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A CONTRACTIBLE LEVI FLAT HYPERSURFACE IN $\mathbb{C}^2$ WHICH IS A DETERMINING SET FOR PLURIHARMONIC FUNCTIONS

by

Franc Forstnerič

An embedded smooth real hypersurface M in a complex manifold X is *Levi flat* if it is foliated by smooth complex hypersurfaces; this foliation will be called the *Levi foliation* of M and its leaves the *Levi leaves*. If M is *real analytic* then for any point $p \in M$ there exists a holomorphic function $f = u+iv \in \mathcal{O}(U)$ in an open neighborhood of p, with $df \neq 0$ on U, such that $M \cap U = \{x \in U : v(x) = 0\}$ and the Levi foliation of $M \cap U$ is given by the level sets of u. In other words, u and v are *pluriharmonic functions* such that v is a local *defining function* for M and $u|_{M\cap U}$ is a local *first integral* for its Levi foliation. One might expect that an oriented, real analytic, Levi flat hypersurface admits a pluriharmonic defining function on sufficiently simple domains, perhaps under an additional analytic asumption such as the existence of a fundamental system of Stein neighborhoods (see e.g. Theorem 2 in [T], p. 298). Here we show that, on the contrary, even a very simple domain in a real analytic Levi flat hypersurface may be a *determining set for pluriharmonic functions.*

THEOREM. *There exist an ellipsoid $B \subset \mathbb{C}^2$ and a closed real analytic Levi flat hypersurface M in a neighborhood of $\overline{B}$ intersecting the boundary bB transversely such that $A = M \cap B$ has the following properties:*

(i) *$\overline{A}$ is diffeomorphic to the three-ball and admits a Stein neighborhood basis.*

(ii) *Any real analytic function on A which is constant on each Levi leaf of A is constant.*

(iii) *Any pluriharmonic function in a connected open neighborhood of A which vanishes on A is identically zero.*

An equivalent formulation of (iii) is that a holomorphic function in a connected open neighborhood of A which is real valued on A is constant. In our example the Levi foliation of A is a *simple foliation* [G_1, p. 79] whose leaves are complex discs. Likely one can also obtain an example with these properties in the ball of $\mathbb{C}^2$. The example shows that it is impossible to obtain a pluriharmonic defining function without taking into account the structure of the Levi foliation at *branch points*, i.e., points where the Levi leaves split or merge.

Moreover, the problem may be *extrinsic* in the sense that it is only detected after analytic continuation (as in our example).

On the other hand, a real analytic Levi flat hypersurface M admits a smooth *asymptotically pluriharmonic defining function* v on any simply connected relatively compact domain $\Omega \subset\subset M$, i.e., such that the pluricomplex Laplacian $dd^c v$ is *flat* on Ω [FL]. This suffices for many applications, especially for the construction of Stein neighborhood basis of certain compact subsets.

Our construction is based on the following.

PROPOSITION. *Let D be the open unit disc in $\mathbb{R}^2$. There exists a real analytic foliation $\mathcal{F}$ of $\mathbb{R}^2$ by closed lines such that any real analytic function on D which is constant on every leaf of the restricted foliation $\mathcal{F}|_D$ is constant.*

REMARK. While we cannot exclude the possibility that an example of this kind is contained in the vast literature on the subject, we could not find a precise reference in some of the standard sources concerning foliations of the plane ([HR], [Ha], [G$_1$], [G$_2$], [CC]). Every smooth foliation of $\mathbb{R}^2$ by lines has a global continuous first integral but in general not one of class $\mathcal{C}^1$, not even in the analytic case (Wazewsky [W]). However, there exists a smooth first integral without critical points on any relatively compact subset (Kamke [K]), and hence there is an essential difference between the smooth and the real analytic case. Our example is elementary and clearly elucidates the main point.

Proof of the Proposition. Let (x, y) be coordinates on $\mathbb{R}^2$. Define subsets $E_1, E_2 \subset \mathbb{R}^2$ by

