by the map $f_0: X \to Z$.

LEMMA 3.5. (Notation as above.) *There is a homotopy η_t $(t \in [0,1])$ of holomorphic sections of E'_V such that η_0 is the zero section and $s'(\eta_t(x)) = f_t(x)$ for $x \in V$ and $t \in [0,1]$.*

Proof. Such η_t is obtained immediatelly by joining the homotopies $\{\xi_t : t \in [t_j, t_{j+1}]\}$ from Lemma 3.4 belonging to different restricted bundles $E|_{V_{t_j}}$. (For details see Proposition 3.6 in [FP1, p. 134].)

We continue with the proof of Theorem 3.3. By the assumption the sprays (E_j, p_j, s_j) on Y are algebraic. An inspection of the construction shows that, consequently, the spray (E', p', s') in Lemma 3.5 is also algebraic. (The intermediate splittings $E = H \oplus H'$ in the proof of Lemma 3.4 were not necessarily algebraic, but they were only used to construct the homotopies ξ_t.) To complete the proof it remains to show that the homotopy $\eta_t : V \to E'_V$ in Lemma 3.5 admits a uniform approximation on K by a homotopy $\widetilde{\eta}_t : X \to E'$ consisting of algebraic sections depending algebraically also on $t \in \mathbb{C}$, with $\widetilde{\eta}_0 = \eta_0$ being the zero section; the map $F(x,t) = s'(\widetilde{\eta}_t(x)) \in Z$ $(x \in X,\ t \in \mathbb{C})$ then satisfies the conclusion of Theorem 3.3.

In the holomorphic case one argues that, as X is Stein, the composed bundle $E' \to X$ is biholomorphic to a vector bundle ([FP1, Corollary 3.5], [Gr3, §1.3.A']), and hence the existence of a holomorphic homotopy $\widetilde{\eta}_t$ as above is insured by the Oka-Weil approximation theorem. However, the construction of a vector bundle structure on E' involves Grauert's theorem to the effect that a holomorphic vector bundle structure is invariant under holomorphic homotopies. We give an alternative proof which avoids transcendental methods. By the construction of the composed bundle E' there is a finite sequence

$$E' = E^{m,0} \longrightarrow E^{m-1,0} \longrightarrow \cdots \longrightarrow E^{1,0} \longrightarrow X \tag{3.2}$$

in which every map $E^{j,0} \to E^{j-1,0}$ is an algebraic vector bundle projection. Since X is affine, the bundle $E^{1,0} \to X$ is generated by finitely many (say n_1) algebraic sections according to Serre's Theorem A ([Se1], p. 237, Théorème 2). This gives a surjective algebraic map $\pi_1: E^{1,1} = X \times \mathbb{C}^{n_1} \to E^{1,0}$ of the trivial bundle onto $E^{1,0}$. Pulling back the sequence (3.2) to the new total space $E^{1,1}$ we obtain a commutative diagram

$$\begin{array}{ccccccccccc}
E^{m,1} & \longrightarrow & E^{m-1,1} & \longrightarrow & \cdots & \longrightarrow & E^{2,1} & \longrightarrow & E^{1,1} & \longrightarrow & X \\
\downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \pi_{m-1}} & & & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \| \\
E^{m,0} & \longrightarrow & E^{m-1,0} & \longrightarrow & \cdots & \longrightarrow & E^{2,0} & \longrightarrow & E^{1,0} & \longrightarrow & X
\end{array}$$

in which all horizontal maps are algebraic vector bundle projections and the vertical maps π_j for $j \ge 2$ are the induced natural maps. More precisely, we begin by letting $E^{2,1} \to E^{1,1}$ be the pull-back of the bundle $E^{2,0} \to E^{1,0}$ in the bottom row by the vertical morphism $\pi_1: E^{1,1} \to E^{1,0}$, and we denote by $\pi_2: E^{2,1} \to E^{2,0}$ the asociated natural map which makes the respective diagram commute. Moving one step to the left, $E^{3,1} \to E^{2,1}$ is the pull-back of the bundle $E^{3,0} \to E^{2,0}$ in the bottom row by the morphism $\pi_2: E^{2,1} \to E^{2,0}$, and $E^{3,1} \to E^{3,0}$ is the induced natural map; etc. There is an algebraic spray map $s^1: E^{m,1} \to Z$ which is the composition of $\pi_m: E^{m,1} \to E^{m,0}$ with the initial spray $s: E^{m,0} = E \to Z$.

We claim that the homotopy τ_t (consisting of holomorphic sections of $E^{m,0}|_V$) lifts to a homotopy τ_t^1 of holomorphic sections of $E^{m,1}|_V$ such that $s^1(\tau_t^1) = f_t$ for $t \in [0,1]$. It suffices to see that the $E^{1,0}$-component of τ_t (i.e., the projection of τ_t under the composed projection $E^{m,0} \to E^{1,0}$) lifts to $E^{1,1}$; the rest of the lifting follows automatically by the natural vertical maps. But this follows from the fact that the surjective algebraic vector bundle map $\pi_1: E^{1,1} \to E^{1,0}$ admits an algebraic splitting, i.e., there is an algebraic vector bundle map $\sigma_1: E^{1,0} \to E^{1,1}$ over X such that $\pi_1 \circ \sigma_1$ is the identity on $E^{1,0}$. In the holomorphic case this is Theorem 7 in [GR, p. 256]. The proof given there also applies in the algebraic case (over an affine algebraic base manifold X) by applying Serre's Theorem B for coherent algebraic sheaves on affine manifolds ([Se1], p. 237, Théorème 2; see also §50 in [Se1], pp. 242-243]).

We now repeat the same argument with the bundle $E^{2,1} \to E^{1,1} = X_1$ over the affine base manifold $X_1 = X \times \mathbb{C}^{n_1}$: the bundle is generated by algebraic sections which gives a surjective algebraic bundle map $E^{2,2} = X_1 \times \mathbb{C}^{n_2} \to E^{2,1}$. As before we lift the top line in the above diagram to a new level

$$E^{m,2} \longrightarrow E^{m-1,2} \longrightarrow \cdots \longrightarrow E^{2,2} \longrightarrow E^{1,1} = X_1.$$

Note that $E^{2,2} = X_1 \times \mathbb{C}^{n_2} = X \times \mathbb{C}^{n_1+n_2}$ (algebraic equivalence). The homotopy τ_t^1 lifts to a homotopy τ_t^2 in $E^{m,2}|_V$ and we have a new spray map $s^2: E^{m,2} \to Z$ satisfying $s^2(\tau_t^2) = f_t$ for all $t \in [0,1]$.

After m steps we obtain a lifting of the homotopy τ_t to a homotopy τ_t^m of sections of $E^{m,m} = X \times \mathbb{C}^N$ ($N = n_1 + n_2 + \cdots + n_m$) over $V \subset X$. We also have an algebraic spray $s^m: E^{m,m} \to Z$ such that $s^m(\tau_t^m) = f_t: V \to Z|_V$. Since the bundle $E^{m,m} = X \times \mathbb{C}^N$ is trivial, we can approximate τ_t^m uniformly on K by a homotopy of algebraic (polynomial) sections $\widetilde{\tau}_t$ which is also polynomial in the variable t. The algebraic map $F(x,t) = s^m(\widetilde{\tau}_t(x)) \in Z$ ($(x,t) \in X \times \mathbb{C}$) satisfies the conclusion of Theorem 3.3.

4. Transversality theorems for holomorphic maps.

Recall [Wh, p. 227] that a *stratification* of a closed complex subvariety A of a complex manifold X is a decomposition of A into a locally finite disjoint union of open connected complex manifolds A_α, called the *strata* of A, such that the boundary of each stratum is a union of lower dimensional strata. Whitney proved that every closed complex subvariety $A \subset X$ admits a stratification which is *regular with respect to tangent planes* [Wh, Theorem 8.5]. We shall not recall this Condition (a) of Whitney but only remark the following consequence which is of importance for transversality.

Given complex manifolds X, Y we denote by $J^k(X,Y)$ the manifold of all k-jets of holomorphic maps $X \to Y$. We have $J^0(X,Y) = X \times Y$ and

$$J^1(X,Y) = \{(x,y,\lambda) \colon x \in X,\ y \in Y,\ \lambda \in \mathrm{Hom}(T_xX, T_yY)\}.$$

For a holomorphic map $f \colon X \to Y$ we denote by $j^k f \colon X \to J^k(X,Y)$ its k-jet extension for any $k \in \mathbb{N}$. In particular, $(j^1 f)_x = j^1_x f = (x, f(x), df_x)$.

Given stratified complex subvarieties $A \subset X$, $B \subset Y$, let $NT_{A,B} \subset J^1(X,Y)$ consist of all $(x,y,\lambda) \in J^1(X,Y)$ such that x belongs to a stratum A_α of A, y belongs to a stratum B_β of B and $\lambda(T_xA_\alpha) + T_yB_\beta \neq T_yY$ (that is, λ restricted to T_xA_α fails to be transverse to T_yB_β inside T_yY). *If the stratifications of A and B are Whitney regular then $NT_{A,B}$ is a closed analytic subvariety of $J^1(X,Y)$.* (The emphasis is on 'closed' which is insured by Whitney regularity; see [Tr], [GM. p. 38].)

Given a map $f \colon X \to Y$, we say that $f|_A$ *is transverse to* B if the range of $j^1 f \colon X \to J^1(X,Y)$ does not intersect the set $NT_{A,B}$. Equivalently,

$$(x \in A_\alpha,\ y = f(x) \in B_\beta) \implies df_x(T_xA_\alpha) + T_y(B_\beta) = T_yY.$$

We say that holomorphic maps $X \to Y$ satisfy the *jet transversality theorem*, abbreviated JTT, if for any $k = 0, 1, 2, \ldots$ and closed Whitney stratified complex subvarieties $A \subset X$, $B \subset J^k(X,Y)$ the set

$$\mathcal{T}_{A,B} = \{f \in \mathcal{H}(X,Y) \colon j^k f|_A \text{ is transverse to } B\}$$

is residual in $\mathcal{H}(X,Y)$. (It follows from JTT that for any sequence of closed Whitney stratified subvarieties $A_i \subset X$, $B_i \subset J^{k_i}(X,Y)$, $k_i \geq 0$, $i = 1, 2, \ldots$, the set of $f \in \mathcal{H}(X,Y)$ such that $j^{k_i} f|_{A_i}$ is transverse to B_i for every i is residual in $\mathcal{H}(X,Y)$.)

We shall base our discussion on the following condition introduced by Gromov [Gr2, pp. 71–73]. He also indicated that it implies the jet transversality theorem if X is Stein [Gr2, p. 73, (C')].

CONDITION Ell_1: For every holomorphic map $f \colon X \to Y$ there is a holomorphic map $F \colon X \times \mathbb{C}^N \to Y$ for some $N \geq \dim Y$ such that $F(\cdot, 0) = f$ and $F(x, \cdot) \colon \mathbb{C}^N \to Y$ has rank equal to $\dim Y$ at $0 \in \mathbb{C}^N$ for every $x \in X$.

