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REPRESENTATIONS OF GRAPHS AND MAPS

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Abstract. *Several methods for representing (drawing) graphs are considered, which minimize different energies of the representations. The idea of representations is generalized to maps. Some well-known operations on maps are defined, using a set of matrices. These matrices can also be used to obtain the representations of the derived maps from representations of original maps.*

Keywords. Representations of graphs, representations of maps, energy, iterative methods, drawings, operations on maps.

1 Introduction

This is a shortened preliminary version of a paper on representations of graphs and maps to be published in a book on topological graph theory. We address a set of related problems that are sometimes called *graph drawing, graph visualization, realization, representation, etc.* The common thread of all these problems is that we are dealing with two mathematical universes, two data structures, if you like, and a mapping that assigns to each object of one universe an object of the other universe. Clearly, only mappings that preserve the essence of the object are considered.

Note that there are different approaches to this problem. The first approach is to map a graph to another mathematical structure. For instance, we assign to each graph a collection of vectors. In this approach we forget about the original graph and study the representation, the image of the graph in the new structure. An alternative is to enrich the graph structure with special vertex- or edge-labels which embed the category of graphs into a richer category, say the category of coordinatized graphs. Coordinatized graphs are used in several computer packages for graph drawings, such as PAJEK [2, 3], VEGA [23, 20, 21], CAGE [6], etc.

The notion of graph representation was first used implicitly in the work of STEINITZ[26], WAGNER[30], STEIN[25] and FÁRY [11]. They produced the following fundamental results in the theory of planar graphs.

Theorem 1.1 (Steinitz). The graph G is planar and 3-connected if and only if G is the 1-skeleton of a convex three-dimensional polyhedron. (The 1-skeleton of a polyhedron is just the graph determined by its vertices and edges.)

Theorem 1.2 (Fáry). Let G be a simple graph. Then G is planar if and only if G can be embeddeded in the plane using straight-line segments for edges.

However, the idea of graph representation was first explicitly considered for planar graphs by TUTTE [29, 28] and there was no need to make a distinction between representation of graphs and representation of maps. It seems that in [18] the notion of vector representation of graphs with no reference to maps was proposed explicitly for the first time in a quite general setting, with additional conditions imposed on the representation.

This paper has no intentions to be encyclopaedic. Extensive literature on the subject is provided in the *Annotated bibliography on graph drawing algorithms* [7] by DI BATTISTA, EADES, TAMASSIA and TOLLIS and their book on graph drawing algorithms [9]. A series of conferences on graph drawing is organized annually, see for example [27], [4], [8], [31], [16].

We focus on methods for representing (drawing) graphs which minimize different energies of the representations. Later on we generalize the idea of representation to maps. Several well-known operations on maps are defined, using a set of matrices. These matrices can then be used to obtain the representations of the new maps.

2 Representations of graphs

Define a *representation* ρ of a graph G in $\mathbb{R}^m$ to be a map ρ from VG into $\mathbb{R}^m$. Clearly this notion can be generalized if $\mathbb{R}^m$ is replaced by a vector space over an arbitrary field, or a module, metric space, etc. When we want to avoid confusion we call such a representation a *vector representation* of graphs. We regard the vectors $\boldsymbol{\rho}(u), u \in VG$, as row vectors and we represent ρ by the $|VG| \times m$ matrix R with the images of the vertices of G as its rows.

Let bary(X) $= \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$ be the *barycenter* of X. A representation ρ is called *balanced* if $\sum_{u \in VG} \boldsymbol{\rho}(u) = 0$. In other words, the barycenter of a balanced representation is the origin.

Edge-extensions of representations. A representation ρ can also be regarded as a drawing of the graph G in $\mathbb{R}^m$. The vertices of the graph are clearly points in $\mathbb{R}^m$, whose position is determined by ρ. To complete the drawing of the graph, we represent the edges of G by subsets of $\mathbb{R}^m$. Specifically, if e is an edge of G with endpoints u and v, then $\rho(e)$ is a connected subset of $\mathbb{R}^m$ containing $\boldsymbol{\rho}(u)$ and $\boldsymbol{\rho}(v)$. Since we are concerned with graphs (and not maps) we allow degeneracies (edge-crossings, extra vertices on the edge segment, etc.).

Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a finite set of points in $\mathbb{R}^m$. The *convex hull* conv(X) of X is the set of all points $\sum_{i=1}^{n} \lambda_i \mathbf{x}_i$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$. Clearly, conv is a closure on $\mathbb{R}^m$.

There are several "natural" edge-extensions of ρ. The most natural representation for edges is $\rho(uv) = \text{conv}(\boldsymbol{\rho}(u), \boldsymbol{\rho}(v))$. This idea was used, for instance by TUTTE [28] under the name of *straight representation*, when embedding planar graphs with straight line segments in the plane $\mathbb{R}^2$. We will use the name *convex representation* (note that Tutte's "convex representation" is stronger).

Other possibilities include projective edge-extension, affine edge-extension and others. One may take a completely different approach by not requiring $\rho(e)$ to be a connected subset but rather just a point in $\mathbb{R}^m$. Any representation $\rho : VG \to \mathbb{R}^m$ that is extended in some arbitrary way to the map $\rho : EG \to \mathbb{R}^m$ is called the *subdivision representation* or *Levi representation* of G. The barycentric representation, $\rho(uv) = \text{bary}(\boldsymbol{\rho}(u), \boldsymbol{\rho}(v))$ is a special case of a Levi representation.

3 Energy of representations

Graph representations are an interesting source of various discrete optimization problems. In order to find the optimal representation from a given class of representations one needs a real-valued criterion function $\mathcal{E}(\rho)$ that can be used for comparing the representations. When the minimum is sought, such a function is called the *energy function*. In this section several energies are introduced. In [14] the energy of the representation ρ is defined to be the value:

$$\mathcal{E}(\rho) = \sum_{uv \in EG} ||\boldsymbol{\rho}(u) - \boldsymbol{\rho}(v)||^2 \qquad (1)$$

This can be generalized as follows. Let $\omega : EG \to \mathbb{R}^+$ be a map defining an edge-weighted graph. Define the energy:

$$\mathcal{E}(\rho) = \sum_{uv \in EG} \omega_{uv} ||\boldsymbol{\rho}(u) - \boldsymbol{\rho}(v)||^2 \qquad (2)$$

Throughout this section n will denote $|VG|$ and m the dimension of representations.

The Laplace method. The $n \times n$ matrix Q with the elements $q_{uv} = -\omega_{uv}, uv \in EG, q_{uv} = 0, uv \notin EG, q_{uu} = -\sum_{uv \in EG} q_{uv}$ is called the *generalized Laplacian* of an edge-weighted graph G.

The representation ρ in $\mathbb{R}^m$ is *orthonormal* if $R^T R = I_m$, where R is the $n \times m$ matrix for ρ. It is clear, that for an orthonormal representation $n \geq m$. Note that the term *orthogonal representation* was used in [18] with a different meaning.

Theorem 3.1 ([22, 14]). Let G be an edge-weighted graph with edge-weights ω and the generalized Laplacian Q. Assume that the eigenvalues of Q are $\lambda_1 \leq \cdots \leq \lambda_n$ and that $\lambda_2 > 0$. The minimum energy of a balanced orthonormal representation of G in $\mathbb{R}^m$ equals $\sum_{i=2}^{m+1} \lambda_i$.

The orthonormal representation ρ of the above Theorem is given by the matrix $[\mathbf{x}_2, \ldots, \mathbf{x}_{m+1}]$ composed of orthonormal eigenvectors corresponding to $\lambda_2, \ldots, \lambda_{m+1}$. For $m = 2$ and $m = 3$ we get a graph representation in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. An example of such a representation is shown on Figure 1. Any procedure that obtains a representation of a graph by solving the eigenvalue and eigenvector problem will be called the *eigenvector method*. In the above case, Theorem 3.1

guarantees that the eigenvector method produces a representation that minimizes the energy, given by (1) and (2).

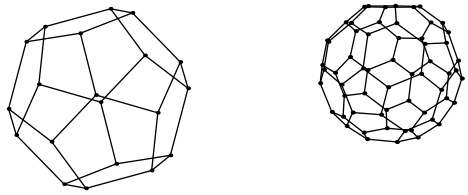

Figure 1: A $\mathbb{R}^2$ representation of the dodecahedron using second and third eigenvectors of the Laplace matrix Q and a $\mathbb{R}^3$ representation of the C_{60} fullerene, using second, third and fourth eigenvectors.

