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TWO-GRAPHS, DOUBLE COVERS OVER COMPLETE GRAPHS, AND ALMOST SELF-COMPLEMENTARY GRAPHS

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Two-graphs, double covers over complete graphs, and almost self-complementary graphs

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Abstract

A graph X is called almost self-complementary with respect to a perfect matching $\mathcal{I}$ in the complement X^{C} if it is isomorphic to the graph obtained from X^{C} by removing the edges of $\mathcal{I}$. In this paper we investigate the relationship between self-complementary two-graphs and double covers over complete graphs that are almost self-complementary with respect to their set of fibres. In particular, we classify all doubly transitive self-complementary two-graphs, and therefore all almost self-complementary graphs with an automorphism group acting 2-transitively on the corresponding perfect matching.

Keywords: Almost self-complementary graph, homogeneously almost self-complementary graph, self-complementary two-graph, graph cover, regular covering projection, lifting automorphisms.

1 Introduction

Almost self-complementary graphs were introduced by Alspach in the 1990s, first studied by Dobson and the second author in [6], and systematically investigated by the present authors in [13, 14]. The initial goal of the research leading to this article was to provide examples and analyze the structure of almost self-complementary graphs that occur as 2-fold covers over the complete graphs. As it turns out, such graphs are closely related to a particular type of two-graphs, namely, self-complementary two-graphs. Consequently, the study of self-complementary two-graphs and their relationship to almost self-complementary

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graphs became the main theme of this article. Two-graphs were introduced by Higman, and later studied by Taylor [15, 16]; Taylor's classification of doubly transitive two-graphs [16] allowed us to determine all almost self-complementary 2-fold covers over the complete graphs which admit a doubly transitive action of an automorphism group on the set of fibres. This result was one of the crucial steps in the classification of all homogeneusly almost self-complementary graphs on $4p$ vertices, where p is a prime, presented in [14]. The main tool for the study of the relationship between almost self-complementary graphs and two-graphs is the theory of lifting automorphisms along regular covering projections, which was developed in [9] and upgraded for a special case of elementary abelian graph covers in [10].

This article is organized as follows. In Section 2 we shall present the necessary background on almost self-complementary graphs and graph covers. In Section 3 we shall talk about two-graphs and their relationship with double covers over complete graphs, while in Section 4 we shall focus on self-complementary two-graphs and their relationship with almost self-complementary double covers. The paper concludes with a classification of doubly transitive self-complementary two-graphs (and hence a classification of almost self-complementary graphs with an automorphism group acting doubly transitive on the associated perfect matching) and two constructions of self-complementary two-graphs.

2 Graphs, almost complements, and graph covers

In this section we shall review the necessary background and terminology from graph theory, in particular, on graph morphisms, almost self-complementary graphs, and graph covers.

2.1 Graphs, graph morphisms and automorphisms

For a set V and a positive integer k, let $V^{(k)}$ denote the set of all k-subsets of V. A *graph* is an ordered pair (V, E), where V is a finite set and E a subset of $V^{(2)}$. If (V, E) is a graph, then V and E are called the *vertex set* and the *edge set* of (V, E), respectively. The *dart set* of a graph (V, E) is the set of ordered pairs $(u, v) \in V^2$ such that $\{u, v\} \in E$. For a graph X, the vertex set, the edge set, and the dart set of X will be denoted by V_X, E_X, and D_X, and their elements will be called the *vertices*, *edges*, and *darts* of X, respectively. Two vertices u and v of X are said to be *adjacent* in X if $\{u, v\}$ is an edge of X. The adjacency relation in X will be denoted by $\sim_X$, or simply by $\sim$ if the graph X is clear from the context. The *neighbourhood* of a vertex u in X is the set $X(u) = \{v \in V_X \mid u \sim_X v\}$ and its cardinality is called the *valency* of u. A graph is *regular of valency k* (or *k-regular*) if the valency of each of its vertices is equal to k. A *void graph* is a regular graph of valency 0, and a regular graph with vertex set V and edge set $V^{(2)}$ is called the *complete graph* and denoted by K_V. If X is a graph and $U \subseteq V_X$, then $X[U]$ denotes the subgraph of X *induced* by X, that is, the subgraph with vertex set U and edge set $E_X \cap U^{(2)}$.

A *graph morphism* $f : \tilde{X} \to X$ is a function $f : V_{\tilde{X}} \to V_X$ such that $f(u) \sim_X f(v)$ whenever $u \sim_{\tilde{X}} v$. A graph morphism is an *epimorphism* (*isomorphism*) if it is surjective (bijective) as a function from $V_{\tilde{X}}$ to V_X and induces a surjective (bijective) function from $E_{\tilde{X}}$ to E_X. If $\wp : \tilde{X} \to X$ is a graph epimorphism and $v \in V_X$, then the preimage $\wp^{-1}(v)$ is called a $\wp$*-fibre*, and the set of all $\wp$-fibres is denoted by $\mathcal{F}_\wp$. An *automorphism* of a graph X is an isomorphism $X \to X$. The set of all automorphisms of a graph X with multiplication defined by $\alpha\beta = \alpha \circ \beta$ is a group called the *automorphism group* of X, denoted by $\mathrm{Aut}(X)$.

One of the central roles in this article is played by double covers over complete graphs, which we now define. A partition of a set V into subsets of size 2 is called a *perfect matching on* V. A graph X is called a *double cover over a complete graph* if there exists a perfect matching $\mathcal{F}$ on V_X such that for any two distinct elements $I_1, I_2 \in \mathcal{F}$ the graph $X[I_1]$ is void and the graph $X[I_1 \cup I_2]$ is isomorphic to $2K_2$, the 1-regular graph on four vertices. If X is a double cover over the complete graph K_n, then we say that $\mathcal{F}$ is a *set of fibres* of X since $\mathcal{F}$ is the set of $\wp$-fibres for the natural morphism $\wp\colon X \to K_n$ identifying the two vertices in each element of $\mathcal{F}$. Note that a set of fibres of a double cover over a complete graph is not necessarily unique, however, every automorphism of the double cover maps any set of fibres to a set of fibres.

2.2 Almost self-complementary graphs and antimorphisms

Recall that a graph is called *self-complementary* if it is isomorphic to its complement X^{C}. If X is a graph and $\mathcal{I}$ a perfect matching in X^{C} (that is, a perfect matching on V_X disjoint from E_X), then the *almost complement* $\mathrm{AC}_\mathcal{I}(X)$ *of* X *with respect to* $\mathcal{I}$ is the graph $(V_X, {V_X}^{(2)} \setminus (E_X \cup \mathcal{I}))$. If X is isomorphic to $\mathrm{AC}_\mathcal{I}(X)$, then we say that X is *almost self-complementary with respect to* $\mathcal{I}$. Clearly, if X is almost self-complementary with respect to $\mathcal{I}$, then there exists an isomorphism from X to $\mathrm{AC}_\mathcal{I}(X)$, that is, a permutation on V_X mapping every element of E_X to an element of $V^{(2)} \setminus (E_X \cup \mathcal{I})$. Such a permutation is called an $\mathcal{I}$*-antimorphism* of X.

More generally, an *antimorphism* of a graph X is a permutation φ on V_X such that $\varphi(E_X) \cap E_X = \emptyset$. An antimorphism φ of a graph X with n vertices is called an i-antimorphism if $\mathrm{val}_X(u) + \mathrm{val}_X(\varphi(u)) = n - 1 - i$ for all $u \in V_X$. It is not difficult to see [13] that any antimorphism of a self-complementary graph is a 0-antimorphism, and any antimorphism of a regular almost self-complementary graph is a 1-antimorphism and therefore an $\mathcal{I}$-antimorphism for some perfect matching $\mathcal{I}$ in X^{C}. Since the perfect matching $\mathcal{I}$ with respect to which a 1-antimorphism is an $\mathcal{I}$-antimorphism is uniquely determined, we shall use the simpler term "antimorphism" instead of "1-antimorphism" or "$\mathcal{I}$-antimorphism" when working with regular almost self-complementary graphs.