For examples see [Gr2, p. 72]. Condition Ell_1 clearly implies that Y is dominable by $\mathbb{C}^p$, $p = \dim Y$. The analogous condition can be defined in the algebraic category. Compare also [GM, p. 39, 1.3.5.].

THEOREM 4.1. *If X is Stein then Condition* Ell_1 *for Y implies the jet transversality theorem for holomorphic maps $X \to Y$. If X and Y are algebraic manifolds and X is affine then the algebraic Condition* Ell_1 *implies the jet transversality theorem on compacts for algebraic maps $X \to Y$.*

The precise meaning of the last statement is that for any compact subset $K \subset X$ the set of algebraic maps $X \to Y$ which satisfy the given (jet) transversality condition at all points of K is open and dense in $\mathcal{O}(X,Y)$. See also Theorem 4.6 for JTT with interpolation on a subvariety $X_0 \subset X$.

PROPOSITION 4.2. *Condition* Ell_1 *holds for holomorphic maps from Stein manifolds to subelliptic manifolds. Algebraic* Ell_1 *holds for algebraic maps from affine algebraic manifolds to algebraically subelliptic manifolds.*

Theorem 1.3 in §1 follows immediately from Theorem 4.1 and Proposition 4.2. To prove Theorem 4.1 one applies Abraham's method [Ab] which is essentially a reduction to Sard's theorem [Sa]. Although this method is well known (see e.g. §1.3.7. in [GM, pp. 39-40] and, in the holomorphic case, [Fo] and [KZ]), we include a sketch of proof because in §5 we shall need a variation of this technique. We certainly do not claim any originality here. Proposition 4.2, which is new, is proved below.

Whitney's condition (a) is essential in the following (well known) lemma.

LEMMA 4.3. *Let X and Y be complex manifolds and let $A \subset X$, $B \subset J^k(X,Y)$ be closed, Whitney stratified, complex subvarieties. For any compact subset $K \subset X$ the set*

$$\mathcal{T}_{A,B,K} = \{f \in \mathcal{H}(X,Y) \colon j^k f|_A \text{ is transverse to } B \text{ on } A \cap K\}$$

is open in $\mathcal{H}(X,Y)$. The analogous result holds for algebraic maps.

Proof. If $(j^k f)|_A \colon A \to J^k(X,Y)$ is transverse to B at every point of $A \cap K$ then $(j^k f)(K) \cap NT_{A,B} = \emptyset$. Since $NT_{A,B}$ is closed in $J^k(X,Y)$, there is a compact set $L \subset X$ with $K \subset \operatorname{int} L$ such that $(j^k f)(L) \cap NT_{A,B} = \emptyset$. If $g \in \mathcal{H}(X,Y)$ is sufficiently uniformly close to f on L then $j^k g$ is close to $j^k f$ on K (by Cauchy estimates) and hence $(j^k g)(K) \cap NT_{A,B} = \emptyset$ for such g.

Proof of Theorem 4.1. It suffices to show that for every compact $K \subset X$ the set $\mathcal{T}_{A,B,K} \subset \mathcal{H}(X,Y)$ (which is open in $\mathcal{H}(X,Y)$ by Lemma 4.3) is dense in $\mathcal{H}(X,Y)$; the JTT then follows by taking the intersection of such sets over a countable family of compacts with union X.

Consider first the basic case $k = 0$, $A = X$, $B \subset Y$. Let $f \colon X \to Y$ and $F \colon X \times \mathbb{C}^N \to Y$ be as in the definition of Ell_1. Let $\pi \colon X \times \mathbb{C}^N \to \mathbb{C}^N$ denote the projection $\pi(x,t) = t$. Fix a compact set K in X. Since $\partial_t F(x,0) \colon T_0\mathbb{C}^N = \mathbb{C}^N \to T_{f(x)}Y$ is surjective for every $x \in X$, there are a small ball $D \subset \mathbb{C}^N$ around the origin and an open set $U \subset X$ containing K such that F is a submersion of $V = U \times D$ to Y. Hence $B' = F^{-1}(B) \cap V$ is a closed Whitney stratified complex analytic subvariety of V (the strata B_β of B pull back by $F|_V$ to the strata B'_β of B'). Set $f_t = F(\cdot\,,t)\colon X \to Y$ for $t \in \mathbb{C}^N$. If $(x,t) \in B'_\beta \subset V$ then $y = f_t(x) \in B_\beta$ and by inspecting definitions we see that the following are equivalent (compare [GM, p. 40]):

(a) $(df_t)_x(T_xX) + T_yB_\beta = T_yY$;

(b) (x,t) is a regular point of the restricted projection $\pi\colon B'_\beta \to D$.

By Sard's theorem [Sa] the set of regular values of all projections in (b) (for every β) is residual in D. Choosing t in this set and close to 0 we get a map $f_t\colon X \to Y$ which is transverse to B on U and approximates $f = f_0$.

In general (when A is a subvariety of X) one applies the above argument with U replaced by the manifold $U \cap A_\alpha$ for a fixed stratum A_α of A. This gives a residual set of t's in $\mathbb{C}^N$ for which $f_t|_{A_\alpha \cap U}$ is transverse to a stratum B_β. Since A and B have at most countably many strata, we find $t \in \mathbb{C}^N$ arbitrarily close to 0 such that $f|_{A\cap U}$ is transverse to B. (Whitney's regularity condition is only needed to insure Lemma 4.3.)

Suppose now that $k > 0$ and $B \subset J^k(X,Y)$. Fix $f \in \mathcal{H}(X,Y)$ and a compact set $K \subset X$. It sufficies to prove that f can be approximated uniformly on K by holomorphic maps $X \to Y$ whose k-jet extension is transverse to B on $A \cap K$. Let $F\colon X \times \mathbb{C}^N \to Y$ be as in Ell_1. We may assume that X is embedded in a Euclidean space $\mathbb{C}^n$. Let $\mathcal{W}$ denote the complex vector space of all polynomial maps $P\colon \mathbb{C}^n \to \mathbb{C}^N$ of degree $\le k$. Consider the map $G\colon X \times \mathcal{W} \to Y$, $G(x,P) = F(x,P(x))$ $(x \in X,\ P \in \mathcal{W})$. For each fixed $P \in \mathcal{W}$ set $G_P = G(\cdot\,,P)\colon X \to Y$. Observe that $G_0(x) = F(x,0) = f(x)$.

LEMMA 4.4. *The map $\Phi\colon X \times \mathcal{W} \to J^k(X,Y)$, $\Phi(x,P) = j^k_x(G_P)$ (the k-jet extension of G_P at $x \in X$) is a submersion in an open neighborhood of $X \times \{0\}$ in $X \times \mathcal{W}$.*

Proof. This argument is local and hence we may assume $X = \mathbb{C}^n$. Write $P = (P_1,\ldots,P_N) \in \mathcal{W}$ and let $t = (t_1,\ldots,t_N)$ be the coordinates on $\mathbb{C}^N$. For every multiindex $I = (i_1,\ldots,i_n)$ we have

$$\partial^I_x(G_P) = \sum_{j=1}^N \frac{\partial}{\partial t_j} F(x,P(x))\, \partial^I_x P_j(x) + H_I(x)$$

where $H_I(x)$ contains only terms $\partial^J_x P$ with $|J| < |I|$ multiplied by various partial derivatives of F. Hence the k-jet map $j^k_x(G_P)$ is lower triangular with respect to the components of $j^k_x P$ and the diagonal terms are nondegenerate at $P = 0$ (since $G_0(x) = F(x,0)$ and $\partial_t F(x,0)$ is nondenegenerate). This proves Lemma 4.4. (For more details see [KZ].)

Sard's theorem [Sa] implies that for most $P \in \mathcal{W}$, $j^k(G_P)|_A$ is transverse to B at every point of $A \cap K$. This concludes the proof of Theorem 4.1.

REMARK. The map F in Condition Ell_1 insures the existence of *global* holomorphic maps $X \to Y$ approximating the initial map f and satisfying the transversality condition on a compact set $K \subset X$. If one's goal is to approximate f on a compact $K \subset X$ by a transverse map which is only defined in a neighborhood $U \subset X$ of K, it suffices to apply the above proof with a holomorphic map $F: U \times D \to Y$ satisfying Ell_1 along $U \times \{0\}$, where $D \subset \mathbb{C}^N$ is a small ball around $0 \in \mathbb{C}^N$. Such U and F always exist (without any condition on Y) provided that K is holomorphically convex in X. Indeed, the set $\{(x, f(x)): x \in K\} \subset X \times Y$ has an open Stein neighborhood $\Omega \subset X \times Y$ according to [Si1], [D1]. There exist holomorphic vector fields $V^1, \ldots, V^N$ in Ω tangent to the fiber Y of the projection $\pi_X: X \times Y \to X$ and generating the tangent space of the fiber at each point. (By Lemma 5 in [Fo, p. 178] one may take $N = p + [\frac{1}{2}(n+1)]$ with $n = \dim X$, $p = \dim Y$.) Let θ^j_t denote the local flow of V^j. For points x in a small open neighborhood $U \subset X$ of K and for $t_1, \ldots, t_N \in \mathbb{C}$ sufficiently small we take

$$F(x, t_1, \ldots, t_N) = \pi_Y \circ \theta^1_{t_1} \circ \cdots \circ \theta^N_{t_N}(x, f(x)).$$

Applying the above proof with this map gives the following result from [KZ] (we state a special case).

THEOREM 4.5 (Kaliman and Zaidenberg [KZ]). *Assume that X is a Stein manifold, Y is a complex manifold and $A \subset X$, $B \subset J^k(X, Y)$ are closed Whitney stratified complex subvarieties. For any $f \in \mathcal{H}(X, Y)$ and compact set $K \subset X$ there is a holomorphic map $g: U \to Y$ in an open neighborhood $U \subset X$ of K such that $j^k g|_{A \cap U}$ is transverse to B and g approximates f as close as desired uniformly on K.*

The analogous result holds for sections of any holomorphic submersion $Z \to X$ onto a Stein manifold X. In general g cannot be chosen holomorphic on X. The result fails on non-Stein X (take X compact and $Y = \mathbb{C}$).

Proof of Proposition 4.2. Fix a holomorphic map $f: X \to Y$. If Y admits a dominating spray (E, p, s) then $f^*E \to X$ is a holomorphic vector bundle and there is a holomorpic injection $\iota: f^*E \to E$ covering f. Then $s \circ \iota: f^*E \to Y$ satisfies Ell_1 except that f^*E need not be a trivial bundle. By Cartan's Theorem A f^*E is generated by finitely many (say N) holomorphic sections over X which gives a surjective complex vector bundle map $\tau: X \times \mathbb{C}^N \to f^*E$. The map $F = s \circ \iota \circ \tau: X \times \mathbb{C}^N \to Y$ satisfies Ell_1.