$$E_1 = \{x < -1\} \cup \{y > 0\}, \quad E_2 = \{x > 1\} \cup \{y > 0\}.$$

Let $\mathcal{F}_j$ denote the restriction of the foliation $\{y = c\}_{c\in\mathbb{R}}$ to E_j $(j = 1, 2)$. Let ψ be a real analytic orientation preserving diffeomorphism of the half line $(0, +\infty)$, so $\lim_{t\downarrow 0} \psi(t) = 0$. (We do not require that ψ extends analytically to a neighborhood of 0; later we shall assume that ψ is flat at the origin.) Then $\phi(x, y) = (x, \psi(y))$ is a real analytic diffeomorphism of the upper half plane $E_{1,2} = E_1 \cap E_2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ onto itself which maps every leaf of $\mathcal{F}_1|_{E_{1,2}}$ to a leaf of $\mathcal{F}_2|_{E_{1,2}}$. Let E be the quotient of the topological (disjoint) sum $E_1 \sqcup E_2$ obtained by identifying a point $(x, y) \in E_1$ belonging to $E_{1,2}$ with the point $\phi(x, y) \in E_2$. The foliations $\mathcal{F}_j$ on E_j amalgamate into a real analytic foliation $\mathcal{F}$ on E. By construction E is a real analytic manifold homeomorphic to $\mathbb{R}^2$, and hence there exists a real analytic diffeomorphism of E onto $\mathbb{R}^2$. (This follows in particular from the classification theorem for simply connected Riemann surfaces.)

We identify E with $\mathbb{R}^2$ and denote the resulting real analytic foliation of $\mathbb{R}^2$ by $\mathcal{F} = \mathcal{F}_\psi$. Let $\pi : \mathbb{R}^2 \to Q = \mathbb{R}^2/\mathcal{F}$ denote the natural projection onto the space of leaves. Q admits the structure of a non-Hausdorff real analytic manifold such that π is a real analytic submersion. (The real analytic structure on Q is obtained by declaring that the restriction of π to any small local analytic transversal ℓ to $\mathcal{F}$ is a diffeomorphism of ℓ onto the open set $\pi(\ell) \subset Q$. The transition maps are given by isotopies along the leaves of $\mathcal{F}$ and hence are real analytic. For details see [HR] or [Ha].) In our case Q is the quotient of the topological sum $\mathbb{R}_1 \sqcup \mathbb{R}_2$ of two copies of the real axis $\mathbb{R}$ obtained by identifying a point $t > 0$ in $\mathbb{R}_1$ with the point $\psi(t) \in \mathbb{R}_2$ (no identifications are made for points $t \le 0$). The only pair of branch points in Q (i.e., points without a pair of disjoint neighborhoods) are those corresponding to $0 \in \mathbb{R}_1$ and $0 \in \mathbb{R}_2$.

LEMMA 1. *If ψ is flat at origin* ($\lim_{t\downarrow 0} \psi^{(k)}(t) = 0$ *for* $k \in \mathbb{N}$) *then every real analytic function on* $\mathbb{R}^2$ *which is constant on every leaf of* $\mathcal{F}_\psi$ *is constant.*

Proof. A real analytic function f on $\mathbb{R}^2$ which is constant on the leaves of $\mathcal{F}_\psi$ is of the form $f = h \circ \pi$ for some real analytic function $h\colon Q \to \mathbb{R}$. Such h is given by a pair of real analytic functions $h_j\colon \mathbb{R}_j \to \mathbb{R}$ $(j = 1,2)$ satisfying $h_1(t) = h_2(\psi(t))$ for $t > 0$. As $t \downarrow 0$, flatness of ψ at 0 implies flatness of h_1 at 0. Hence h_1 (and therefore also h_2) must be constant which proves Lemma 1.

Fix ψ and consider the following pair of subsets of E_1 resp. E_2:

$$\begin{aligned} D_1 &= \{-2 < x < -1,\ -1 < y < +2\}, \\ D_2 &= \{1 < x < 2,\ -1 < y < \psi(2)\} \cup \\ &\qquad \cup \{-2 < x < 2,\ \psi(1) < y < \psi(2)\}. \end{aligned}$$

Let D be the subset of $E = \mathbb{R}^2$ obtained from $D_1 \sqcup D_2$ by identifying pairs of points via ϕ. Note that a point $(x,y) \in D_1$ with $1 < y < 2$ gets identified with $(x, \psi(y)) \in D_2$ and no other identifications occur. Clearly D is a simply connected domain in $\mathbb{R}^2$ with compact closure and the space of leaves $Q_D = D/\mathcal{F}$ of the restricted foliation $\mathcal{F}|_D$ is a non-Haudorff manifold with two sheets and a simple branch at $t = 1 \in \mathbb{R}_1$ resp. $\psi(1) \in \mathbb{R}_2$.

LEMMA 2. *If ψ is flat at the origin then every real analytic function f on the domain $D \subset\subset \mathbb{R}^2$ constructed above which is constant on every leaf of $\mathcal{F}_\psi|_D$ is constant.*

Proof. As in Lemma 1 such f is of the form $f = h \circ \pi$ for some real analytic h on $Q_D = D/\mathcal{F}_\psi$. Such h is given by a pair of real analytic functions $h_1\colon (-1,2) \to \mathbb{R}$, $h_2\colon (-1,\psi(2)) \to \mathbb{R}$ satisfying $h_1(t) = h_2(\psi(t))$ for $1 < t < 2$. By analyticity the relation persists on the largest interval on which both sides are defined, which is $(0,2)$. By flatness of ψ at 0 we conclude as in Lemma 1 that h_1 and h_2 must be constant. This proves Lemma 2.