The Laplace representation method, i.e. the eigenvector method applied to the Laplace matrix, choosing 2nd, 3rd and 4th eigenvectors, sometimes does not produce the best results. One of the reasons lies in eigenvalue degeneracies. That is why it makes sense to study the combinatorial structure of the eigenspaces. This can be done in several ways. There are at least two ideas involved. The first one represents *oriented matroids*, [32], and the other one the idea of a *nodal domain*, [13].

The Tutte method. A cycle C of G is called *peripheral* if no edge not in C joins two vertices in C and $G \setminus VC$ is connected. For example, any face of a 3-connected planar graph can be shown to be a peripheral cycle.

We say, that a representation ρ of G is *barycentric* relative to a subset S of VG if for each $u \notin S$ the vector $\boldsymbol{\rho}(u)$ is the barycenter of the images of neighbours of u. The following Lemma and Theorem are adapted from Tutte's article [29].

Lemma 3.2. Let G be a connected graph, let S be a nonempty subset of vertices of G, and let σ be a map from S into $\mathbb{R}^m$. If $G \setminus S$ is connected, there is a unique $m-$dimensional representation ρ of G that extends σ and is barycentric relative to S.

Theorem 3.3 (Tutte). Let C be a peripheral cycle in a 3-connected graph G. Let σ be a mapping from VC to the vertices of a convex $|VC|-$gon in $\mathbb{R}^2$ such that adjacent vertices in C are adjacent in the polygon. The unique barycentric representation determines a drawing of G in $\mathbb{R}^2$. This drawing has no crossings if and only if the graph is planar.

A barycentric drawing based on this theorem is obtained by solving the system of equations

$$\boldsymbol{\rho}(v) = \frac{1}{deg(v)} \sum_{u \in N(v)} \boldsymbol{\rho}(u), \;\; v \in VG \setminus S. \quad (3)$$

The drawings given by Theorem 3.3 are sometimes called the *Tutte drawing* of a graph, see [14]. They are related to the notion of the *Schlegel diagram* of a polyhedron, i.e. a graph that is obtained by stereographic projection or more generally by a central projection of the 1-skeleton of a k-dimensional polyhedron to one of its faces from a point $\mathbf{x}$ very near the face but outside the polygon. An algorithm for drawing Schlegel diagrams was developed by BOR PLESTENJAK, see [24].

Here is a similar approach. Let S be a subset of vertices VG. For each vertex $v \in VG$ let $\delta(v)$ denote the distance from S, that is, the length of the shortest path from v to a vertex of S. On the edges not having both endpoints in S, define $\omega_{uv} = \phi(\delta(u), \delta(v))$, for a suitable symmetric function ϕ such as $\omega_{uv} = 1 + |\delta(u) - \delta(v)|^p$, or $\omega_{uv} = \frac{1}{\max(\delta(u),\delta(v))^p}$, for some parameter $p \in \mathbb{R}$. Let $\omega_{vv} = \sum_{uv \in EG} w_{uv}$, $v \in VG \setminus S$. Select a map σ from S into $\mathbb{R}^m$. The corresponding weighted barycentric representation ρ is called the *Schlegel representation* of G with respect to S. It is defined by

$$\boldsymbol{\rho}(v) = \frac{1}{\omega_{vv}} \sum_{u \in N(v)} \omega_{uv} \boldsymbol{\rho}(u), \;\; v \in VG \setminus S.$$

Figure 2 shows Tutte and Schlegel drawings of graphs. For the definition of the leapfrog operation `Le` see Section 4. Since ρ can be explicitely computed by the Tutte or Schlegel method, it is not hard to find suitable energies $\mathcal{E}$ which are minimized at those two representations.

Iterative Methods. In the "graph drawing community" the main problem is to devise algorithms to draw graphs that come from real-life. Such graphs are usually large and lack internal structure. Even those that possess a structure (such as graphs of printed boards or genealogical trees) very often require special considerations. An interested reader is referred to [7] for extensive information on the subject.