An $\mathcal{I}$-antimorphism that preserves the perfect matching $\mathcal{I}$ is called $\mathcal{I}$*-fair*, and the set of all $\mathcal{I}$-fair antimorphisms of X is denoted by $\mathrm{Ant}_\mathcal{I}(X)$. A graph is called $\mathcal{I}$-fairly almost self-complementary if it admits an $\mathcal{I}$-fair antimorphism. Note that for some almost self-complementary graphs $\mathrm{Ant}_\mathcal{I}(X)$ can be empty (see [13] for examples). Similarly, an automorphism of X preserving a perfect matching $\mathcal{I}$ is called $\mathcal{I}$*-fair*, and the group of all $\mathcal{I}$-fair automorphisms of X is denoted by $\mathrm{Aut}_\mathcal{I}(X)$. Note that $\mathrm{Ant}_\mathcal{I}(X)$ normalizes $\mathrm{Aut}_\mathcal{I}(X)$ and $\varphi_1\varphi_2 \in \mathrm{Aut}_\mathcal{I}(X)$ for every $\varphi_1, \varphi_2 \in \mathrm{Ant}_\mathcal{I}(X)$. Hence $\mathrm{Aut}_\mathcal{I}(X)$ is a (normal) subgroup of index 2 in $\langle \mathrm{Aut}_\mathcal{I}(X), \mathrm{Ant}_\mathcal{I}(X) \rangle$ whenever $\mathrm{Aut}_\mathcal{I}(X) \neq \mathrm{Ant}_\mathcal{I}(X)$, that is, whenever X has more than 2 vertices.

A graph X that is almost self-complementary with respect to $\mathcal{I}$ is said to be *homogeneously almost self-complementary with respect to* $\mathcal{I}$ if $\mathrm{Ant}_\mathcal{I}(X)$ is not empty (that is, if

X admits an $\mathcal{I}$-fair antimorphism) and $\mathrm{Aut}_{\mathcal{I}}(X)$ acts transitively on V_X. Note that there are vertex-transitive almost self-complementary graphs with $\mathrm{Ant}_{\mathcal{I}}(X) \neq \emptyset$ that are not homogeneously almost self-complementary; the smallest example is the cycle C_6 (see more about this in [13]). On the other hand, it is still an open question whether there exists a graph X that is almost self-complementary with respect to a perfect matching $\mathcal{I}$, has $\mathrm{Aut}_{\mathcal{I}}(X)$ acting transitively on V_X, but has no $\mathcal{I}$-fair antimorphisms. We should mention that homogeneously almost self-complementary graphs are equivalent to index-2 homogeneous factorizations of the "cocktail party graph" $K_{2n} - nK_2$. The reader is referred to [7] for more information on homogeneous factorizations.

A possible approach, and indeed a natural one, to the structural analysis of almost self-complementary graphs is to consider the graphs called *bricks* induced by pairs of elements of a perfect matching with respect to which the graph is almost self-complementary. (Note that bricks depend on the choice of the perfect matching.) Bricks are clearly bipartite graphs on four vertices with the elements of the perfect matching as bipartition sets, and in general, any bipartite graph on four vertices can occur as a brick. An almost self-complementary graph may contain different types of bricks. However, in this article we shall consider only those almost self-complementary graphs with all bricks isomorphic to the graph $2K_2$. Note that every such graph is a double cover over a complete graph and the fibres of the double cover form a perfect matching with respect to which the graph is almost self-complementary. Conversely, if X is a double cover over a complete graph with the set of fibres $\mathcal{I}$ and if X is almost self-complementary with respect to $\mathcal{I}$, then the bricks corresponding to $\mathcal{I}$ are all isomorphic to $2K_2$. Note, however, that there exist double covers over complete graphs that are almost self-complementary graphs but are not almost self-complementary with respect to any set of fibres of the double cover; the smallest such graph is the cycle C_6. To distinguish between the two cases, we shall reserve the term *almost self-complementary double cover* for double covers that are almost self-complementary with respect to some set of fibres. Thus C_6 is an almost self-complementary graph and a double cover over K_3, but it is not an almost self-complementary double cover. As we shall see in Corollary 4, a double cover over a complete graph that is almost self-complementary with respect to a set of fibres $\mathcal{F}$ is necessarily $\mathcal{F}$-fairly almost self-complementary.

2.3 Graph covers

In this section we shall summarize and adapt for our purposes the theory of lifting automorphisms along regular covering projections, which was developed in [9] and [10]. Throughout this section we shall assume that X is a connected graph.

A graph epimorphism $\wp\colon \tilde{X} \to X$ is a *covering projection* if the restriction of $\wp$ to $\tilde{X}(\tilde{v})$ is a bijection from $\tilde{X}(\tilde{v})$ to $X(\wp(\tilde{v}))$ for every $\tilde{v} \in V_{\tilde{X}}$. Hence, if $\wp\colon \tilde{X} \to X$ is a covering projection, then any two vertices in a $\wp$-fibre have equal valency and are at distance at least 3, whence the subgraph of $\tilde{X}$ induced by any two fibres is either void or 1-regular. Therefore, since X is assumed to be connected, the size of a fibre $\wp^{-1}(v)$ of a covering projection $\wp$ is independent of the choice of the vertex v. A covering projection with fibres of size r is called an *r-fold covering projection.*

Let $\wp_1\colon \tilde{X}_1 \to X_1$ and $\wp_2\colon \tilde{X}_2 \to X_2$ be covering projections, and let $\alpha\colon X_1 \to X_2$ be a graph isomorphism. If there exists an isomorphism $\tilde{\alpha}\colon \tilde{X}_1 \to \tilde{X}_2$ such that $\wp_2 \circ \tilde{\alpha} = \alpha \circ \wp_1$, then we say that α *lifts along* $(\wp_1, \wp_2)$, that $\tilde{\alpha}$ is a *lift of* α *along* $(\wp_1, \wp_2)$, and that $(\alpha, \tilde{\alpha})\colon \wp_1 \to \wp_2$ is an *isomorphism of covering projections*. In particular, if α is an automorphism of the graph X and $\wp\colon \tilde{X} \to X$ a covering projection, and if there exists an automorphism $\tilde{\alpha}$ of $\tilde{X}$ such that $\wp \circ \tilde{\alpha} = \alpha \circ \wp$, then we say that α *lifts along* $\wp$ and that $\tilde{\alpha}$ is a *lift of* α *along* $\wp$. If G is a subgroup of $\mathrm{Aut}(X)$, then the set of all lifts along $\wp$ of the elements in G is a group called the *lift of* G *along* $\wp$. The lift of the trivial group $\{\mathrm{id}_X\}$ is denoted by $\mathrm{CT}(\wp)$ and called the *group of covering transformations* of $\wp$. If $\wp_1\colon \tilde{X}_1 \to X$ and $\wp_2\colon \tilde{X}_2 \to X$ are covering projections onto the same graph, then isomorphisms of the form $(\mathrm{id}_X, \tilde{\alpha})\colon \wp_1 \to \wp_2$ are called *equivalences of covering projections*. Two covering projections are called *isomorphic* (*equivalent*) if there exists an isomorphism (equivalence) between them. It is not difficult to see that isomorphism and equivalence of covering projections are equivalence relations. The following lemma follows directly from the above definitions, hence the proof is omitted.