Let $\mathcal{F} = \mathcal{F}_\psi$ be the foliation of $\mathbb{R}^2$ constructed above with the diffeomorphism $\psi(t) = te^{-1/t}$ of $(0,+\infty)$ (which is flat at 0), and let $D \subset\subset \mathbb{R}^2$ satisfy the conclusion of Lemma 2. Choose a disc containing D; clearly Lemma 2 still holds for this disc, and by an affine change of coordinates on $\mathbb{R}^2$ we may assume this to be the unit disc. This completes the proof of the Proposition.

REMARK. The Proposition holds for any foliation $\mathcal{F}_\psi$ constructed above for which the diffeomorphism ψ of $(0,+\infty)$ is such that $h\circ\psi$ does not extend as a real analytic function to a neighborhood of 0 for any real analytic function h near 0. An example is t^α for an irrational $\alpha>0$. The foliation of $\mathbb{R}^2$ determined by the algebraic 1-form $\omega=(\alpha-x)(1+x)dy-xdx$ has the space of leaves $\mathcal{C}^1$-diffeomorphic to the 'simple branch' Q determined by $\psi(t)=t^\alpha$ [G_2, p. 120]; hence it might be possible to find a disc $D\subset\mathbb{R}^2$ satisfying the Proposition for this foliation. These examples suggest that a real analytic foliation of $\mathbb{R}^2$ only rarely admits real analytic first integrals on large relatively compact subsets.

Proof of the Theorem. Let $\mathcal{F}$ be a real analytic foliation of $\mathbb{R}^2$ furnished by the Proposition such that any real analytic function on $D=\{x_1^2+x_2^2<1\}\subset\mathbb{R}^2$ which is constant on the leaves of $\mathcal{F}|_D$ is constant. Denote by (x_1+iy_1,x_2+iy_2) the coordinates on $\mathbb{C}^2$ and identify $\mathbb{R}^2$ with the plane $\{y_1=0,y_2=0\}\subset\mathbb{C}^2$. Complexifying the leaves of $\mathcal{F}$ we obtain the Levi foliation of a closed, real analytic, Levi flat hypersurface M in an open tubular neighborhood $\Omega\subset\mathbb{C}^2$ of $\mathbb{R}^2$. Set $B=\{x_1^2+x_2^2+c(y_1^2+y_2^2)<1\}$ where $c>0$ is chosen sufficiently large such that $\overline{B}\subset\Omega$. Note that $B\cap\mathbb{R}^2=D$. A generic choice of c insures that M intersects bB transversely (since transversality holds along $bD\cap M$). Set $A=M\cap B\subset\subset M$. If B is sufficiently thin (i.e., c is sufficiently large) then clearly $\overline{A}$ is diffeomorphic to the closed three-ball. If a real analytic function $u\in\mathcal{C}^\omega(A)$ is constant on every Levi leaf of A then $u|_D$ is constant on every leaf of $\mathcal{F}|_D$ and hence is constant. Thus A satisfies property (ii) in the Theorem.

The foliation $\mathcal{F}$ of $\mathbb{R}^2$ is transversely orientable and hence admits a transverse real analytic vector field ν. Its complexification is a holomorphic vector field w in an open neighborhood of $\mathbb{R}^2$ in $\mathbb{C}^2$ such that the field iw is transverse to M in a neighborhood of $\overline{B}$ provided that B is chosen sufficiently thin. By moving M off itself to either side using a short time flow of iw in a neighborhood of $\overline{B}$ we obtain thin neighborhoods of $\overline{A}$ with two Levi flat boundary components; intersecting these with rB for $r>1$ close to 1 gives a fundamental system of Stein neighborhoods of $\overline{A}$.

Suppose that v is a real pluriharmonic function in a connected open neighborhood of A such that $v|_A=0$. For every point $x\in A$ there is an open connected neighborhood $U_x\subset B$ and a pluriharmonic function u_x on U_x, determined up to a real constant, such that u_x+iv is holomorphic on U_x. Since A is contractible, we have $H^1(A,\mathbb{R})=0$ and hence the collection $\{u_x\}_{x\in A}$ can be assembled into a pluriharmonic function u in a neighborhood of A such that $u+iv$ is holomorphic. Since $v|_A=0$, u must be constant on every Levi leaf on A and hence constant by property (ii) of A. It follows that v is constant and thus identically zero. This proves the Theorem.

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