In general Y admits a dominating family of sprays (E_j, p_j, s_j) for $j = 1, \ldots, k$ and we perform the above procedure several times. (In essence we use dominating composed sprays as in §3 above.) Let $E'_1 = f^*E_1 \to X$ be the pull-back of $\pi_1: E_1 \to Y$ and define $\sigma_1: E'_1 \to Y$ by $\sigma_1(x, e) = s_1(f(x), e)$. As before there is a surjective complex vector bundle map $X \times \mathbb{C}^{n_1} \to E'_1$ for some $n_1 \in \mathbb{N}$, and composing it with σ_1 we obtain $f_1: X_1 = X \times \mathbb{C}^{n_1} \to Y$ satisfying $f_1(x, 0) = f(x) = y \in Y$ and

$$\partial_t f_1(x, t)|_{t=0}(T_0\mathbb{C}^{n_1}) = (ds_1)_y(E_{1,y}) \subset T_y Y.$$

Repeating the construction with $f_1\colon X_1 \to Y$ and the spray $s_2\colon E_2 \to Y$ we find $n_2 \in \mathbb{N}$ a holomorphic map $f_2\colon X_2 = X_1 \times \mathbb{C}^{n_2} = X \times \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \to Y$ satisfying $f_2(x,t,0) = f_1(x,t)$ (so $f_2(x,0,0) = f(x) = y$) and

$$\partial_u f_2(x,0,u)|_{u=0}(T_0\mathbb{C}^{n_2}) = (ds_2)_y(E_{2,y}).$$

After k steps we obtain a map $F\colon X \times \mathbb{C}^N \to Y$ ($N = n_1 + \cdots + n_k$) satisfying the following for every $x \in X$, $y = f(x) \in Y$:

$$F(x,0) = f(x), \quad \partial_t F(x,0)(T_0\mathbb{C}^N) = \sum_{j=1}^{k} (ds_j)_y(E_{j,y}) = T_y Y.$$

The last equality is the domination property (2.1). This completes the proof in the holomorphic case.

In the algebraic case the bundle $E_1' = f^*E_1 \to X$ and the map $\sigma_1\colon E_1' \to Y$ are algebraic. By Serre's Theorem A [Se1, p. 237, Théorème 2] E_1' is generated by finitely many algebraic sections which gives a surjective algebraic vector bundle morphism $X_1 = X \times \mathbb{C}^{n_1} \to E_1'$ for some $n_1 \in N$. Proceeding as before we see that the final map F is algebraic.

Proof of Theorem 1.4. Assume that X is Stein and maps $X \to Y$ satisfy the Oka principle (§1). Let $f\colon X \to Y$ be a holomorphic map. Choose compact $\mathcal{H}(X)$-convex subsets $K \subset L$ of X, with K contained in the interior of L. By Theorem 4.5 we can approximate f uniformly on L by a holomorphic map $g\colon U \to Y$, defined on an open set $U \supset L$, such that $j^k g|_{U \cap A}$ is transverse to B. If the approximation is sufficiently close, there is a smooth map $\widetilde{g}\colon X \to Y$ which agrees with g in a neighborhood of L and with f on $X \backslash U$. By the assumed Oka principle $\widetilde{g}$ can be approximated uniformly on L by holomorphic maps $\widetilde{f}\colon X \to Y$. If the approximation is sufficiently close then $\widetilde{f}$ still satisfies the desired transversality condition on K according to Lemma 4.3. This shows the density of transverse maps on compacts in X, thus completing the proof.

We also give an interpolation version of Theorem 1.4 (compare [KZ]). Given a closed complex subvariety X_0 of X, $f_0 \in \mathcal{H}(X,Y)$ and $r \in \mathbb{N}$ the set

$$\mathcal{H}(X,Y;X_0,f_0,k) = \{f \in \mathcal{H}(X,Y)\colon j^r f|_{X_0} = j^r f_0|_{X_0}\}$$

is a closed metric subspace of $\mathcal{H}(X,Y)$ and hence a Baire space. We say that maps $X \to Y$ satisfy the *Oka principle with jet interpolation on* X_0 if for any continuous map $f_0\colon X \to Y$ which is holomorphic in an open neighborhood of $X_0 \cup K$ (where $K \subset X$ is compact $\mathcal{H}(X)$-convex) and for any $r \in \mathbb{N}$ there is a homotopy $\{f_t\}_{t\in[0,1]} \in \mathcal{H}(X,Y;X_0,f_0,r)$ satisfying the Oka property (§1). For results in this direction see [FP3] and [F1].

THEOREM 4.6. *Let X be a Stein manifold. Assume that maps $X \to Y$ satisfy the Oka principle with jet interpolation on a subvariety $X_0 \subset X$. Let $A \subset X$ and $B \subset J^k(X,Y)$ be closed Whitney stratified complex subvarieties. If $f_0 \in \mathcal{H}(X,Y)$ is such that $j^k f_0|_A$ is transverse to B at all points of $A \cap X_0$ then for any $r \in \mathbb{N}$ there is a residual set of $f \in \mathcal{H}(X,Y;X_0,f_0,r)$ for which $j^k f|_A$ is transverse to B.*

Proof. It suffices to prove that we can approximate f_0 uniformly on any compact holomorphically convex subset $K \subset X$ by $f \in \mathcal{H}(X,Y;X_0,f_0,r)$ such that $j^k f|_A$ is transverse to B at every point of K. Assume first that maps $X \to Y$ satisfy Condition Ell_1 and let $F: X \times \mathbb{C}^N \to Y$ be as in the definition of Ell_1, with $F(\cdot\,,0) = f_0$. Consider the basic case $k=0$, $B \subset Y$. There exist functions $g_1,\ldots,g_l \in \mathcal{H}(X)$ defining X_0 and vanishing (at least) to order $r+1$ on X_0. For every $x \in X$ let $\sigma_x: \mathbb{C}^{Nl} \to \mathbb{C}^N$ be defined by

$$\sigma_x(t_1,\ldots,t_l) = \sum_{j=1}^{l} t_j g_j(x), \qquad t_j \in \mathbb{C}^N,\ j=1,2,\ldots,l.$$

Clearly σ_x is surjective if $x \in X\backslash X_0$ and is the zero map if $x \in X_0$. The map $\widetilde{F}: X \times \mathbb{C}^{Nl} \to Y$ defined by $\widetilde{F}(x,t) = \widetilde{F}(x,t_1,\ldots,t_l) = F(x,\sigma_x(t_1,\ldots,t_l))$ is a submersion with respect to t (at $t=0$) if $x \in X\backslash X_0$ and is degenerate if $x \in X_0$. Hence the proof of Theorem 4.1 applies over $X\backslash X_0$. Let $f_t = \widetilde{F}(\cdot\,,t): X \to Y$ for $t \in \mathbb{C}^{Nl}$. By construction $j^r f_t|_{X_0} = j^r f_0|_{X_0}$ for every t. Choose a compact set $K \subset X$. By the assumption $f_0|_A$ is transverse to B on $A \cap X_0$. Hence there is an open neighborhood $U \subset X$ of $A \cap X_0 \cap K$ such that $f_t|_{A\cap U}$ is transverse to B for every t sufficiently close to 0. (Here we need Whitney regularity.) The set $K' = K\backslash U \subset X\backslash X_0$ is compact and hence for most values of t the map $f_t|_A$ is transverse to B on $A \cap K'$. Thus $f_t|_A$ is transverse to B on $K \cap A$ for most t close to 0 which concludes the proof.

The same argument gives a semiglobal version analogous to Theorem 4.5 without any restriction on Y. The proof of the general case is completed as in Theorem 1.4 by applying the Oka principle with jet interpolation on X_0.

In the algebraic category we are unable to obtain a global transversality theorem since $\mathcal{O}(X,Y)$ need not be a Baire space. However, the global result holds under the following stronger assumption on Y.

PROPOSITION 4.7. *If Y is an algebraic manifold with an algebraic spray $s: E \to Y$ such that $s: E_y \to Y$ is a submersion for every $y \in Y$ then for every affine algebraic manifold X the regular algebraic maps $X \to Y$ satisfy the (global) jet transversality theorem.*

Proof. Let $f_0: X \to Y$ be an algebraic map. Pulling back the spray $s: E \to Y$ by f_0 gives an *algebraic submersion* $F: X \times \mathbb{C}^N \to Y$ satisfying $f_0 = F(\cdot\,,0)$. For a generic choice of $t \in \mathbb{C}^N$ the algebraic map $f_t|_A = F(\cdot\,,t)|_A$

is transverse to a given subvariety $B \subset Y$ by Sard [Sa]. (Indeed this holds for all but finitely many values of t.) Similarly we obtain the jet transversality by considering maps $x \to F(x, P(x))$ for a suitably chosen polynomial P (compare with the proof of Theorem 4.1).

5. The homotopy principle for holomorphic submersions.

In this section we prove the homotopy principle for holomorphic submersions from Stein manifolds to a certain class of complex manifolds (Theorem 5.1). The proof relies on [F3], [F4] and the tools developed in this paper.

Recall the setup from [F4]. For a pair of complex manifolds X, Y we denote by $\mathcal{S}(X, Y)$ the set of all (f, ι) where $f\colon X \to Y$ is a continuous map and $\iota\colon TX \to TY$ is a fiberwise surjective complex vector bundle map such that the diagram in Figure 1 commutes.

$$\begin{array}{ccc} TX & \stackrel{\iota}{\longrightarrow} & TY \\ \big\downarrow & & \big\downarrow \\ X & \stackrel{f}{\longrightarrow} & Y \end{array}$$

Figure 1: The space $\mathcal{S}(X, Y)$

Note that ι is the composition of a surjective complex vector bundle map $TX \to f^*TY$ with the natural map $f^*TY \to TY$. Observe that

— the existence of ι in Fig. 1 is invariant under homotopies of the base map;

— if $f\colon X \to Y$ is a holomorphic submersion then $(f, Tf) \in \mathcal{S}(X, Y)$.

Hence for every continuous map $f\colon X \to Y$ which is homotopic to a holomorphic submersion $X \to Y$ there exists a ι such that $(f, \iota) \in \mathcal{S}(X, Y)$. When does the converse hold ? By Theorem II in [F3] it holds if X is Stein and $Y = \mathbb{C}^p$, $p < \dim X$: *Every surjective complex vector bundle map $TX \to X \times \mathbb{C}^p$ is homotopic to the tangent map of a holomorphic submersion $X \to \mathbb{C}^p$.* A positive answer to this question for a given manifold Y, applied with $X = \mathbb{C}^n$ ($n \ge p = \dim Y$) and the constant map $f\colon \mathbb{C}^n \to y_0 \in Y$, implies that Y is dominable by $\mathbb{C}^p$.