Lemma 1 *For each $i \in \{1, 2\}$, let $\wp_i\colon \tilde{X}_i \to X_i$ and $\wp_i'\colon \tilde{X}_i' \to X_i$ be equivalent covering projections. Then a graph isomorphism $\alpha\colon X_1 \to X_2$ lifts along $(\wp_1, \wp_2)$ if and only if it lifts along $(\wp_1', \wp_2')$. In other words, $\wp_1$ and $\wp_2$ are isomorphic if and only if $\wp_1'$ and $\wp_2'$ are isomorphic.*

Recall that the action of a group G on a set Ω is said to be *regular* if for any points $a, b \in \Omega$ there exists a unique element of G mapping a to b.

A covering projection $\wp\colon \tilde{X} \to X$ is said to be *regular* if the group of covering transformations $\mathrm{CT}(\wp)$ acts regularly on a fibre of $\wp$. Note that since X is a connected graph and the subgraph of $\tilde{X}$ induced by any two $\wp$-fibres is either void or 1-regular, $\mathrm{CT}(\wp)$ acts regularly on one $\wp$-fibre if and only if it acts regularly on all $\wp$-fibres. If we want to emphasize the isomorphism class of $\mathrm{CT}(\wp)$ as an abstract group, then we say that a regular covering projection $\wp$ is a *K-regular covering projection*, where K is an abstract group isomorphic to $\mathrm{CT}(\wp)$. Note that if $\tilde{X}$ is connected or the $\wp$-fibres have size 2, then $\wp$ is regular if and only if $\mathrm{CT}(\wp)$ acts transitively on a $\wp$-fibre. Moreover, if $\wp$ is a 2-fold covering projection, then $\wp$ is necessarily regular.

Every regular covering projection can be obtained, up to equivalence, by a voltage assignment construction, which we now describe. Let X be a graph and K an abstract group. A *K-voltage assignment* on X is a mapping $\zeta\colon D_X \to K$ satisfying the condition $\zeta(u, v)^{-1} = \zeta(v, u)$ for every dart $(u, v) \in D_X$. If ζ is a K-voltage assignment on X, then the *derived cover* $\tilde{X}_\zeta = \mathrm{Cov}(X; \zeta)$ is the graph with vertex set $V_X \times K$, where $(u, g) \sim_{\tilde{X}_\zeta} (v, h)$ if and only if $u \sim_X v$ and $g^{-1}h = \zeta(u, v)$. The corresponding covering projection $\wp_\zeta\colon \tilde{X}_\zeta \to X$ defined by $\wp_\zeta(u, g) = u$ for every $(u, g) \in V_X \times K$ is called the *derived covering projection associated with* ζ. We say that voltage assignments $\zeta_1, \zeta_2\colon D_X \to K$ are *isomorphic* (*equivalent*) if the derived covering projections $\wp_{\zeta_1}$ and $\wp_{\zeta_2}$ are isomorphic (equivalent).

Note that if $\wp_\zeta$ is a derived covering projection, then $\mathrm{CT}(\wp_\zeta)$ necessarily acts transitively on a $\wp_\zeta$-fibre, and hence it is a regular covering projection if and only if the corresponding derived covering graph is connected or a 2-fold cover. Conversely, every K-regular covering projection onto X is equivalent to a derived covering projection associated with a K-voltage assignment on X, and the derived covering graph is connected or $K \cong \mathbb{Z}_2$. Moreover, the is easy to verify (or see [9]).

Proposition 2 *If $\wp\colon \tilde{X} \to X$ is a K-regular covering projection and T a spanning tree of X, then there exists a voltage assignment $\zeta\colon D_X \to K$ which is trivial on the darts of T such that the derived covering projection $\wp_\zeta$ is equivalent to $\wp$.*

Note that a graph is a double cover (in the sense of Subsection 2.1) over the complete graph K_Ω if and only if it is isomorphic to a derived cover associated with a $\mathbb{Z}_2$-voltage assignment on K_Ω. In general, an isomorphism of covering graphs $\mathrm{Cov}(X;\zeta_1)$ and $\mathrm{Cov}(X;\zeta_2)$ does not necessarily induce an isomorphism of the corresponding covering projections $\wp_{\zeta_1}$ and $\wp_{\zeta_2}$. However, as we shall see below, in the very special situation of $\mathbb{Z}_2$-regular covering projections onto complete graphs, isomorphism of covering graphs $\mathrm{Cov}(K_\Omega;\zeta_1)$ and $\mathrm{Cov}(K_\Omega;\zeta_2)$ implies the existence of an isomorphism of these graphs that induces an isomorphism of the two covering projections. But first observe that if $\zeta_1,\zeta_2\colon D_X \to \mathbb{Z}_2$ are voltage assignments on a graph X, then $\wp_{\zeta_1}$ and $\wp_{\zeta_2}$, viewed as functions from $V_X \times \mathbb{Z}_2$ to V_X, coincide. In particular, they have the same set of fibres $\mathcal{F} = \mathcal{F}_{\wp_{\zeta_1}} = \mathcal{F}_{\wp_{\zeta_2}}$. The following lemma will be crucial for establishing a correspondence between two-graphs and double covers over complete graphs.

Lemma 3 *Let Ω be a set, $\zeta_1,\zeta_2\colon D_{K_\Omega} \to \mathbb{Z}_2$ voltage assignments on K_Ω, and $\mathcal{F}$ the set of fibres of $\wp_{\zeta_1}$ and therefore also of $\wp_{\zeta_2}$. Then the following holds.*

(i) *An isomorphism $\tilde{\alpha} : \mathrm{Cov}(K_\Omega;\zeta_1) \to \mathrm{Cov}(K_\Omega;\zeta_2)$ preserves $\mathcal{F}$ if and only if it is a lift along $(\wp_{\zeta_1},\wp_{\zeta_2})$ of some permutation α on Ω.*

(ii) *If $\zeta_1 = \zeta_2 = \zeta$, then the group $\tilde{G} = \mathrm{Aut}_{\mathcal{F}}(\mathrm{Cov}(K_\Omega;\zeta))$ of $\mathcal{F}$-preserving automorphisms of $\mathrm{Cov}(K_\Omega;\zeta)$ is the lift of the largest subgroup $\mathcal{L}(\zeta)$ of $\mathrm{Sym}(\Omega)$ that lifts along $\wp_\zeta$, and its permutation representation $\tilde{G}^{\mathcal{F}}$ (in its action on $\mathcal{F}$) is permutation isomorphic to $\mathcal{L}(\zeta)$.*

(iii) *Covering graphs $\mathrm{Cov}(K_\Omega;\zeta_1)$ and $\mathrm{Cov}(K_\Omega;\zeta_2)$ are isomorphic if and only if they are isomorphic via an $\mathcal{F}$-preserving isomorphism, that is, if and only if p_{ζ_1} and p_{ζ_2} are isomorphic covering projections.*

Proof. Statement (i) is easy to see: a permutation $\tilde{\alpha} \in \mathrm{Sym}(\Omega\times\mathbb{Z}_2)$ preserves $\mathcal{F}$ if and only if there exists a permutation $\alpha \in \mathrm{Sym}(\Omega)$ such that $\wp_{\zeta_1} \circ \tilde{\alpha} = \alpha \circ \wp_{\zeta_2}$. Moreover, such a permutation α (if it exists) is uniquely determined by $\tilde{\alpha}$, and the mapping $\wp^*\colon \tilde{\alpha} \mapsto \alpha$ is a group homomorphism from the largest subgroup of $\mathrm{Sym}(\Omega \times \mathbb{Z}_2)$ preserving $\mathcal{F}$ to $\mathrm{Sym}(\Omega)$.

Assuming $\zeta_1 = \zeta_2 = \zeta$, it then easily follows that $\tilde{G}$ is the lift of $\mathcal{L}(\zeta)$. Moreover, the image and the kernel of the restriction $\wp^*|_{\tilde{G}}$ of the group homomorphism $\wp^*$ to $\tilde{G}$ are $\mathcal{L}(\zeta)$ and $\mathrm{CT}(\wp_\zeta)$, respectively. Hence, $\wp^*|_{\tilde{G}}$ induces an isomorphism of permutation groups $\tilde{G}^{\mathcal{F}}$ and $\mathcal{L}(\zeta)$, and (ii) follows.