One of the most practical approaches is the so-called *spring embedders* or a *force-directed placement*. The main idea is to consider edges as springs,

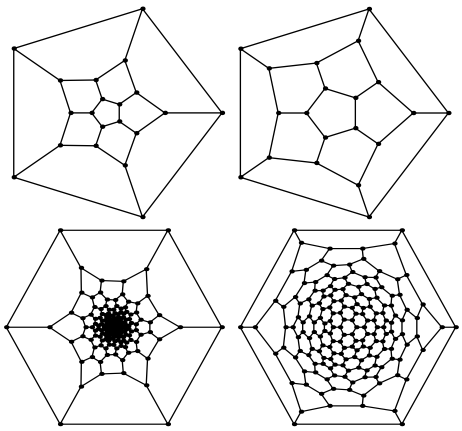

Figure 2: A $\mathbb{R}^2$ representation of the dodecahedron - Tutte method and Schlegel method with $\omega_{uv} = \frac{1}{\max(\delta(u),\delta(v))}$ and a $\mathbb{R}^2$ representation of $\texttt{Le}(C_{60})$ - Tutte method and Schlegel method with $\omega_{uv} = \frac{1}{\max(\delta(u),\delta(v))^{2.5}}$.

define two forces, one pulling vertices apart and another one attracting adjacent vertices. The forces depend on the actual distances between the two vertices. Such methods include [10], [12], [5], [24], and [6]. An energy function can be defined that takes into account the contributions of all pairs of vertices and all pairs of adjacent vertices. The problem of energy minimization is then solved by local optimization or simulated annealing techniques, see for example [15]. The method has many variations and adaptations. For instance, positions of some vertices may be fixed or instead of the plane one may consider 3-dimensional space or even higher dimensional space. All these methods define a vector representation of graphs where lines are represented as the convex closure of its endpoints.

One of the approaches with an explicit energy function is the one by KAMADA AND KAWAI, [15]. Let $d(u, v)$ denote the graph distance between the vertices u and v in G. Then the energy of the representation is given by:

$$\mathcal{E}(\rho) = \sum_{u,v \in V} \left(\frac{||\boldsymbol{\rho}(u) - \boldsymbol{\rho}(v)|| - d(u,v)}{d(u,v)} \right)^2 \quad (4)$$

Molecular mechanics tries to find molecular coordinates for a molecular graph and uses very similar ideas to those of the graph drawing community. It uses knowledge from chemistry, such as angles between two incident edges. The most common form of the energy can be described as follows:

$$\mathcal{E}(\rho) = \sum_{edges} k_b(r - r_0)^2 + \sum_{angles} k_\theta(\theta - \theta_0)^2 + \\ + \sum_{torsions} A(1 + \cos(n\tau - \varphi)) + \\ + \sum_i \sum_j \frac{-A_{i,j}}{r_{i,j}^6} + \frac{B_{i,j}}{r_{i,j}^{12}} + \sum_i \sum_j \frac{q_i q_j}{r_{i,j}} \quad (5)$$

The first term represents stretching energy, the second one bending energy, the third torsion energy, and the last two terms, non-bonded interaction energy, composed of the van der Waals term and the electrostatic term. Constants are determined by comparison with the results obtained by more accurate ab initio quantum mechanics methods.

The energy functions from previous subsections admit closed form optimization solutions. The energy models in the present subsection usually require iterative heuristic algorithms that find near-optimal solutions. We conclude this section with a generic algorithm that can be adapted and fine-tuned for a chosen energy function model.

Generic iterative graph representation algorithm with cooling schedule.

Input: G - graph, m - dimension of representation, $\mathcal{E}$ - energy function, S - set of fixed vertices of G, σ - prescribed coordinate vectors for vertices in S.

Parameters: d - maximum displacement, N - number of iterations, $1 = T(0) \geq T(1) \geq ... \geq T(N) > 0$ - temperature cooling schedule.

Output: ρ - representation of G in $\mathbb{R}^m$ with small, near-optimal energy $\mathcal{E}(\rho)$.