DEFINITION 2. *An algebraic manifold Y is of Class $\mathcal{A}$ if every point of Y admits a Zariski open neighborhood which is biregularly isomorphic to $\mathbb{C}^p \backslash A$ for some algebraic subvariety $A \subset \mathbb{C}^p$ with $\dim A \le p - 2$.*

Examples of manifolds of Class $\mathcal{A}$ and their quotients can be found in §6 (Subsections 3 and 4). The following is our main result.

THEOREM 5.1. *Assume that a complex manifold Y admits an unramified holomorphic covering $\widetilde{Y} \to Y$ where $\widetilde{Y}$ is of Class $\mathcal{A}$. Let X be a Stein manifold with $\dim X > \dim Y$ and $K \subset X$ a compact $\mathcal{H}(X)$-convex subset.*

(a) *For any* $(f,\iota)\in\mathcal{S}(X,Y)$ *the map* f *is homotopic to a holomorphic submersion* $f_1\colon X\to Y$. *If in addition* $f|_K\colon K\to Y$ *is a holomorphic submersion and* $\iota|_K=Tf|_K$ *then* f_1 *can be chosen such that* $f_1|_K$ *approximates* $f|_K$ *uniformly on* K.

(b) *Holomorphic submersions* $f_0,\ f_1\colon X\to Y$ *are homotopic through holomorphic submersions* $X\to Y$ *if and only if* (f_0,Tf_0) *and* (f_1,Tf_1) *belong to the same path connected component of* $\mathcal{S}(X,Y)$.

(c) *If* $\dim X\ \ge\ 2\dim Y-1$ *then every continuous map* $X\to Y$ *is homotopic to a holomorphic submersion; if* $\dim X\ge 2\dim Y$ *then every pair of holomorphic submersions* $X\to Y$ *is regularly homotopic.*

Part (c) of Theorem 5.1 follows from (a) and (b) by topological reasons (Corollary 2.3 in [F4]). We shall reduce parts (a) and (b) to Theorem 2.1 in [F4]. Let $z=(z_1,\ldots,z_n)$, $z_j=x_j+iy_j$, be the coordinates on $\mathbb{C}^n$. Set

$$P=\{z\in\mathbb{C}^n\colon |x_j|\le 1,\ |y_j|\le 1,\ j=1,\ldots,n\}. \tag{5.1}$$

A *special convex set* in $\mathbb{C}^n$ is a compact convex subset of the form

$$K=\{z\in P\colon y_n\le h(z_1,\ldots,z_{n-1},x_n)\} \tag{5.2}$$

where h is a smooth concave function with values in $(-1,1)$.

DEFINITION 3. *Let* d *be a distance function induced by a Riemannian metric on a complex manifold* Y.

(a) Y *satisfies* Property S_n *if for every holomorphic submersion* $f\colon K\to Y$ *on a special convex set* K *(5.2) and for every* $\epsilon>0$ *there is a holomorphic submersions* $\widetilde{f}\colon P\to Y$ *satisfying* $\sup_{x\in K} d(f(x),\widetilde{f}(x))<\epsilon$.

(b) Y *satisfies* Property HS_n *if for every homotopy of holomorphic submersions* $f_t\colon K\to Y$ $(t\in[0,1])$ *such that* f_0 *and* f_1 *extend to holomorphic submersions* $P\to Y$ *there exists for every* $\epsilon>0$ *a homotopy of holomorphic submersions* $\widetilde{f}_t\colon P\to Y$ $(t\in[0,1])$ *satisfying* $\widetilde{f}_0=f_0$, $\widetilde{f}_1=f_1$, *and* $\sup_{x\in K,\ t\in[0,1]} d(f_t(x),\widetilde{f}_t(x))<\epsilon$.

According to Theorem 2.1 in [F4] the conclusion of (a) (respectively of (b)) in Theorem 5.1 holds provided that Y (or some holomorphic covering space of Y) satisfies Property S_n (respectively Property HS_n) with $n=\dim X$. Hence Theorem 5.1 reduces to the following.

PROPOSITION 5.2. *A manifold* Y *of Class* $\mathcal{A}$ *satisfies Properties* S_n *and* HS_n *for every* $n>\dim Y$.

In the proof of Proposition 5.2 we shall need the following lemmas.

LEMMA 5.3. *Every manifold of Class* $\mathcal{A}$ *is algebraically subelliptic.*

Proof. A manifold of Class $\mathcal{A}$ is covered by Zariski open sets U_j biregularly isomorphic to $\mathbb{C}^p\backslash A_j$ where $A_j\subset\mathbb{C}^p$ is an algebraic subvariety with $\dim A_j\le$

$p-2$. Since $\mathbb{C}^p \backslash A_j$ admits a dominating algebraic spray (see [Gr3], 0.5.B. (iii), or Example C in [FP1, p. 119]), Proposition 1.3 in [F1] implies that Y is algebraically subelliptic. (Proposition 1.3 in [F1] is stated only for semi-projective manifolds, but the proof is valid for general algebraic manifolds. See also Lemmas 3.5.B. and 3.5.C. in [Gr3].)

LEMMA 5.4. *Every p-dimensional manifold Y of Class $\mathcal{A}$ is of the form $Y = \widehat{Y} \backslash A$ where $\widehat{Y}$ is a compact algebraic manifold which is Zariski locally equivalent to $\mathbb{C}^p$ and $A \subset \widehat{Y}$ is a closed algebraic subvariety of dimension $\leq p-2$.*

Proof. Cover Y by Zariski open sets U_j admitting algebraic isomorphisms $\phi_j : U_j \to \mathbb{C}^p \backslash A_j$ where $\dim A_j \leq p-2$. Fix a pair of indices which we denote 0 and 1 to simplify the notation. Let $V = U_0 \cap U_1 \subset Y$. For $j = 0, 1$ we have $\phi_j(V) = \mathbb{C}^p \backslash (A_j \cup Z_j)$ for some algebraic subvariety $Z_j \subset \mathbb{C}^p$. The transition map $\psi = \phi_1 \circ \phi_0^{-1} : \mathbb{C}^p \backslash (A_0 \cup Z_0) \to \mathbb{C}^p \backslash (A_1 \cup Z_1)$ extends holomorphically across every point of $A_0 \backslash Z_0$ (since $\operatorname{codim} A_0 \geq 2$) to a map $\widehat{\psi} : \mathbb{C}^p \backslash Z_0 \to \mathbb{C}^p \backslash Z_1$. Similarly ψ^{-1} extends to $\widehat{\psi}^{-1} : \mathbb{C}^p \backslash Z_1 \to \mathbb{C}^p \backslash Z_0$. Clearly the extensions are algebraic inverses of each other; hence $\widehat{\psi}$ is a regular algebraic isomorphism of $\mathbb{C}^p \backslash Z_0$ onto $\mathbb{C}^p \backslash Z_1$ mapping $A_0 \backslash Z_0$ onto $A_1 \backslash Z_1$.

The extended transition maps $\widehat{\psi}_{i,j}$ obtained in this way (for any pair of charts ϕ_i, ϕ_j) satisfy the 1-cocyle condition and hence define an algebraic manifold $\widehat{Y}$ which is Zariski locally isomorphic to $\mathbb{C}^p$. Explicitly, we obtain $\widehat{Y}$ by taking for every index j a copy of $\mathbb{C}^p$, and we glue the copies corresponding to indices i, j by the isomorphism $\widehat{\psi}_{i,j} : \mathbb{C}^p \backslash Z_i \to \mathbb{C}^p \backslash Z_j$. Let $\widehat{U}_j \subset \widehat{Y}$ be the subset corresponding to the j-th copy of $\mathbb{C}^p$, and denote by $\widehat{\phi}_j : \widehat{U}_j \to \mathbb{C}^p$ the tautological map (this is a local chart on $\widehat{Y}$). Let $A \subset \widehat{Y}$ be defined by the condition $\widehat{\phi}_j(A \cap \widehat{U}_j) = A_j$ for every j. By construction, A is a closed complex subvariety of codimension ≥ 2 and Y can be identified with $\widehat{Y} \backslash A$. This proves Lemma 5.4.

Proof of Proposition 5.2. We first prove Property S_n. Let $f_0 : K \to Y$ be a holomorphic submersion where K is a special compact convex set (5.2) in $\mathbb{C}^n$. By Lemma 5.3 and Corollary 3.2 we can approximate f_0 uniformly on K by algebraic maps $\mathbb{C}^n \to Y$; hence we may assume that $f_0 : \mathbb{C}^n \to Y$ is algebraic and $f_0|_K : K \to Y$ is a submersion. The critical locus $\Sigma_0 \subset \mathbb{C}^n$ of f_0 (the set of non-submersion points) is an algebraic subvariety of $\mathbb{C}^n$ which does not intersect K.

Case 1: $\dim \Sigma_0 \leq n-2$. Lemma 3.4 in [F3] provides a holomorphic automorphism ψ of $\mathbb{C}^n$ which approximates the identity map in a neighborhood of K and satisfies $\psi(P) \cap \Sigma_0 = \emptyset$. The map $\widetilde{f} = f_0 \circ \psi : P \to Y$ is a holomorphic submersion approximating f on K. This proves Property S_n.

Case 2: $\dim \Sigma_0 = n-1$. We shall reduce to Case 1 by inductively removing all $(n-1)$-dimensional irreducible components from Σ_0 (we change the map at every step to make it less singular).

Choose an irreducible component $\Sigma_0' \subset \Sigma_0$ of dimension $n-1$ and a point $z_0 \in \Sigma_0'$ which does not belong to any other irreducible component of Σ_0. Write $Y = \widehat{Y} \backslash A$ as in Lemma 5.4. Choose a Zariski open set $U \subset \widehat{Y}$ isomorphic to $\mathbb{C}^p$ and containing the point $y_0 = f(z_0)$. Let $s_0 \colon U \times \mathbb{C}^p \to U$ denote the spray $s_0(y,t) = y + t$, where we identify U with $\mathbb{C}^p$. Choose a closed algebraic subvariety $Y_0 \subset \widehat{Y}$ of pure dimension $p-1$ such that $\widehat{Y} \backslash U \subset Y_0$ and $y_0 \notin Y_0$. Let $L = [Y_0]^{-1}$ where $[Y_0] \to \widehat{Y}$ is the line bundle defined by the divisor of Y_0. Let $\tau_p = \widehat{Y} \times \mathbb{C}^p$. For sufficiently large $m \in \mathbb{N}$ the spray $s_0 \colon \tau_p|_U = U \times \mathbb{C}^p \to U$ extends to a regular algebraic spray $s \colon E = \tau_p \otimes L^{\otimes m} \to \widehat{Y}$ which is degenerate over Y_0 (i.e., $s(y,t) = y$ for all $y \in Y_0$ and $t \in E_y$) and equals s_0 over $\widehat{Y} \backslash Y_0 \subset U$ (using the identification $E|_U \simeq \tau_p|_U$). For the construction of s see Proposition 1.3 in [F1, p. 541] or §3.5.B. and §3.5.C. in [Gr3].