To prove the "only if" part of (iii), assume that $Y_1 = \mathrm{Cov}(K_\Omega;\zeta_1)$ and $Y_2 = \mathrm{Cov}(K_\Omega;\zeta_2)$ are isomorphic graphs. Let $\tilde{\alpha}$ be an isomorphism with the smallest number of fibres in $\mathcal{F}$

not mapped to fibres in $\mathcal{F}$. Suppose this number is positive. Then there exist pairwise distinct vertices $v, u, w \in \Omega \times \mathbb{Z}_2$ such that $\{u, v\} \in \mathcal{F}$ and $\{\tilde{\alpha}(u), \tilde{\alpha}(w)\} \in \mathcal{F}$. Since $\{u, v\}$ is a fibre in a double cover over a complete graph, every vertex in $\Omega \times \mathbb{Z}_2 \setminus \{u, v\}$ is adjacent to either u or v in Y_1. Similarly, every vertex in $\Omega \times \mathbb{Z}_2 \setminus \{\tilde{\alpha}(u), \tilde{\alpha}(w)\}$ is adjacent to either $\tilde{\alpha}(u)$ or $\tilde{\alpha}(w)$ in Y_2, whence every vertex in $\Omega \times \mathbb{Z}_2 \setminus \{u, w\}$ is adjacent to either u or w in Y_1. It follows that v and w have exactly the same neighbours in Y_1 that lie in the set $\Omega \times \mathbb{Z}_2 \setminus \{v, w\}$, and hence the permutation $\tilde{\beta}$ on $\Omega \times \mathbb{Z}_2$ that swaps v with w and leaves all other vertices fixed is an automorphism of Y_1. From this it follows that $\tilde{\alpha}\tilde{\beta}$ is an isomorphism from Y_1 to Y_2 that contradicts the assumption on $\tilde{\alpha}$. Hence $\tilde{\alpha}$ preserves $\mathcal{F}$. From (i) it then follows that $\tilde{\alpha}$ is a lift of some permutation α on Ω and, therefore, that p_{ζ_1} and p_{ζ_2} are isomorphic covering projections. The "if" part of (iii) is obvious. ■

Corollary 4 *Let X be an almost self-complementary double cover over a complete graph, that is, a double cover over a complete graph that is almost self-complementary with respect to a set of fibres $\mathcal{F}$. Then X is an $\mathcal{F}$-fairly almost self-complementary graph.*

Proof. It is easy to see that the almost complement of $X = \mathrm{Cov}(K_\Omega; \zeta)$ with respect to the set $\mathcal{F}$ of $\wp_\zeta$-fibres is $\mathrm{Cov}(K_\Omega; 1 + \zeta)$. The rest follows immediately from Part (iii) of Lemma 3. ■

The next lemma shows that the observations of Lemma 3 can be simplified further in the case of double covers over complete graphs with the property that the group of automorphisms preserving the set $\mathcal{F}$ of fibres acts 2-transitively on $\mathcal{F}$.

Lemma 5 *Let X be a connected double cover over a complete graph and $\mathcal{F}$ a set of fibres of X. If $\mathrm{Aut}_{\mathcal{F}}(X)$ acts 2-transitively on $\mathcal{F}$, then $\mathcal{F}$ is the unique set of fibres of X and hence $\mathrm{Aut}(X) = \mathrm{Aut}_{\mathcal{F}}(X)$.*

Proof. Suppose, on the contrary, that $\mathcal{F}'$ is a set of fibres of the double cover X distinct from $\mathcal{F}$. Then there exist vertices $u, v, v' \in V_X$ such that $\{u, v\} \in \mathcal{F} \setminus \mathcal{F}'$ and $\{u, v'\} \in \mathcal{F}' \setminus \mathcal{F}$. Hence both v and v' are at distance at least 3 from u in the graph X. Let $u' \in V_X$ be such that $\{u', v'\} \in \mathcal{F}$. Then $u \sim_X u'$, and since $\mathrm{Aut}_{\mathcal{F}}(X)$ acts 2-transitively on $\mathcal{F}$, every fibre in $\mathcal{F}$ other than $\{u, v\}$ contains a vertex at distance 1 and a vertex at distance at least 3 from u. But then X contains no vertex at distance 2 from u, a contradiction. Hence $\mathcal{F}$ is the unique set of fibres of X and clearly $\mathrm{Aut}(X) = \mathrm{Aut}_{\mathcal{F}}(X)$. ■

In what follows we shall restrict our consideration to *elementary abelian covering projections*, that is, K-regular covering projections where K is an elementary abelian group. These covering projections were extensively studied in [10] and we shall present just a brief overview of the relevant results. The most important of these results for our purposes is a simple criterion for an automorphism $\alpha \in \mathrm{Aut}(X)$ to have a lift along an elementary abelian covering projection $\wp\colon \tilde{X} \to X$. However, before stating this criterion we need to introduce a few more terms.

Let R be a commutative ring with identity 1. For any set S, let RS denote the free R-module on S, that is, the set of all formal sums $\sum_{x \in S} a_x x$ with coefficients a_x from R. As in the rest of this section, let X be a connected graph. Then $\partial\colon RD_X \to RV_X$ will denote the

R-linear transformation defined by the rule $\partial(u,v) = v - u$ for every dart $(u,v) \in D_X$. The kernel of ∂, denoted by $H_1(X;R)$, is called the *first homology group of X with coefficients in R*, and the elements of $H_1(X;R)$ are called *cycles* of X.

For any spanning tree T of X, let $\epsilon_T\colon E_X \setminus E_T \to D_X \setminus D_T$ be an *orientation function* of X, that is, a function satisfying $\epsilon_T(\{u,v\}) \in \{(u,v),(v,u)\}$ for every edge $\{u,v\} \in E_X \setminus E_T$. Then for every $e \in E_X \setminus E_T$ there exists a unique cycle $C_e \in H_1(X;R)$ of the form $\epsilon_T(e) + \sum_{x \in D_T} a_x x$. The set $\mathcal{B}_{T,\epsilon_T} = \{C_e \mid e \in E_X \setminus E_T\}$ is a basis for the R-module $H_1(X;R)$ called the *cycle basis associated with T and ϵ_T*.

Assume from now on that p is a prime, K a group isomorphic to the elementary abelian group $\mathbb{Z}_p^d$ viewed as a vector space over $\mathbb{Z}_p$, and $R = \mathbb{Z}_p$. Then every K-voltage assignment $\zeta\colon D_X \to K$ extends uniquely to a $\mathbb{Z}_p$-linear mapping from $\mathbb{Z}_p D_X$ to K, and therefore also to its restriction $\zeta^*\colon H_1(X;\mathbb{Z}_p) \to K$ on $\mathrm{Ker}(\partial) = H_1(X;\mathbb{Z}_p)$. Similarly, every automorphism $\alpha \in \mathrm{Aut}(X)$ induces a permutation on D_X, which extends uniquely to a $\mathbb{Z}_p$-linear mapping $\alpha^*\colon H_1(X;\mathbb{Z}_p) \to \mathbb{Z}_p D_X$. Note that the image of α^* is $H_1(X;\mathbb{Z}_p)$, whence α^* can be viewed as an automorphism of $H_1(X;\mathbb{Z}_p)$.

With ζ^* defined, the following nice characterization of voltage assignments that give rise to connected derived covering graphs, and therefore to regular covering projections, can be stated. The proof is straightforward and is left to the reader.