Pseudocode:

1. For each vertex $v \in S$ let $\boldsymbol{\rho}(v) = \boldsymbol{\rho}'(v) := \boldsymbol{\sigma}(v)$.
2. For each vertex $v \in VG \setminus S$ choose a vector $\boldsymbol{\rho}(v) \in \mathbb{R}^m$.
3. For $k = 1, 2, ..., N$ do
   (a) For each vertex $v \in VG \setminus S$ repeat
      i. Choose some vector $\boldsymbol{\delta}(v) \in \mathbb{R}^m$.
      ii. Choose some number $\lambda(v) \in \mathbb{R}$ such that $|\lambda(v)| \leq dT(k)$.
      iii. Let $\boldsymbol{\rho}'(v) := \boldsymbol{\rho}(v) + \lambda(v)\boldsymbol{\delta}(v)$
   (b) If $\mathcal{E}(\rho') < \mathcal{E}(\rho)$ let $\rho = \rho'$
4. Return ρ

The above algorithm can be easily adapted to specific energy functions or constraints in a way that may improve its speed considerably. The reader is referred to the original articles where further details of implementations can be found. A random choice of $\boldsymbol{\delta}(v) \in \mathbb{R}^m$ and $\lambda(v) \in \mathbb{R}$ will

give poor results while an intelligent choice may slow down each step of computation. For $S = \emptyset$ we get the usual drawings, such as Kamada-Kawai. For nonempty S one gets various Tutte representations, or Schlegel diagram drawings. Anyone trying to implement such an algorithm should select a suitable projection from $\mathbb{R}^m$ to the screen and should be warned about the possibility of the graph drifting away, collapsing to a single point, for instance for $S = \emptyset$ if no repulsive forces are present, or expanding to infinity if the graph is not connected.

4 Maps

Three involutions and flags. A map is a cellular embedding of a graph in a surface, with the emphasis on the incidence structure of vertices, edges and faces. We first divide the surface into "right" triangles by joining the center of each face to the midpoint of each incident edge and to each incident vertex, see Figure 3. The resulting triangles are called *flags*.

Abstractly, a map M is defined by a set of *flags* ΦM and three fixed-point free involutions ν, ε and ϕ on ΦM with the following properties.

- $\nu\phi = \phi\nu$ and $\nu\phi$ is fixed-point free.
- The group $\langle \nu, \varepsilon, \phi \rangle$ acts transitively on Φ.

Involutions can be viewed as instructions for glueing together the flags to make the map. The orbits VM of $\langle \varepsilon, \phi \rangle$ are called *vertices*, the orbits EM of $\langle \nu, \phi \rangle$ are called *edges*, the orbits FM of $\langle \nu, \varepsilon \rangle$ are called *faces*. Compare, for instance [14]. This defines three natural maps:

$$v : \Phi M \to VM, e : \Phi M \to EM, f : \Phi M \to FM$$

that can be combined into a single map:

$$(v, e, f) : \Phi M \to VM \times EM \times FM$$

The pair (VM, EM) determines a graph $G = \mathrm{skel(M)}$, the so-called 1-skeleton of the map.

If there are no loops in G and its dual (dual to be defined later in this section), the three equivalence relations on Φ have the property that no two distinct elements φ_1 and φ_2 from Φ fall in the same three equivalence classes. In such a case the above combined map is an injection (one-to-one) and we can visualize ΦM as a set $\Phi M \subseteq VM \times EM \times FM$.

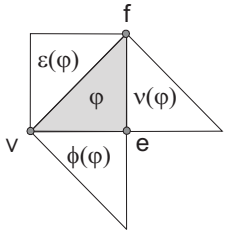

Figure 3: A flag φ may be viewed as a triangle with vertices $v(\varphi), e(\varphi), f(\varphi)$. The flags $\phi(\varphi), \epsilon(\varphi), \nu(\varphi)$ are represented by the three adjacent triangles.

In general, ΦM has to be considered as a multiset. In this model $\nu(v, e, f) = (v', e, f)$, $\varepsilon(v, e, f) = (v, e', f)$, $\phi(v, e, f) = (v, e, f')$ for appropriate $v' \in V, e' \in E, f' \in F$. On the disjoint union $V \sqcup E \sqcup F$ we can define an incidence relation $\sim$ as follows. Two elements, say v and e are incident if and only if they belong to the same triple $\phi = (v, e, f) \in \Phi$. This graph is in fact a 3-colored triangulation of the surface obtained by a barycentric subdivision of the map. We call this graph the *Levi graph* of the map M and denote it by $\mathcal{L}(M)$.