Let $Z_0 = f_0^{-1}(Y_0) \subset \mathbb{C}^n$. Set $E' = f_0^*(E) \to \mathbb{C}^n$ and denote by $F \colon E' \to \widehat{Y}$ the composition of the natural map $E' \to E$ (covering f_0) with the spray $s \colon E \to \widehat{Y}$. By Quillen and Suslin the bundle $E' \to \mathbb{C}^n$ is algebraically trivial, $E' \simeq \mathbb{C}^n \times \mathbb{C}^p$. (One can avoid this result by using Serre's Theorems A and B to show that E' is the quotient of a trivial bundle over $\mathbb{C}^n$, and pulling back F to this trivial bundle.) Thus we obtain an algebraic map $F \colon \mathbb{C}^n \times \mathbb{C}^p \to \widehat{Y}$ satisfying

(a) $F(z,0) = f_0(z)$ $(z \in \mathbb{C}^n)$,

(b) $F(z,t) = f_0(z)$ $(z \in Z_0,\ t \in \mathbb{C}^p)$, and

(c) $\partial_t F(z,t) \colon T_t\mathbb{C}^p \to T_{F(z,t)}\widehat{Y}$ is an isomorphism for $z \in \mathbb{C}^n \backslash Z_0$, $t \in \mathbb{C}^p$.

Write $V = \mathbb{C}^n \backslash Z_0$. Applying the transversality arguments in §4 with the map F (see especially Theorem 4.6 and Proposition 4.7 which is relevant since F is a submersion over V) we obtain an algebraic map $f_1 \colon \mathbb{C}^n \to \widehat{Y}$ satisfying

(i) f_1 approximates f_0 uniformly on K,

(ii) f_0 and f_1 agree along the subvariety Z_0,

(iii) $f_1|_V$ is transverse to the subvariety $A \subset \widehat{Y}$, and hence the algebraic subvariety $B = (f_1|_V)^{-1}(A) \subset V$ has codimension at least 2, and

(iv) the ramification locus $C \subset V$ of $f_1|_V$ has dimension $\leq n-2$.

The set $\Sigma_1 = (\Sigma_0 \cap Z_0) \cup \overline{B} \cup \overline{C}$ is an algebraic subvariety of $\mathbb{C}^n$ which does not intersect K and has less $(n-1)$-dimensional irreducible components than Σ_0. (Indeed, since $\overline{B}$ and $\overline{C}$ contain no such components, the only ones are the $(n-1)$-dimensional components of Σ_0 which are contained in Z_0. Since $z_0 \in \Sigma_0' \backslash Z_0$, Σ_0' is not among them.) The map $f_1 \colon \mathbb{C}^n \backslash \Sigma_1 \to \widehat{Y}$ has range in $Y = \widehat{Y} \backslash A$ and is of maximal rank (a submersion) at every point.

Repeating the same argument with (f_1, Σ_1) and continuing inductively we obtain in finitely many steps a submersion $\widetilde{f} \colon \mathbb{C}^n \backslash \Sigma \to Y$ where Σ is an algebraic subvariety with $\dim \Sigma \leq n-2$. This reduces the proof to Case 1 and hence establishes Property S_n of Y.

The above proof shows the following result which we state for future references.

PROPOSITION 5.5. *If Y_0 is a compact algebraic manifold of class $\mathcal{A}_0$ and $A \subset Y_0$ is a closed algebraic subvariety then every holomorphic map $f: K \to Y_0$ on a compact convex set $K \subset \mathbb{C}^n$ can be approximated uniformly on K by algebraic maps $\widetilde{f}: \mathbb{C}^n \to Y_0$ such that the subvariety $\widetilde{f}^{-1}(A) \subset \mathbb{C}^n$ has codimension equal to the codimension of A in Y_0.*

It remains to prove that a manifold Y of Class $\mathcal{A}$ also satisfies Property HS_n for $n > \dim Y$. We shall need the following lemma on algebraic approximation of the initial homotopy f_t.

LEMMA 5.5. *Let $K \subset P \subset \mathbb{C}^n$ be as in (5.1), (5.2). Let $f_t: K \to Y$ for $t \in [0,1]$ be a homotopy of holomorphic maps such that f_0, f_1 extend to holomorphic maps $P \to Y$. Fix a metric d on Y compatible with the manifold topology. For every $\epsilon > 0$ there is an algebraic map $F: \mathbb{C}^{n+1} \to Y$ such that $d(F(x,t), f_t(x)) < \epsilon$ for all $(x,t) \in (K \times [0,1]) \cup (P \times \{0,1\}) \subset \mathbb{C}^n \times \mathbb{C}$.*

Proof. By Corollary 3.2 we may assume that f_0 is algebraic. By Theorem 4.5 in [FP1] we can approximate the homotopy f_t uniformly on K by a homotopy consisting of holomorphic maps $\widetilde{f}_t: P \to Y$ ($t \in [0,1]$) such that $\widetilde{f}_t = f_t$ for $t = 0, 1$. Hence we may assume that the original homotopy f_t is defined and holomorphic on P, and f_0 is algebraic. Lemma 5.5 now follows from Theorem 3.1 applied with the compact set P in $X = \mathbb{C}^n$.

We continue with the proof of Proposition 5.2. Assume that the map f_t in Lemma 5.5 is a submersion $K \to Y$ for $t \in [0,1]$ and a submersion $P \to Y$ for $t \in \{0,1\}$. If the approximation of f_t by the algebraic map $F_t = F(\cdot\,,t): \mathbb{C}^n \to Y$, furnished by Lemma 5.5, is sufficiently uniformly close on the respective sets then we may assume the same properties for F_t (after shrinking K and P slightly). Let $\Sigma \subset \mathbb{C}^{n+1}$ be the algebraic subvariety consisting of all point $(x,t) \in \mathbb{C}^{n+1}$ at which the partial differential $\partial_x F_t: T_x\mathbb{C}^n \to T_{F(x,t)}Y$ has rank $< \dim Y$. The transversality argument in the proof of Property S_n shows $\dim \Sigma \le n-1$ for a generic choice of F. Fix such F. For all but finitely many $t \in \mathbb{C}$ the set $\Sigma_t = \{x \in \mathbb{C}^n : (x,t) \in \Sigma\}$ then satisfies $\dim \Sigma_t \le n-2$. By a small deformation $\tau: [0,1] \to \mathbb{C}$ of the parameter interval $[0,1] \subset \mathbb{R} \subset \mathbb{C}$ inside $\mathbb{C}$ we can avoid this exceptional set of t's and obtain a homotopy of algebraic submersions $F_{\tau(t)}: \mathbb{C}^n \backslash \Sigma_{\tau(t)} \to Y$ ($t \in [0,1]$) with $\dim \Sigma_{\tau(t)} \le n-2$ ($t \in [0,1]$). Since $F_{\tau(t)}$ approximates f_t on K (resp. on P for $t = 0,1$), $\Sigma_{\tau(t)} \cap K = \emptyset$ for all $t \in [0,1]$ and $\Sigma_{\tau(t)} \cap P = \emptyset$ for $t = 0,1$. (We shrink K and P slightly.)

Applying Lemma 3.4 in [F3] we obtain a family of holomorphic automorphisms $\psi_t \in \mathrm{Aut}\mathbb{C}^n$ depending smoothly on $t \in [0,1]$, with ψ_0, ψ_1 being the identity map, such that $\psi_t(P) \cap \Sigma_{\tau(t)} = \emptyset$ for every $t \in [0,1]$. The homotopy $\widetilde{f}_t = F_{\tau(t)} \circ \psi_t : P \to Y$ ($t \in [0,1]$) consisting of holomorphic submersions approximates the initial homotopy f_t on K (resp. on P for $t = 0,1$). If $\widetilde{f}_0$ is uniformly sufficiently close to f_0 on P then we can join the two maps by a homotopy of submersions $P \to Y$; the same holds at $t = 1$. This shows that Y satisfies Property HS_n, thus concluding the proof of Proposition 5.2.

REMARK. We expect that Theorem 5.1 is valid when Y is an algebraically subelliptic manifold, but our proof breaks down since the jet transversality theorem for algebraic maps only holds on compacts. The problem is to find for a given $n > \dim Y$ an algebraic map $\widetilde{f}\colon \mathbb{C}^n \to Y$, approximating a given initial holomorphic map $f\colon K \to Y$, such that the ramification locus of $\widetilde{f}$ has codimension ≥ 2 in $\mathbb{C}^n$. Such $\widetilde{f}$ exists for example if Y admits a submersive algebraic spray (Proposition 4.7 in §4).

6. Examples.

This section contains a fairly complete list of manifolds which are known to satisfy one or more of the holomorphic flexibility properties considered in the paper. Complex surfaces are considered in Subsection 6.

(6.1) *Elliptic manifolds and submersions.* Ellipticity here means the existence of a dominating spray and should not be confused with the more customary use of this term as in 'elliptic fibrations'. Both ellipticity and subellipticity are stable under passing to an unramified covering space (Proposition 1.6 in [F1] or (∗∗) on p. 883 of [Gr3]), but it is unknown whether they pass to holomorphic quotients. Both properties are stable for taking Cartesian products ([F1], Lemma 2.5), but are not invariant under general bimeromorphic (or birational) isomorphisms.

(6.1.1) Every complex Lie group and homogeneous space is elliptic since the exponential map induces a dominating spray. Unipotent algebraic Lie groups are algebraically elliptic since the exponential map is algebraic.

(6.1.2) The complement $\mathbb{C}^p \backslash A$ of a tame complex subvariety $A \subset \mathbb{C}^p$ of codimension at least two is elliptic (Lemma 7.1 in [FP2]); if A is algebraic then $\mathbb{C}^p \backslash A$ is algebraically elliptic ([F1], Proposition 1.2). In particular, if A is a tame discrete set in $\mathbb{C}^p$ (in the sense of [RR]) for $p > 1$ then $\mathbb{C}^p \backslash A$ is elliptic, and even algebraically elliptic if A is finite.

(6.1.3) More generally, any complex manifold whose tangent bundle is generated by finitely many $\mathbb{C}$-complete holomorphic vector fields is elliptic; a dominating spray is obtained by composing their flows ([Gr3]; [FP1], p. 119, (∗)). In particular, a compact complex manifold whose tangent bundle is generated by global sections is elliptic (examples are the complex tori).