Lemma 6 *Let $\zeta\colon D_X \to \mathbb{Z}_p^d$ be a voltage assignment on a connected graph X and ζ^* its unique $\mathbb{Z}_p$-linear extension to $H_1(X;\mathbb{Z}_p)$. Then the derived covering graph* $\mathrm{Cov}(X;\zeta)$ *is connected if and only if ζ^* is surjective. Consequently, $\wp_\zeta$ is a regular covering projection if and only if ζ^* is surjective or $(p,d) = (2,1)$.*

The following criterion for an automorphism to have a lift can be deduced from [10, Theorem 3.2] with the aid of the discussion preceding [10, Proposition 5.1].

Proposition 7 *Let X be a connected graph and $K \cong \mathbb{Z}_p^d$ an elementary abelian group. For $i = 1,2$, let $\zeta_i\colon D_X \to K$ be a voltage assignment on X with the derived regular covering projection $\wp_{\zeta_i}$ and the unique $\mathbb{Z}_p$-linear extension ζ_i^* to $H_1(X;\mathbb{Z}_p)$. For an automorphism $\alpha \in \mathrm{Aut}(X)$, let α^* denote the unique $\mathbb{Z}_p$-linear extension of α (as a permutation on D_X) to $H_1(X;\mathbb{Z}_p)$. Then α lifts along $(\wp_{\zeta_1}, \wp_{\zeta_2})$ if and only if there exists $\alpha^\# \in \mathrm{Aut}(K)$ such that $\alpha^\# \circ \zeta_1^* = \zeta_2^* \circ \alpha^*$.*

For our purposes it suffices to consider the case $K = \mathbb{Z}_2$. Observe that in this case $\mathrm{Aut}(K)$ is trivial, and $\wp_{\zeta_1}$, $\wp_{\zeta_2}$ are automatically regular covering projections.

Corollary 8 *Let X be a connected graph. For $i = 1,2$, let $\zeta_i\colon D_X \to \mathbb{Z}_2$ be a voltage assignment on X with the derived covering projection $\wp_{\zeta_i}$ and the unique $\mathbb{Z}_2$-linear extension ζ_i^* to $H_1(X;\mathbb{Z}_2)$. For an automorphism $\alpha \in \mathrm{Aut}(X)$, let α^* denote the unique $\mathbb{Z}_2$-linear extension of α (as a permutation on D_X) to $H_1(X;\mathbb{Z}_2)$. Then α lifts along $(\wp_{\zeta_1}, \wp_{\zeta_2})$ if and only if $\zeta_1^* = \zeta_2^* \circ \alpha^*$.*

The covering graphs $K_{p^k+1}^2$ defined in the important construction below are essentially due to Taylor [15, Example 6.2]. In the following lemma, which illustrates the criterion stated in Corollary 8, we shall prove the essential properties that will later show that the covering graphs $K^2_{p^k+1}$ are homogeneously almost self-complementary graphs with the group of fair automorphisms acting 2-transitively on a corresponding perfect matching.

Construction 9 For an odd prime p and a positive integer k such that $p^k \equiv 1 \pmod 4$, let $\mathbb{F} = GF(p^k)$ be a finite field of cardinality p^k, and let $\mathrm{S}\mathbb{F}$ and $\mathrm{N}\mathbb{F}$ be the sets of all squares and all non-squares, respectively, in the multiplicative group $\mathbb{F}^*$. Observe that $\mathrm{S}\mathbb{F} = -\mathrm{S}\mathbb{F}$ and $\mathrm{N}\mathbb{F} = -\mathrm{N}\mathbb{F}$. Let $\Omega = \mathbb{F} \cup \{\infty\}$ (note that Ω can be thought of as the set of points of the projective line $\mathrm{PG}(1, p^k)$), and let $\zeta \colon D_{K_\Omega} \to \mathbb{Z}_2$ be the voltage assignment on K_Ω defined by

$$\zeta(x,y) = \begin{cases} 0 & \text{if } \infty \in \{x,y\} \\ 0 & \text{if } x,y \in \mathbb{F} \text{ and } x-y \in \mathrm{S}\mathbb{F} \\ 1 & \text{if } x,y \in \mathbb{F} \text{ and } x-y \in \mathrm{N}\mathbb{F} \end{cases}.$$

Then define $K^2_{p^k+1}$ as the derived covering graph $\mathrm{Cov}(K_\Omega, \zeta)$.

Lemma 10 *Let Ω, ζ, and $K^2_{p^k+1}$ be defined as in Construction 10 above. Then:*

(i) *The covering projections $\wp_\zeta$ and $\wp_{1+\zeta}$ are isomorphic.*

(ii) *The 2-transitive group $\mathrm{P\Sigma L}(2, p^k)$ is the largest subgroup of $\mathrm{Aut}(K_\Omega)$ that lifts along $\wp_\zeta$.*

Proof. We define the following permutations on Ω:

- $\iota(0) = \infty$, $\iota(\infty) = 0$, and $\iota(x) = -x^{-1}$ for $x \in \mathbb{F}^*$;
- $\sigma(\infty) = \infty$ and $\sigma(x) = x^p$ for $x \in \mathbb{F}$;
- $\alpha_t(\infty) = \infty$ and $\alpha_t(x) = tx$ for $x \in \mathbb{F}$ and $t \in \mathbb{F}^*$; and
- $\beta_c(\infty) = \infty$ and $\beta_c(x) = x + c$ for $x \in \mathbb{F}$ and $c \in \mathbb{F}$.

Let r be a generator of the group $\mathbb{F}^*$. We shall prove the following:

- Permutations ι, σ, α_{r^2}, and β_c (for every $c \in \mathbb{F}$) lift along $\wp_\zeta$.
- Permutation α_r does not lift along $\wp_\zeta$, however, it lifts along the pair of covering projections $(\wp_{\zeta'}, \wp_\zeta)$, where $\zeta'(e) = 1 + \zeta(e)$ for every $e \in D_{K_\Omega}$.

Let T be the spanning tree (that is, a star) of K_Ω containing all edges incident with the vertex ∞, let $\epsilon_T \colon E_{K_\Omega} \setminus E_T \to D_{K_\Omega} \setminus D_T$ be an orientation function, and let $\mathcal{B} = \mathcal{B}_{T,\epsilon_T}$ be the corresponding basis for the first homology group $H_1(K_\Omega; \mathbb{Z}_2)$. By Corollary 8, an automorphism $\varphi \in \mathrm{Aut}(K_\Omega)$ lifts along $\wp_\zeta$ if and only if $\zeta^* \circ \varphi^* = \zeta^*$, where ζ^* and φ^* are the unique $\mathbb{Z}_2$-linear extensions of ζ and φ, respectively, to the $\mathbb{Z}_2$-module $H_1(K_\Omega; \mathbb{Z}_2)$. Clearly, it suffices to check the validity of this equality on the elements of $\mathcal{B}$, which are all possible cycles of the form $(\infty, v) + (v, u) + (u, \infty)$ for $v, u \in \Omega \setminus \{\infty\}$. Therefore, $\varphi \in \mathrm{Aut}(K_\Omega)$ lifts along $\wp_\zeta$ if and only if

$$\zeta(\varphi(\infty), \varphi(v)) + \zeta(\varphi(v), \varphi(u)) + \zeta(\varphi(u), \varphi(\infty)) = \zeta(v, u) \tag{1}$$

for every pair of distinct vertices $v, u \in \Omega \setminus \{\infty\}$.