The flag graph. There is a dual viewpoint that works for most maps. In the Levi graph of M the flags are represented as triangles while the vertices fall in three color classes: $V = VM \sqcup EM \sqcup FM$. In the cubic *flag graph* $\mathcal{F}(M)$ the vertices are flags and the edges are colored with the three involutions.

The flags form the vertex set of the so-called *flag graph*. The edges of the flag graph are determined by the three involutions. Each edge is colored by the appropriate involution ν, ε, ϕ.

The subgraph of ΦM determined by the edges colored with ν, ϕ consists of disjoint 4-cycles, and the flag graph is connected.

The edges of each color class form a so-called *one-factor*. Such a one-factor can be viewed as a fixed-point free involution. Hence, we may view $\nu, \varepsilon, \phi : \Phi M \to \Phi M$, as three fixed-point free involutions with an additional condition, namely that $(\nu\phi)^2 = 1$.

Representations of maps The pioneering work of Tutte on planar graph representations in $\mathbb{R}^2$ is, in fact, dealing with planar map representations, specifically *convex representations*, where the edges are represented as line segments and the faces are convex polygons in the plane apart from

the "infinite" face whose complement is a convex set.

On the other hand, uniform polyhedra [17] can be viewed as representations of some highly regular maps with planar, but not necessarily convex faces with self-intersections allowed.

The easiest way to introduce representations to maps is via graphs. There are at least two alternatives: one may define the representation of M as a representation of its Levi graph or as a representation of its flag graph.

In addition to edge-extensions we need also *face-extensions*. For each face $f \in F$ we can define $\rho(f)$ to be the convex (or some other) closure of the vertices incident with f. Sometimes it is convenient to use the barycenter of the vertices of f for $\boldsymbol{\rho}(f)$.

We define the extended representation ρ to be *normal* if the extensions of edges are 1-dimensional, the extensions of faces are 2-dimensional and for any two non-incident elements e, f of M

$$\dim(\rho(e) \cap \rho(f)) < \min(\dim(\rho(e)), \dim(\rho(f)))$$

For any real representation ρ of M we may define the energy $\mathcal{E}(\rho)$ paraphrasing the CAGE-method [6] as follows:

$$\mathcal{E}(\rho) := \sum_{v \in V M'} \frac{\sum (A(\rho, Q) ||c(\rho, Q) - \boldsymbol{\rho}(v)||)^2}{\sum A(\rho, Q)^2},$$

where M' is two dimensional subdivision of $M, M' := \texttt{Su2}(M)$, $Q = \{u, v, w\}$ is a triangular face in FM' whose represented triangle $\rho(Q)$ has the area $A(\rho, Q)$ and the barycenter $c(\rho, Q) := \frac{1}{3}(\boldsymbol{\rho}(u) + \boldsymbol{\rho}(v) + \boldsymbol{\rho}(w))$. The definition of two dimensional subdivision `Su2` is given later in this section. The inner sums run through the set of flags $Q \in FM', v \in Q$. DELGADO FRIEDRICHS [6] does not specify the energy explicitly and uses it only for 2-dimensional representations of planar maps where the area is calculated using vector products.$A(\rho, Q) = \frac{1}{2}||(\boldsymbol{\rho}(v) - \boldsymbol{\rho}(u)) \times (\boldsymbol{\rho}(w) - \boldsymbol{\rho}(u))||$. However, there is no reason why it should not be used in a more general setting. For representations in higher dimensions the calculation of $A(\rho, Q)$ should be performed without vector products.

Operations on Maps. Using the formal definition of a map in terms of flags, we can now define several transformations or operations on maps. The most important among them are the dual, the truncation, the medial and the one dimensional subdivision.

The medial map. For a definition of the medial operation, see for instance [1, 14]. Here is an intuitive description: If M is realized as a polyhedron, then think of cutting away each vertex v by a separate plane Π_v in such a way, that for each edge e with the endvertices u and v the corresponding planes Π_u and Π_v intersect at the midpoint of e. The resulting map is fourvalent, it resides in the same surface as M and has two types of faces: those corresponding to the original faces and those corresponding to the original vertices. In particular, a vertex v of valence k gives rise to a $k-$gonal face. For example, the medial of a tetrahedron is an octahedron, and the medial of an octahedron is a cuboctahedron.