(6.1.4) Let $\mathbf{T}^p = \mathbb{C}^p/\Gamma$ where $\Gamma \subset \mathbb{C}^p$ is a lattice of real rank $2p$ ($p \geq 2$). For any finite set of points $q_1, \ldots, q_m \in \mathbb{C}^2$ the set $\Gamma_0 = \cup_{j=1}^m (\Gamma + q_j)$ is tame in $\mathbb{C}^p$ (Proposition 4.1 in [BL], [Bu]) and hence $\widetilde{Y} = \mathbb{C}^p \backslash \Gamma_0$ is elliptic by (6.1.2). Note that $\widetilde{Y}$ covers $\mathbf{T}^p$ with m points removed (since the inverse image of a point in $\mathbf{T}^p$ equals $\Gamma + q$ for some $q \in \mathbb{C}^p$). Since the Oka property is inherited by holomorphic quotients by Corollary 1.5, we conclude

PROPOSITION 6.1.4. *The complement of finitely many points in a complex torus of dimension ≥ 2 enjoys the Oka property and the jet transversality theorem for holomorphic maps from Stein manifolds.*

In [BL] and [Bu] it was shown that one can even remove finitely many small balls from a torus $\mathbf{T}^p$ and still have a surface dominable by $\mathbb{C}^p$. We do not know whether the Oka property is preserved under such removal.

(6.1.5) (Sub-)ellipticity of a holomorphic submersion $\pi\colon Y \to Z$ was defined in §2 in terms of dominating (families of) fiber-sprays over small open subsets in Z. Given a projective fiber bundle $\pi\colon E \to Z$ with fiber $\mathbb{P}^n$ and a closed subvariety $\Sigma \subset E$ with (algebraic) fibers of dimension $\leq n-2$, the restricted submersion $\pi\colon Y = E\backslash\Sigma \to Z$ is subelliptic [F1]; if E is the projective closure of a holomorphic vector bundle $E_0 \to Z$ then $\pi\colon E_0\backslash\Sigma \to Z$ is elliptic [Gr3], [FP2]. In general this fails if we remove complex hypersurfaces. However, we show the following.

PROPOSITION 6.1.5.1. *If g is a meromorphic function on a complex manifold Z then the submersion $\pi\colon Y = \{(z,w) \in Z \times \mathbb{C}\colon w \neq g(z)\} \to Z$, $\pi(z,w) = z$, is elliptic outside the indeterminacy set of g.*

Proof. The intedeterminacy set of g is a closed complex subvariety $A \subset Z$ of complex codimension at least two. Each point $z_0 \in Z\backslash A$ has an open neighborhood $U \subset Z\backslash A$ such that $g|_U = g_0/g_1$ where g_0 and g_1 are holomorphic functions without common zeros in U. The vector field

$$V(z,w) = \bigl(g_1(z)w - g_0(z)\bigr)\frac{\partial}{\partial w}$$

on $U \times \mathbb{C}$ is tangent to the fibers of π, it vanishes precisely on the graph of $g|_U$, and is complete with the flow $\phi_t(z,w) = (z, \psi(z,w,t))$ $(t \in \mathbb{C})$ where

$$\begin{aligned}\psi(z,w,t) &= g(z)\bigl(1 - e^{tg_1(z)}\bigr) + w\, e^{tg_1(z)} \\ &= -g_0(z)\bigl(t + t^2 g_1(z)/2 + \cdots\bigr) + w\, e^{tg_1(z)}.\end{aligned}$$

The restriction of ϕ to $Y|_U$ is a dominating fiber-spray for the restricted submersion $\pi\colon Y|_U \to U$.

By the homotopy principle for sections of elliptic submersions [Gr3], [FP2] we obtain the following.

COROLLARY 6.1.5.2. *If g is a meromorphic function without point of indeterminacy on a Stein manifold X then functions $f\colon X \to \mathbb{C}$ satisfying $f(x) \neq g(x)$ $(x \in X)$ enjoy the Oka principle (every continuous function with this property is homotopic to a holomorphic function, etc.)*

Corollary 6.1.5.2 applies in particular to any meromorphic function g on an open Riemann surface X. Picard's theorem shows that this is false if g admits essential singularities.

QUESTION: Does the manifold Y in Proposition 6.1.5.1 enjoy the Oka property of the base Z does ? In particular, does the complement of a meromorphic graph in $\mathbb{C}^2$ enjoy the Oka property ?

(6.2) *Subelliptic manifolds.* Our main examples (besides the elliptic ones) are the manifolds of Class $\mathcal{A}$ (see below). Here we only discuss the general properties of this class.

(6.2.1) Algebraic subellipticity is a Zariski-local property: *If every point of Y admits an algebraically subelliptic Zariski open neighborhood then Y is algebraically subelliptic* (Proposition 1.3 in [F1]). It is not known whether the same is true for holomorphic (sub-)ellipticity. The following seems a good test case since Y is a union of Fatou-Bieberbach domains:

QUESTION: Does $Y = \mathbb{C}^p\backslash$(closed ball) for $p \geq 2$ admit a nontrivial spray ? Is Y subelliptic ? Does it satisfy the Oka property ?

(6.2.2) In [Gr3, §3.5.C] it is claimed that algebraic subellipticity is preserved by removal of codimension ≥ 2 algebraic subvarieties. We don't know how to prove this; compare with Proposition 1.4 and Lemma 5.1 in [F1].

(6.2.3) Is every subelliptic manifold also elliptic ? A good test case may be $\mathbb{P}^q\backslash A$ where $A \subset \mathbb{P}^q$ is a thin algebraic subvariety; $\mathbb{P}^q\backslash A$ is algebraically subelliptic by Proposition 1.2 in [F1].

(6.3) *Classes $\mathcal{A}_0 \subset \mathcal{A}$.* By Lemma 5.3 every manifold of Class $\mathcal{A}$ is algebraically subelliptic.

(6.3.1) The basic examples of Class $\mathcal{A}_0$ manifolds are the affine spaces, the projective spaces $\mathbb{P}^q$ and the complex Grassmanians. Further examples are the total spaces W of holomorphic fiber bundles $\pi: W \to Y$ where the base Y is of Class $\mathcal{A}_0$, the fiber is $F = \pi^{-1}(y) \in \{\mathbb{C}^m, \mathbb{P}^m\}$ and the structure group is $GL_m(\mathbb{C})$ respectively $PGL_m(\mathbb{C})$. Every such bundle is algebraic by GAGA [Se2] and its restriction to any affine Zariski open set $U \simeq \mathbb{C}^q \subset Y$ is algebraically trivial, $\pi^{-1}(U) \simeq U \times F$. It follows that W is covered by Zariski open sets biregular to $\mathbb{C}^{q+m}$ and hence is of Class $\mathcal{A}_0$.

(6.3.2) A special case of (6.3.1) are the *Hirzebruch surfaces H_l* where $l = 0, 1, 2, \ldots$; these are fiber bundles over $\mathbb{P}^1$ with fiber $\mathbb{P}^1$. H_0 equals $\mathbb{P}^1 \times \mathbb{P}^1$, and H_1 is $\mathbb{P}^2$ blown up at one point. Every H_l is birationally equivalent to $\mathbb{P}^2$.

(6.3.3) Manifolds of Class $\mathcal{A}$ have been considered by Gromov [Gr3, §3.5.D] in connection with the Oka-Grauert principle. He called such manifolds *Ell-regular* and showed that this class is stable under blowing up points [Gr3, §3.5.D"]. In our terminology this means the following.

PROPOSITION 6.3.3. (Gromov [Gr3]) *If Y is of Class $\mathcal{A}$ (respectively $\mathcal{A}_0$) and $\widetilde{Y}$ is obtained from Y by blowing up finitely many points then $\widetilde{Y}$ is also of Class $\mathcal{A}$ (respectively $\mathcal{A}_0$).*

Proof. By localization it suffices to show that the manifold L obtained by blowing up $\mathbb{C}^q$ at the origin is of Class $\mathcal{A}$. L is the total space of a holomorphic line bundle $\pi: L \to \mathbb{P}^{q-1}$ (the universal bundle); L is trivial over the complement of each hyperplane $\mathbb{P}^{q-2} \subset \mathbb{P}^{q-1}$ (which equals $\mathbb{C}^{q-1}$), and hence each point in L has a Zariski neighborhood of the form $\pi^{-1}(\mathbb{P}^{q-1} \backslash \mathbb{P}^{q-2})$ which is biregular to $\mathbb{C}^q$.

(6.4) *Quotients of Class $\mathcal{A}$ manifolds.* These satisfy the Oka property (Corollary 1.5) and the homotopy principle for submersions (Theorem 5.1).

(6.4.1) Quotients $Y = \mathbb{C}^p/\text{lattice} = \mathbf{T}^k \times (\mathbb{C}^*)^l \times \mathbb{C}^m$ where $\mathbf{T}^k$ is a complex k-dimensional torus and $k+l+m = p$. These are Lie groups and hence elliptic. The same argument as in Corollary 6.1 shows that the complement of any finite set in Y enjoys the Oka property.

(6.4.2) Let $\pi: W \to Y$ be a holomorphic fiber bundle whose base Y is a quotient of a Class $\mathcal{A}_0$ manifold, the fiber $\pi^{-1}(y)$ is $\mathbb{C}^m$ respectively $\mathbb{P}^m$, and the structure group is $GL_m(\mathbb{C})$ respectively $PGL_m(\mathbb{C})$. Then W is also a quotient of a Class $\mathcal{A}_0$ manifold. To see this, let $\rho: \widetilde{Y} \to Y$ be a holomorphic covering with $\widetilde{Y}$ of Class $\mathcal{A}_0$. Pulling back $\pi: W \to Y$ by ρ gives a holomorphic fiber bundle $\widetilde{\pi}: \widetilde{W} \to \widetilde{Y}$ with the same fiber and structure group.

$$\begin{array}{ccc} \widetilde{W} & \stackrel{\iota}{\longrightarrow} & W \\ \big\downarrow \widetilde{\pi} & & \big\downarrow \pi \\ \widetilde{Y} & \stackrel{\rho}{\longrightarrow} & Y \end{array}$$

By (6.3.2) $\widetilde{W}$ is of Class $\mathcal{A}_0$. Clearly the natural map $\iota: \widetilde{W} \to W$ is a holomorphic covering and the claim follows.

(6.4.3) The *Hopf surfaces*: these are quotients of $\mathbb{C}^p \backslash \{0\}$ $(p \geq 2)$ by an infinite cyclic group, for example the one generated by $z \to 2z$ [BH, p. 225]. They are non-algebraic and non-Kählerian; their Kodaira dimension is $-\infty$.

(6.5) *Riemann surfaces.* We have the following precise result. The equivalence of (d), (e) and (f) is well known and is stated only for completeness.

PROPOSITION 6.5. *The following are equivalent for a Riemann surface:*

(a) it admits a dominating spray,

(b) it enjoys the Oka property;

(c) it satisfies the conclusion of Theorem 5.1;

(d) it is dominable by $\mathbb{C}$;

(e) it is non-hyperbolic;

(f) it is one of the surfaces $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^*$ or a torus $\mathbb{C}/\Gamma$.