By (1), ι lifts if and only if $\zeta(0, -x^{-1}) + \zeta(-x^{-1}, -y^{-1}) + \zeta(-y^{-1}, 0) = \zeta(x, y)$ for every pair of distinct elements $x, y \in \mathbb{F}^*$, and $\zeta(-y^{-1}, 0) = \zeta(0, y)$ for every $y \in \mathbb{F}^*$. To see that the latter condition holds, observe that $y \in \mathrm{S}\mathbb{F}$ if and only if $-y^{-1} \in \mathrm{S}\mathbb{F}$. To check the former condition, observe that for $x, y \in \mathbb{F}^*$ we have that $\zeta(-x^{-1}, -y^{-1}) = 0$ if and only if $-y^{-1} - (-x^{-1}) = -\frac{x-y}{xy} \in \mathrm{S}\mathbb{F}$, which is true if and only if $x-y$ and xy are either both squares or both non-squares. Next, observe that xy is a square if and only if $\zeta(0, x) + \zeta(0, y) = 0$. Therefore, $\zeta(-x^{-1}, -y^{-1}) = 0$ if and only if $\zeta(x, y) + \zeta(0, x) + \zeta(0, y) = 0$, and hence $\zeta(-x^{-1}, -y^{-1}) = \zeta(x, y) + \zeta(0, x) + \zeta(0, y)$. But $\zeta(0, x) = -\zeta(0, -x^{-1})$ and $\zeta(0, y) = -\zeta(0, -y^{-1})$, so indeed, $\zeta(0, -x^{-1}) + \zeta(-x^{-1}, -y^{-1}) + \zeta(-y^{-1}, 0) = \zeta(x, y)$. Therefore, ι lifts along $\wp_\zeta$.

If $\varphi \in \{\sigma, \alpha_r, \alpha_{r^2}\}$ or $\varphi = \beta_c$ for $c \in \mathbb{F}$, then $\infty^\varphi = \infty$, and (1) is equivalent to the condition $\varphi(u) - \varphi(v) \in \mathrm{S}\mathbb{F} \Leftrightarrow u - v \in \mathrm{S}\mathbb{F}$. This condition is clearly fulfilled for $\varphi = \beta_c$ (where $c \in \mathbb{F}$), $\varphi = \sigma$, and $\varphi = \alpha_{r^2}$. Hence these automorphisms of K_Ω lift along $\wp_\zeta$.

Finally, let $\varphi = \alpha_r$. For $C = (\infty, x) + (x, y) + (y, \infty) \in \mathcal{B}$ we have $\zeta^*(\varphi^*(C)) = \zeta(rx, ry) \neq \zeta(x, y)$ since r is a non-square. On the other hand, $\zeta'^*(C) = (1 + \zeta(\infty, x)) + (1 + \zeta(x, y)) + (1 + \zeta(y, \infty)) = 1 + \zeta(x, y)$. Since multiplication by r interchanges the sets $\mathrm{S}\mathbb{F}$ and $\mathrm{N}\mathbb{F}$, $\zeta(rx, ry) = 1 + \zeta(x, y)$ for every pair $\{x, y\} \subseteq \mathbb{F}$. Hence $\zeta^* \circ \varphi^* = \zeta'^*$, and thus by Corollary 8, the permutation α_r lifts along $(\wp_{\zeta'}, \wp_\zeta)$, but it does not lift along $\wp_\zeta$. In particular, we have shown (i), that is, that $\wp_\zeta$ and $\wp_{\zeta'} = \wp_{1+\zeta}$ are isomorphic covering projections.

To prove statement (ii) of the lemma, observe first that the group generated by the permutations $\iota, \sigma, \alpha_{r^2}$ and β_c (for all $c \in \mathbb{F}$) is the 2-transitive group $\mathrm{P\Sigma L}(2, p^k)$ acting on the points of the projective line $\Omega = \mathrm{PG}(1, p^k)$. Now, let H be the largest subgroup of $\mathrm{Aut}(K_\Omega) = \mathrm{Sym}(\Omega)$ that lifts along $\wp_\zeta$. Since it contains $\mathrm{P\Sigma L}(2, p^k)$, it acts 2-transitively on Ω. Moreover, since the degree n of H (that is, the size of $|\Omega|$) is congruent 2 modulo 4 and $n - 1$ is a prime power, the classification of 2-transitive permutation groups (see for example [5, Theorem 5.3(S)]) implies that the socle T of H is either the alternating group A_{1+p^k}, or $\mathrm{PSL}(2, p^k)$, or $\mathrm{PSU}(3, q)$ with $q = p^l$ and $3l = k$. If $T \cong A_{1+p^k}$, then H is 3-transitive, which contradicts the fact that H preserves the set of triples $\{u, v, w\} \subseteq \Omega$ with $\zeta(u, v) + \zeta(v, w) + \zeta(w, u) = 0$. If $T \cong \mathrm{PSL}(2, p^k)$, then $H \leq \mathrm{P\Gamma L}(2, p^k)$. Howerever, the only subgroup of $\mathrm{P\Gamma L}(2, p^k)$ containing $\mathrm{P\Sigma L}(2, p^k)$ but not containing α_r is $\mathrm{P\Sigma L}(2, p^k)$, implying that $H \cong \mathrm{P\Sigma L}(2, p^k)$, as claimed. Finally, if $T \cong \mathrm{PSU}(3, q)$ with $q = p^l$ and $k = 3l$, then $H \leq \mathrm{P\Gamma U}(3, q)$, and therefore $|H| \leq lp^{3l}(p^{2l} - 1)(p^{3l} + 1)$. On the other hand, $|H| \geq |\mathrm{P\Sigma L}(2, p^k)| = \frac{1}{2} 3lp^{3l}(p^{3l} + 1)(p^{3l} - 1)$, implying that $3(p^{3l} - 1) \leq 2(p^{2l} - 1)$, a contradiction. ∎

As we shall see in Lemmas 19 and 22, isomorphism of covering projections $\wp_\zeta$ and $\wp_{1+\zeta}$ implies that the covering graph $K^2_{p^k+1}$ is almost self-complementary with respect to the perfect matching $\mathcal{F} = \{{\wp_\zeta}^{-1}(v) \mid v \in \Omega\}$. Since a 2-transitive group lifts along $\wp_\zeta$, the automorphism group of $K^2_{p^k+1}$ acts 2-transitively on the perfect matching $\mathcal{F}$.

3 Two-graphs

In [15] Taylor writes that "regular 2-graphs were introduced by Professor G. Higman in his Oxford lectures as a means of studying Conway's sporadic simple group ·3 in its doubly transitive representation of degree 276." Subsequently, two-graphs were extensively studied by Taylor in [15, 16] and, in particular, all two-graphs with a 2-transitive group of automorphisms were classified. Later, two-graphs were considered mostly in the context of antipodal distance-regular graphs (see, for example, [2, 8, 17]). The purpose of this section is to show that a two-graph on n vertices corresponds to a double cover over the complete graph K_n and, as we shall see in Section 4, self-complementary two-graphs correspond to almost self-complementary double covers. In this correspondence, the automorphism group of the two-graph coincides with the maximal subgroup of $\mathrm{Aut}(K_n)$ that lifts. Consequently, Taylor's classification of doubly transitive two-graphs [16] yields a classification of almost self-complementary graphs admitting a group of automorphisms acting 2-transitively on the corresponding perfect matching.

3.1 Definition of two-graphs and basic properties

To convey the charming flavour of algebraically influenced combinatorics in the 1970s we shall adopt the terminology of [15]. A *two-graph* on a vertex set Ω is a function $\Phi\colon \Omega^{(3)} \to \mathbb{Z}_2$ that satisfies the rule

$$\Phi(\{v,w,z\}) + \Phi(\{u,w,z\}) + \Phi(\{u,v,z\}) + \Phi(\{u,v,w\}) = 0 \tag{2}$$

for every quadruple of pairwise distinct vertices $u, v, w, z \in \Omega$. The size of the vertex set Ω is called the *order* of Φ. Note that Condition (2) for a function $\Phi\colon \Omega^{(3)} \to \mathbb{Z}_2$ is equivalent to the requirement that each element of $\Omega^{(4)}$ contain an even number of elements $C \in \Omega^{(3)}$ with $\Phi(C) = 1$. An element $C \in \Omega^{(3)}$ such that $\Phi(C) = 1$ is called a *coherent triple* of Φ. The set of coherent triples of Φ will be denoted by $\mathcal{T}_\Phi$. For two distinct vertices $\omega, \omega' \in \Omega^{(2)}$ we let $r_\Phi(\omega)$ and $\lambda_\Phi(\{\omega, \omega'\})$ denote the number of coherent triples of Φ containing ω and $\{\omega, \omega'\}$, respectively.