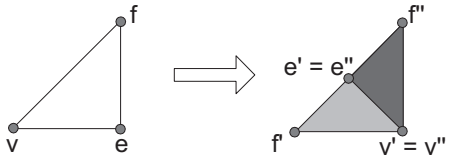

Figure 4: The medial operation Me can be defined as a subdivision of flag triangles.

Now we define the medial operation `Me` using flags. Let (v, e, f) be the vertices of a triangle, corresponding to a flag φ. We subdivide this triangle into two triangles: (v', e', f') corresponding to φ' and (v'', e'', f''), corresponding to φ'' where $v' = v'' = e, e' = e'', f' = v, f'' = f$. The new vertex $e' = e''$ is located on the edge vf of the original triangle. If placed in the barycenter, one can write this in the form: $e' = e'' = (v + f)/2$. We treat the entries v, e, and f as formal variables that can be added which gives a shorter description of this subdivision.

$$\begin{bmatrix} v' \\ e' \\ f' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ e \\ f \end{bmatrix},$$

$$\begin{bmatrix} v'' \\ e'' \\ f'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ e \\ f \end{bmatrix}$$

If we define the two matrices:

$$M' = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, M'' = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

then the medial operation admits a description by a set of two matrices $\{M', M''\}$.

Operations and their matrices. The following table shows definitions of some of the well-known operations.

Operation	Example	Matrices
Identity Id		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Medial Me		$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$
Dual Du		$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
Truncation Tr		$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
One dimensional subdivision Su1		$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Combined operations. If we combine some of these transformations some other interesting transformations result. The operation $\mathtt{Le}(M) := \mathtt{Tr}(\mathtt{Du}(M))$ is sometimes called the *leapfrog transformation* in chemistry. For instance $\mathtt{Le}(Dodecahedron)$ is the well-known Buckminsterfullerene, a model for the pure carbon molecule discovered in the 1980's. An interesting operation is the *two dimensional subdivision* $\mathtt{Su2}(M) := \mathtt{Du}(\mathtt{Tr}(\mathtt{Du}(M)))$. The well-known *barycentric subdivision* plays an important role in mathematics. It can be defined as follows. $\mathtt{BS}(M) := \mathtt{Su2}(\mathtt{Su1}(M))$. Its dual $\mathtt{Co}(M) := \mathtt{Du}(\mathtt{BS}(M))$ was used in [19]. We would like to stress the fact that $\mathtt{Co}(M)$ is in fact the flag graph of M. Note that $\mathtt{Co}(M)$ has a quadrilateral 2-factor.

The following result is quite useful for automatic determination of the set of matrices for combined operations.

Theorem 4.1. Let $\{M_1, ..., M_m\}$ be the matrices defining operation S and let $\{N_1, ..., N_n\}$ define operation T. Then the combined operation ST is defined by the mn product matrices $\{M_1N_1, M_1N_2, ..., M_1N_n, M_2N_1, ..., M_mN_n\}$.

Using this result it can be easily shown that $\mathtt{Du}(\mathtt{Du}(M)) = M$ and that $\mathtt{Me}(M) = \mathtt{Me}(\mathtt{Du}(M))$.

Operations and representations. The matrices that are used for defining operations on maps can be used to obtain representations of new map from representations of the old one.

As an example, we consider the medial. If we are given a representation $\rho : VM \sqcup EM \sqcup F \to \mathbb{R}^m$ with row vectors, one may define:

$$\begin{bmatrix} \boldsymbol{\rho}'(v') \\ \boldsymbol{\rho}'(e') \\ \boldsymbol{\rho}'(f') \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho}(v) \\ \boldsymbol{\rho}(e) \\ \boldsymbol{\rho}(f) \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{\rho}'(v'') \\ \boldsymbol{\rho}'(e'') \\ \boldsymbol{\rho}'(f'') \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho}(v) \\ \boldsymbol{\rho}(e) \\ \boldsymbol{\rho}(f) \end{bmatrix}$$

In practice one gets better results if the new representation ρ'' is defined from ρ' by keeping the vertex part of the representation: $\rho''_V := rho'_V$ and extending ρ'' to edges and faces by barycentric extension. On other occasions one can use spring embedders or other methods to get the vertex part of the representation.

Operations such as the *Petrie dual* cannot be described in this way and other tools have to be used to produce meaningful representations for them.

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