Proof. The universal covering of any Riemann surface is $\mathbb{P}^1$, $\mathbb{C}$ or $U = \{z \in \mathbb{C}: |z| < 1\}$. $\mathbb{P}^1$ has no nontrivial quotients while the quotients of $\mathbb{C}$ are $\mathbb{C}^*$ and the complex tori $\mathbb{C}/\Gamma$. All of these are homogeneous and hence elliptic (admit a dominating spray). They also satisfy Theorem 5.1 since $\mathbb{P}^1$ and $\mathbb{C}$ are of Class $\mathcal{A}$. The disc and its quotients are hyperbolic.

(6.6) *Compact complex surfaces.* Recall (§1) that the Oka property implies dominability by $\mathbb{C}^2$, and by [Kd], [CG], [KO] a connected compact complex surface which is dominable by $\mathbb{C}^2$ has the Kodaira dimension $\kappa \in \{-\infty, 0, 1\}$. By the Enriques-Kodaira classification (Chapter VI in [BH]; see also [BL] for a very readable survey) every such surface is obtained from one of the following *minimal surfaces* by blowing up finitely many points:

(6.6.1) a holomorphic $\mathbb{P}^1$-bundle over a curve C; $\kappa = -\infty$.

(6.6.2) A torus (quotient $\mathbb{C}^2/\Gamma$ where Γ is a lattice of rank four); $\kappa = 0$.

(6.6.3) A K3 surface (a simply connected compact surface Y with trivial canonical bundle $K_Y = \Lambda^2(T^*Y)$); $\kappa = 0$.

(6.6.4) A minimal surface with the structure of an elliptic fibration. Here, κ can be any of the numbers $-\infty, 0, 1$.

The above list is arranged according to the results of Buzzard and Lu [BL]. All surfaces on the list are minimal in the sense that they do not contain any smooth rational curves $\mathbb{P}^1$ with self-intersection number -1 (any such can be blown down), and every compact complex surface X is obtained from one of them (or from a minimal surface of general type if $\operatorname{kod} X = 2$) by blowing up points. Furthermore, if $\operatorname{kod} X \geq 0$ then X is obtained from a unique minimal Y. Recall that blowing up of points preserves the Class $\mathcal{A}_0$ (6.3.3).

Tori (6.6.2) are complex Lie groups; they satisfy the Oka property and the conclusion of Theorem 5.1. See also (6.1.4).

For the surfaces of type (6.6.1) and for certain holomorphic fibrations (6.6.3) we have the following result in which the base C need not be compact.

PROPOSITION 6.6. *Assume that $\pi: Y \to C$ is either a holomorphic fiber bundle with fiber $\mathbb{P}^1$ or an unramified elliptic fibration without multiple fibers over a Riemann surface C. Then the following are equivalent:*

(a) Y satisfies the Oka property,

(b) Y is dominable by $\mathbb{C}^2$,

(c) C is non-hyperbolic.

Proof. (a)⇒(b) holds in general (see §1), (b)⇒(c) is obvious, and (c)⇒(a) follows from Corollary 1.5 and Proposition 6.5 (in this case C is one of the Riemann surfaces $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^*$ or a torus $\mathbb{C}/\Gamma$, and Y is semisubelliptic).

Observe also that the total space of a fiber bundle $Y \to C$ with fiber $\mathbb{P}^1$ and base $C \in \{\mathbb{P}^1, \mathbb{C}, \mathbb{C}^*\}$ is of Class $\mathcal{A}$, and if C is a torus then it is a quotient of a Class $\mathcal{A}$ manifold. Theorem 5.1 applies to all such manifolds.

Assume now that $\pi: Y \to C$ is an elliptic fibration without multiple fibers but with a nonempty (finite) ramification locus $\operatorname{br}\pi \subset Y$. Although we are unable to settle this case, we describe the reduction to a semilocal problem around the branching locus. As before dominability of Y by $\mathbb{C}^2$ implies that C is either $\mathbb{P}^1$ or a torus. Setting $C' = \pi(\operatorname{br}\pi)$ (a finite set in C) and $Y' =$

$\pi^{-1}(C') \subset Y$, the restricted map $\pi: Y \backslash Y' \to C \backslash C'$ is a subelliptic submersion and a smooth fiber bundle. Let X be a Stein manifold, $f: X \to Y$ a continuous map and $g = \pi \circ f: X \to C$.

$$\begin{array}{ccccc} & & Y & \supset & Y' \\ & \overset{f}{\nearrow} & \big\downarrow \pi & & \big\downarrow \pi \\ X & \overset{g}{\longrightarrow} & C & \supset & C' \end{array}$$

Since C is elliptic, there is a homotopy $g_t: X \to C$ $(t \in [0,1])$ from $g = g_0$ to a holomorphic map g_1.

PROBLEM. Find an open set $U \subset C$ containing C' and a homotopy of continuous maps $f_t: g_t^{-1}(U) \to Y$ satisfying $f_0 = f$ and $g_t = \pi \circ f_t$ for all $t \in [0,1]$), such that f_1 is holomorphic in $V := g_1^{-1}(U)$.

If such $\{f_t\}$ exists then by the homotopy lifting property applied to the fiber bundle $\pi: Y \backslash Y' \to C \backslash C'$ we can extend f_t continuously to X such that $g_t = \pi \circ f_t$ on X for all t. In particular, $f_1: X \to Y$ is a lifting of the holomorphic map $g_1: X \to C$ such that $f_1|_V$ is holomorphic. Now Theorem 1.3 from [F2] gives a homotopy $f_t: X \to Y$ for $t \in [1,2]$, beginning at $t = 1$ with f_1 and ending at $t = 2$ with a holomorphic map $f_2: X \to Y$, such that every f_t is holomorphic in a neighborhood of $g_1^{-1}(C') \subset X$ and $\pi \circ f_t = g_1$ for all $t \in [1,2]$. This gives a homotopy from $f = f_0$ to a holomorphic map f_2 provided the above problem is solvable. (The approximation on compact $\mathcal{H}(X)$-convex subset of X does not cause complications, nor does the interpolation along a subvariety in X.)

We shall not treat the above problem here.

The last case to discuss is when π has multiple fibers. We follow the exposition in [BL, §3.2] where the relevant references can be found. For each $z \in C$ denote by $n_z \in \mathbb{N}$ the multiplicity of the fiber $\pi^{-1}(z) \subset Y$. The rational divisor $\mathcal{D} = \sum_{z \in C}(1 - 1/n_z)z$ endows C with an orbifold structure with the Euler characteristic

$$\chi = \chi(C, \mathcal{D}) = 2 - 2g(C) - \sum_{z \in C} \left(1 - \frac{1}{n_z}\right).$$

With the possible exception of the case when $C = \mathbb{P}^1$ and the support of $\mathcal{D}$ consists of one or two points there exists a universal orbifold covering $\rho: \widetilde{C} \to C$ of the of the pair $(C, \mathcal{D})$ where $\widetilde{C}$ equals $\mathbb{P}^1$, $\mathbb{C}$ or the disc $\{z \in \mathbb{C}: |z| < 1\}$ according as $\chi > 0$, $\chi = 0$ or $\chi < 0$. Let $\widetilde{\pi}: \widetilde{Y} \to \widetilde{C}$ be the pull-back by ρ of the fibration $\pi: Y \to C$.

$$\begin{array}{ccc} \widetilde{Y} & \overset{\iota}{\longrightarrow} & Y \\ \big\downarrow \widetilde{\pi} & & \big\downarrow \pi \\ \widetilde{C} & \overset{\rho}{\longrightarrow} & C \end{array}$$

The fibration $\widetilde{Y} \to \widetilde{C}$ has simple fibers and the natural map $\iota: \widetilde{Y} \to Y$ is an unramified holomorphic covering. Clearly Y is dominable by $\mathbb{C}^2$ if and only if $\widetilde{Y}$ is, and by the earlier discussion this is so if and only if $\chi(C, \mathcal{D}) \geq 0$, i.e., $\widetilde{C}$ is either $\mathbb{P}^1$ or $\mathbb{C}$. Assume this to be the case. If the fibration $\widetilde{\pi}: \widetilde{Y} \to \widetilde{C}$ is unramified then $\widetilde{Y}$ is semisubelliptic, and in this case both $\widetilde{Y}$ and its quotient Y satisfy the Oka property by Corollary 1.5. We do not know whether the same is true in the exceptional case mentioned above, or if $\widetilde{\pi}$ has a nontrivial ramification locus. This concludes our discussion of elliptic fibrations.

We do not know which K3 surfaces (other then the elliptic fibrations discussed above) enjoy the Oka property. In [BL] it is proved that every compact complex surface bimeromorphic to an elliptic or a Kummer K3 surface is holomorphically dominable by $\mathbb{C}^2$ (Propositions 4.4 and 4.5 in [BL]), but validity of the Oka property seems a much more difficult problem in these classes.

Equally difficult is to determine whether the Oka property is preserved after the removal of a certain complex hypersurface from the given manifold. We have already mentioned in (6.2) that the complement $\mathbb{P}^n \backslash A$ of a codimension ≥ 2 subvariety is subelliptic and hence enjoys the Oka property, but removing a hypersurface often leads to Kobayashi hyperbolicity. For this very active area of research we refer to [D2], [D3], [DE], [DZ], [SY1], [SY2].

QUESTION: Does the complement $\mathbb{P}^2 \backslash C$ of a smooth cubic curve $C \subset \mathbb{P}^2$ enjoy the Oka property ?

By Proposition 5.1 in [BL] $\mathbb{P}^2 \backslash C$ is dominable by $\mathbb{C}^2$. Indeed, there is a finite branched covering $Y_0 \to \mathbb{P}^2 \backslash C$ by a complex manifold Y_0 which admits an unramified holomorphic covering $Y \to Y_0$ by the complement of a meromorphic graph in $\mathbb{C}^2$, and the latter manifold Y is easily seen to be dominable by $\mathbb{C}^2$ (Theorem 5.2 in [BL]; this also follows from Proposition 6.1.5.1 above). We do not know whether Y enjoys the Oka property.

Acknowledgement. I wish to thank G. Buzzard for the communication regarding the domination property, F. Lárusson for pointing out the connection with his paper [L2], and D. Trotman for pointing out some relevant references concerning the jet transversality theorem. This research has been supported by a grant from the Ministry of Science and Education of the Republic of Slovenia.

References

[Ab] ABRAHAM, R., Transversality in manifolds of mappings. *Bull. Amer. Math. Soc.*, 69 (1963), 470–474.

[BH] BARTH, W., HULEK, K., PETERS, C. A. M., VAN DE VEN, A., *Compact Complex Surfaces.* 2nd ed., Springer-Verlag, Berlin, 2004.

[Be] BERTINI, E., *Introduzione alla geometria proiettiva degli iperspazi.* Enrico Spoerri, Pisa, 1977.

[Bi] BISHOP, E., Mappings of partially analytic spaces. *Amer. J. Math.*, 83 (1961), 209–242.