A two-graph Φ is called *regular* if there exists an integer λ_Φ such that $\lambda_\Phi(e) = \lambda_\Phi$ for every $e \in \Omega^{(2)}$. It is easy to prove that if Φ is regular, then for every $\omega \in \Omega$ we have that $r_\Phi(\omega) = r_\Phi$, where $r_\Phi = \lambda_\Phi(|\Omega| - 1)/2$. We remark that for a regular two-graph Φ on a set Ω, the pair $(\Omega, \mathcal{T}_\Phi)$ is a *triple system* (see [3] for the definition and further information on triple systems) with the additional property that every 4-element subset of the vertex set contains an even number of blocks (triples) in $\mathcal{T}_\Phi$. Conversely, the set of blocks of every triple system with this additional property is the set of coherent triples of a regular two-graph. We shall adopt the usual terminology of design theory and call the parameters r_Φ and λ_Φ of a regular two-graph Φ the *replication number* and the *index*, respectively.

A bijection α between vertex sets of two-graphs Φ_1 and Φ_2 is an *isomorphism of two-graphs* if it induces a bijection of the sets of coherent triples $\mathcal{T}_{\Phi_1}$ and $\mathcal{T}_{\Phi_2}$; that is, if $\Phi_2 \circ \alpha = \Phi_1$. The group of automorphisms of a two-graph Φ (that is, isomorphisms from Φ to Φ) will be denoted by the usual symbol $\mathrm{Aut}(\Phi)$.

For a two-graph Φ on a set Ω and a subgroup $M \leq \mathrm{Aut}(\Phi)$ we say that Φ is *M-vertex-transitive* (*M-block-transitive*) if M acts transitively on Ω ($\mathcal{T}_\Phi$, respectively). In particular, Φ is *vertex-transitive* (*block-transitive*) if it is $\mathrm{Aut}(\Phi)$-vertex-transitive ($\mathrm{Aut}(\Phi)$-block-transitive, respectively). Since every regular two-graph Φ on a set Ω gives rise to a triple system $(\Omega, \mathcal{T}_\Phi)$, and since any block-transitive automorphism group of a pairwise balanced block design (and therefore of a triple system) is also point-transitive [1, Theorem 4.16 of Chapter 3], we can deduce the following.

Proposition 11 *If Φ is a regular M-block-transitive two-graph for some $M \leq \mathrm{Aut}(\Phi)$, then Φ is also M-vertex-transitive.*

Note that the assumption on regularity in the above proposition is essential since there exist non-regular two-graphs that are block-transitive but not vertex-transitive; the smallest example is the two-graph on vertex set $\{1, 2, 3, 4\}$ with the set of coherent triples $\{\{1, 2, 3\}, \{1, 2, 4\}\}$.

3.2 Two-graphs and double covers over complete graphs

Let **Volt**, **DCov**, and **TGrph** denote the sets of all $\mathbb{Z}_2$-voltage assignments over complete graphs, double covers over complete graphs, and two-graphs, respectively. Let $[\zeta]$, $[X]$, and $[\varphi]$ denote the isomorphism classes of $\zeta \in \mathbf{Volt}$, $X \in \mathbf{DCov}$, and $\Phi \in \mathbf{TGrph}$, respectively. Furthermore, let $[\mathbf{Volt}] = \{[\zeta] : \zeta \in \mathbf{Volt}\}$, $[\mathbf{DCov}] = \{[X] : X \in \mathbf{DCov}\}$, and $[\mathbf{TGrph}] = \{[\Phi] : \Phi \in \mathbf{TGrph}\}$ be the corresponding sets of isomorphism classes on **Volt**, **DCov**, and **TGrph**, respectively. Part (iii) of Lemma 3 shows that there is a bijection between the set [**DCov**] of isomorphism classes of double covers over complete graphs and the set [**Volt**] of isomorphism classes of voltage assignments over complete graphs. In this section we shall establish a bijection between [**Volt**] and [**TGrph**], and therefore between the set [**DCov**] of isomorphism classes of double covers over complete graphs and the set [**TGrph**] of isomorphism classes of two-graphs.

We begin by defining a mapping $T : \mathbf{Volt} \to \mathbf{TGrph}$ that will give rise to a bijection $[\mathbf{Volt}] \to [\mathbf{TGrph}]$. For a voltage assignment $\zeta\colon D_{K_\Omega} \to \mathbb{Z}_2$, let $\Phi_\zeta\colon \Omega^{(3)} \to \mathbb{Z}_2$ be the mapping defined by the rule

$$\Phi_\zeta(\{u, v, w\}) = \zeta(u, v) + \zeta(v, w) + \zeta(w, u) \tag{3}$$

for every triple of pairwise distinct elements $u, v, w \in \Omega$. Observe that Φ_ζ is indeed a well-defined function. A straightforward calculation shows that Φ_ζ satisfies (2), whence it is a two-graph. Let $T : \mathbf{Volt} \to \mathbf{TGrph}$ be defined by $T(\zeta) = \Phi_\zeta$. First we show that T is surjective.

Let Φ be a two-graph on a set Ω. For any vertex $\omega \in \Omega$ we define a voltage assignment $\zeta_{\omega,\Phi} : D_{K_\Omega} \to \mathbb{Z}_2$ by

$$\zeta_{\omega,\Phi}(u, v) = \begin{cases} \Phi(\{u, v, \omega\}) & \text{if } \omega \notin \{u, v\} \\ 0 & \text{if } \omega \in \{u, v\}. \end{cases} \tag{4}$$

An easy calculation shows the following.

Lemma 12 *For any two-graph* Φ *on a set* Ω *and any* $\omega \in \Omega$ *we have that* $\Phi_{\zeta_{\omega,\Phi}} = \Phi$. *Consequently, the mapping* $T : \mathbf{Volt} \to \mathbf{TGrph}$ *defined by* $T(\zeta) = \Phi_\zeta$ *is surjective.*

Next we show that T maps equivalent voltage assignments to the same two-graph, and isomorphic voltage assignments to isomorphic two-graphs.

Lemma 13 *For a set* Ω*, let* $\zeta_1, \zeta_2 \colon D_{K_\Omega} \to \mathbb{Z}_2$ *be voltage assignments, and let* α *be an automorphism of* K_Ω*. Then* α *is an isomorphism from* Φ_{ζ_1} *to* Φ_{ζ_2} *if and only if it lifts along the pair of covering projections* $(\wp_{\zeta_1}, \wp_{\zeta_2})$.