[Br] BRODY, R., Compact manifolds and hyperbolicity. *Trans. Amer. Math. Soc.*, 235 (1978), 213–219.

[Bu] BUZZARD, G., Tame sets, dominating maps, and complex tori. *Trans. Amer. Math. Soc.*, 355 (2002), 2557–2568.

[BL] BUZZARD, G., LU, S. S. Y., Algebraic surfaces holomorphically dominable by $\mathbb{C}^2$. *Invent. Math.*, 139 (2000), 617–659.

[CG] CARLSON, J., GRIFFITHS, P., A defect relation for equidimensional holomorphic mappings between algebraic varieties. *Ann. Math.*, (2) 95 (1972), 557–584.

[Ca] CARTAN, H., Espaces fibrés analytiques, in *Symposium Internat. de topologia algebraica*, 97–121, Mexico, 1958. (Also in Oeuvres, Vol. 2, Springer-Verlag, New York, 1979.)

[Cy] CAUTY, R., Sur les ouverts des CW-complexes et les fibrés de Serre. *Colloq. Math.*, 63 (1992), 1–7.

[D1] DEMAILLY, J.-P., Cohomology of q-convex spaces in top degrees. *Math. Z.*, 204 (1990), 283–295.

[D2] ——— , Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. *Algebraic Geometry – Santa Cruz* 1995, 285–360, *Proc. Symp. Pure Math.*, 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[D3] ——— , Variétés projectives hyperboliques et équations différentielles algébriques. *Journée en l'Honneur de Henri Cartan,* 3–17, *SMF Journ. Annu.*, 1997, Soc. Math. France, 1997.

[DE] DEMAILLY, J.-P., El GOUL, J., Hyperbolicity of generic surfaces of high degrees in projective 3-space. *Amer. J. Math.*, 122 (2000), 515–546.

[DL] DEMAILLY, J.-P., LEMPERT, L., SHIFFMAN, B., Algebraic approximations of holomorphic maps from Stein domains to projective manifolds. *Duke Math. J.*, 76 (1994), 333–363.

[DZ] DETHLOFF, G., ZAIDENBERG, M., Plane curves with hyperbolic and C-hyperbolic complements. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 23 (1996), 749–778.

[EG] ELIASHBERG, Y., GROMOV, M., Embeddings of Stein manifolds. *Ann. Math.* 136 (1992), 123–135.

[Fo] FORSTER, O., Plongements des variétés de Stein. *Comment. Math. Helv.*, 45 (1970), 170–184.

[FR] FORSTER, O., RAMSPOTT, K. J., Analytische Modulgarben und Endromisbündel. *Invent. Math.*, 2 (1966), 145–170.

[F1] FORSTNERIČ, F., The Oka principle for sections of subelliptic submersions. *Math. Z.*, 241 (2002), 527–551.

[F2] ——— , The Oka principle for multivalued sections of ramified mappings. *Forum Math.*, 15 (2003), 309–328.

[F3] ——— , Noncritical holomorphic functions on Stein manifolds. *Acta Math.*, 191 (2003), 143–189. [arXiv: math.CV/0211112]

[F4] ——— , Holomorphic submersions from Stein manifolds. *Ann. Inst. Fourier*, to appear. [arXiv: math.CV/0309093]

[F5] ——— , The homotopy principle in complex analysis: A survey, in *Explorations in Complex and Riemannian Geometry: A Volume dedicated to Robert E. Greene* (J. Bland, K.-T. Kim, and S. G. Krantz, eds., 73–99). Contemporary Mathematics, 332, American Mathematical Society, Providence, 2003.

[F6] ——— , Convex approximation implies Oka's property. Preprint, February 2004. [arXiv: math.CV/0402278]

[FP1] FORSTNERIČ, F., PREZELJ, J., Oka's principle for holomorphic fiber bundles with sprays. *Math. Ann.*, 317 (2000), 117-154.

[FP2] ——— , Oka's principle for holomorphic submersions with sprays. *Math. Ann.*, 322 (2002), 633-666.

[FP3] ——— , Extending holomorphic sections from complex subvarieties. *Math. Z.*, 236 (2001), 43–68.

[Go] GORESKY, M., Whitney stratified chains and cochains. *Trans. Amer. Math. Soc.*, 261 (1981), 175–196.

[GM] GORESKY, M., MacPHERSON, R., *Stratified Morse Theory.* Springer-Verlag, Berlin-New York, 1988.

[G1] GRAUERT, H., Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen. *Math. Ann.*, 133 (1957), 139–159.

[G2] ——— , Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen. *Math. Ann.*, 133 (1957), 450–472.

[G3] ——— , Analytische Faserungen über holomorph-vollständigen Räumen. *Math. Ann.*, 135 (1958), 263–273.

[GG] GREEN, M., GRIFFITHS, P., Two applications of algebraic geometry to entire holomorphic mappings. *The Chern Symposium 1979* (*Proc. Internat. Sympos., Berkeley, CA,* 1979, pp. 41–74), Springer-Verlag, New York, 1980.

[Gr1] GROMOV, M., Stable maps of foliations into manifolds. *Izv. Akad. Nauk, S.S.S.R.*, 33 (1969), 707–734.

[Gr2] GROMOV, M., *Partial Differential Relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 9. Springer-Verlag, Berlin-Heidelberg-New York, 1986.

[Gr3] ——— , Oka's principle for holomorphic sections of elliptic bundles. *J. Amer. Math. Soc.*, 2 (1989), 851-897.

[GR] GUNNING, R. C., ROSSI, H., *Analytic functions of several complex variables.* Prentice–Hall, Englewood Cliffs, 1965.

[Ha] HARTSHORNE, R., *Algebraic Geometry.* Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[HK] HEINZNER, P., KUTZSCHEBAUCH, F., An equivariant version of Grauert's Oka principle. *Invent. Math.*, 119 (1995), 317–346.

[KZ] KALIMAN, SH., ZAIDENBERG, M., A transversality theorem for holomorphic mappings and stability of Eisenman-Kobayashi measures. *Trans. Amer. Math. Soc.*, 348 (1996), 1–12.

[Ko1] KOBAYASHI, S., *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1970.

[Ko2] Intrinsic distances, measures and geometric function theory. *Bull. Amer. Math. Soc.* 82 (1976), 357–416.

[KO] KOBAYASHI, S., OCHIAI, T., Meromorphic mappings onto compact complex spaces of general type. *Invent. Math.*, 31 (1975), 7–16.

[Kd] KODAIRA, K., Pluricanonical systems on algebraic surfaces of general type. *J. Math. Soc. Japan*, 20 (1968), 170–192.

[L1] LÁRUSSON, F., Excision for simplicial sheaves on the Stein site and Gromov's Oka principle. *Internat. J. Math.*, 14 (2003), 191–209.

[L2] ——— , Model structures and the Oka principle. Preprint, 2003. [http://www.math.uwo.ca/ larusson/papers/prestack/prestack.pdf]

[Le] LEITERER, J., Holomorphic Vector Bundles and the Oka-Grauert Priciple. *Encyclopedia of Mathematical Sciences*, 10, 63–103, Several Complex Variables IV, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

[Lm] LEMPERT, L., Algebraic approximations in analytic geometry. *Invent. Math.*, 121 (1995), 335–354.

[MT] MUROLO, C., TROTMAN, D. J. A., DU PLESSIS, A. A., Stratified transversality by isotopy. *Trans. Amer. Math. Soc.*, 355 (2003), 4881–4900.

[Ns] NARASIMHAN, R., Imbedding of holomorphically complete complex spaces. *Amer. J. Math.*, 82 (1960), 917–934.

[Na] NASH, J., Real algebraic manifolds. *Ann. Math.*, 56 (1952), 405–421.

[O] OKA, K., Sur les fonctions des plusieurs variables. III: Deuxième problème de Cousin. J. Sc. Hiroshima Univ. **9**, 7–19 (1939).

[P] PHILLIPS, A., Submersions of open manifolds. *Topology*, 6 (1967), 170–206.

[PS] POLETSKY, E. A., SHABAT, B. V., Invariant metrics. *Encyclopedia of Mathematical Sciences*, 9, 63–111, Several Complex Variables III, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

[R] REMMERT, R., Sur les espaces analytiques holomorphiquement séparables et holomorphiquement convexes. *C. R. Acad. Sci. Paris*, 243 (1956), 118–121.

[RR] ROSAY, J.-P., RUDIN, W., Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$. *Trans. Amer. Math. Soc.*, 310 (1988), 47–86.

[Sa] SARD, A., The measure of critical values of differentiable maps. *Bull. Amer. Math. Soc.*, 48 (1942), 883–890.

[Sch] SCHÜRMANN, J., Embeddings of Stein spaces into affine spaces of minimal dimension. Math. Ann. **307**, 381–399 (1997).

[Se1] SERRE, J.-P., Faisceaux algébriques cohérents. *Ann. Math.*, 61 (1955), 197–278.

[Se2] ——— , Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier*, 6 (1956), 1–42.

[Si1] SIU, J.-T., Every Stein subvariety admits a Stein neighborhood. *Invent. Math.*, 38 (1976), 89–100.

[Si2] ——— , Recent techniques in hyperbolicity problems. *Several Complex Variables* (Berkeley, CA, 1995-1996, 429–508), *Math. Sci. Res. Inst. Publ.*, 37, Cambridge Univ. Press, Cambridge, 1999.

[SY1] SIU, Y.-T., YEUNG, S.-K., A generalized Bloch's theorem and the hyperbolicity of the complements of an ample divisor in an abelian variety. *Math. Ann.*, 306 (1996), 743–758; Erratum, *Math. Ann.*, 326 (2003), 205–207.

[SY2] ——— , Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic surfaces of low degrees. *Amer. J. Math.*, 119 (1997), 1139–1172; Addendum, *Amer. J. Math.*, 125 (2003), 441–448.

[St] STEIN, K., Analytische Funktionen mehrerer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem. *Math. Ann.*, 123 (1951), 201–222.

[T1] THOM, R., Les singularités des applications differentiables. *Ann. Inst. Fourier*, 6 (1955), 43–87.

[T2] ——— , Un lemme sur les applications différentiables. *Bol. Soc. Mat. Mexicana*, (2) 1 (1956), 59–71.

[Tr] TROTMAN, D., Stability of transversality to a stratification implies Whitney (a)-regularity. *Invent. Math.*, 50 (1979), 273–277.

[Wh] WHITNEY, H., Local properties of analytic varieties. *Differentiable and Combinatorial Topology, A Symposium in honor of Marston Morse* (S. Cairns, ed., 205–244), Princeton University Press, Princeton, N.J., 1965.

[Wi] WINKELMANN, J., The Oka-principle for mappings between Riemann surfaces. *Enseign. Math.*, 39 (1993), 143–151.

[Za] ZALCMAN, L., A heuristic principle in complex function theory. *Amer. Math. Monthly*, 82 (1975), 813–817.

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