Proof. For an arbitrary element $\{u, v, w\}$ of $\Omega^{(3)}$ consider the cycle $C = (u, v) + (v, w) + (w, u)$ in the first homology group $H_1(K_\Omega; \mathbb{Z}_2)$. Recall that ζ_i^* is the unique $\mathbb{Z}_2$-linear extension of ζ_i to $H_1(K_\Omega; \mathbb{Z}_2)$. By the definition (3) of Φ_{ζ_1} and Φ_{ζ_2} we have that $\zeta_1^*(C) = \Phi_{\zeta_1}(\{u, v, w\})$ and $(\zeta_2^* \circ \alpha^*)(C) = \Phi_{\zeta_2}(\alpha(\{u, v, w\}))$. Hence ζ_1^* and $\zeta_2^* \circ \alpha^*$ coincide on the set $\mathcal{C}_3$ of 3-cycles in $H_1(K_\Omega; \mathbb{Z}_2)$ if and only if α is an isomorphism from Φ_1 to Φ_2. Since $\mathcal{C}_3$ spans $H_1(K_\Omega; \mathbb{Z}_2)$, α is an isomorphism from Φ_1 to Φ_2 if and only if $\zeta_1^* = \zeta_2^* \circ \alpha^*$ (on $H_1(K_\Omega; \mathbb{Z}_2)$), and by Corollary 8, the latter statement holds if and only if α lifts along the pair of covering projections $(\wp_{\zeta_1}, \wp_{\zeta_2})$. ■

Corollary 14 *If* $\zeta_1, \zeta_2 \colon D_{K_\Omega} \to \mathbb{Z}_2$ *are voltage assignments, then* Φ_{ζ_1} *and* Φ_{ζ_2} *are isomorphic two-graphs if and only if* ζ_1 *and* ζ_2 *are isomorphic voltage assignments, and* $\Phi_{\zeta_1} = \Phi_{\zeta_2}$ *if and only if* ζ_1 *and* ζ_2 *are equivalent voltage assignments.*

Proof. The first statement of the corollary follows directly from Lemma 13. If $\Phi_{\zeta_1} = \Phi_{\zeta_2}$, then id_Ω is an isomorphism between Φ_{ζ_1} and Φ_{ζ_2}. By Lemma 13, id_Ω lifts along $(\wp_{\zeta_1}, \wp_{\zeta_2})$, whence ζ_1 and ζ_2 are equivalent. Conversely, if ζ_1 and ζ_2 are equivalent, then id_Ω lifts along $(\wp_{\zeta_1}, \wp_{\zeta_2})$, and by By Lemma 13, id_Ω is an isomorphism between Φ_{ζ_1} and Φ_{ζ_2}. Therefore, $\Phi_{\zeta_1} = \Phi_{\zeta_2}$. ■

We mention that the last statement of Corollary 14 has been observed previously in a different form; see for example [4, 12].

It is now clear that $T : \mathbf{Volt} \to \mathbf{TGrph}$ induces a bijection from the set of equivalence classes of $\mathbb{Z}_2$-voltage assignments over complete graphs to the set of all two-graphs, and also a bijection from the set [**Volt**] of isomorphism classes of voltage assignments over complete graphs to the set [**TGrph**] of isomorphism classes of two-graphs.

Moreover, since since any double cover over a complete graph is isomorphic to a derived cover $\mathrm{Cov}(K_\Omega; \zeta)$ for some voltage assignment $\zeta \colon D_{K_\Omega} \to \mathbb{Z}_2$, and since by Lemma 3, derived covers $\mathrm{Cov}(K_\Omega; \zeta_1)$ and $\mathrm{Cov}(K_\Omega; \zeta_2)$ are isomorphic if and only if the voltage assignments ζ_1 and ζ_2 are isomorphic, the mapping T also implies a bijection from [**DCov**] to [**TGrph**], and therefore from [**TGrph**] to [**DCov**]. The next theorem explains this more precisely.

Theorem 15 *The rule* $\underline{\mathrm{F}}(\Phi) = \mathrm{Cov}(K_\Omega; \zeta_{\omega,\Phi})$ *for any two-graph* Φ *on the vertex set* Ω *and any* $\omega \in \Omega$ *defines a bijection* $\underline{\mathrm{F}} : [\mathbf{TGrph}] \to [\mathbf{DCov}]$. *Moreover, if* Φ *is a two-graph on a set* Ω*,* ω *a fixed vertex in* Ω*, and* $\mathcal{F}$ *the set of* $\wp_{\zeta_{\omega,\Phi}}$*-fibres of the double cover* $\underline{\mathrm{F}}(\Phi) = \mathrm{Cov}(K_\Omega; \zeta_{\omega,\Phi})$*, then the permutation representation* $\mathrm{Aut}_{\mathcal{F}}(\underline{\mathrm{F}}(\Phi))^{\mathcal{F}}$ *of the group* $\mathrm{Aut}_{\mathcal{F}}(\underline{\mathrm{F}}(\Phi))$ *of* $\mathcal{F}$*-preserving automorphisms of* $\underline{\mathrm{F}}(\Phi)$ *(in its action on* $\mathcal{F}$*) is permutation isomorphic to* $\mathrm{Aut}(\Phi)$.

Proof. Let Φ_1 and Φ_2 be two-graphs on a set Ω, and ω_1, ω_2 fixed vertices in Ω. By Lemma 12, $\Phi_i = \Phi_{\zeta_{\omega_i,\Phi_i}}$, and hence Corollary 14 tells us that two-graphs Φ_1 and Φ_2 are isomorphic if and only if voltage assignments ζ_{ω_1,Φ_1} and ζ_{ω_2,Φ_2} are isomorphic. However, by Part (iii) of Lemma 3, voltage assignments ζ_{ω_1,Φ_1} and ζ_{ω_2,Φ_2} are isomorphic if and only if the derived covering graphs $\mathrm{Cov}(K_\Omega; \zeta_{\omega_1,\Phi_1})$ and $\mathrm{Cov}(K_\Omega; \zeta_{\omega_2,\Phi_2})$ are isomorphic. Thus two-graphs Φ_1 and Φ_2 are isomorphic if and only if the derived covering graphs $\mathrm{Cov}(K_\Omega; \zeta_{\omega_1,\Phi_1})$ and $\mathrm{Cov}(K_\Omega; \zeta_{\omega_2,\Phi_2})$ are isomorphic, implying both that $\underline{\mathrm{F}}$ is a well-defined mapping from $[\mathbf{TGrph}]$ to $[\mathbf{DCov}]$ and that it is injective.

Let $X \in \mathbf{DCov}$ be any double cover over a complete graph. Then there exist a set Ω and a voltage assignment $\zeta : D_{K_\Omega} \to \mathbb{Z}_2$ such that $X \cong \mathrm{Cov}(K_\Omega; \zeta)$. Let $\Phi = \Phi_\zeta$ and fix some vertex $\omega \in \Omega$. Lemma 12 shows that $\Phi_\zeta = \Phi = \Phi_{\zeta_{\omega,\Phi}}$, and so voltage assignments ζ and $\zeta_{\omega,\Phi}$ are equivalent by Corollary 14. But then X and $\mathrm{Cov}(K_\Omega; \zeta_{\omega,\Phi})$ are isomorphic double covers and $\underline{\mathrm{F}}([\Phi]) = [X]$, implying that $\underline{\mathrm{F}} : [\mathbf{TGrph}] \to [\mathbf{DCov}]$ is also surjective.

Let Φ be a two-graph on a set Ω, ω a fixed vertex in Ω, and $\zeta = \zeta_{\omega,\Phi}$. Note that $\Phi = \Phi_\zeta$ by Lemma 12, and hence by Lemma 13, a permutation $\alpha \in \mathrm{Sym}(\Omega)$ is an automorphism of Φ if and only if it lifts along $\wp_\zeta$. Thus $\mathrm{Aut}(\Phi)$ is the largest subgroup $\mathcal{L}(\Omega)$ of $\mathrm{Sym}(\Omega)$ that lifts along $\wp_\zeta$. Let $\underline{\mathrm{F}}(\Phi) = \mathrm{Cov}(K_\Omega; \zeta)$ and let $\mathcal{F}$ be the set of $\wp_\zeta$-fibres. Now by Lemma 3, the group $\mathrm{Aut}_{\mathcal{F}}(\underline{\mathrm{F}}(\Phi))$ of $\mathcal{F}$-preserving automorphisms of $\underline{\mathrm{F}}(\Phi)$ is the lift of $\mathcal{L}(\Omega)$ and its permutation representation $\mathrm{Aut}_{\mathcal{F}}(\underline{\mathrm{F}}(\Phi))^{\mathcal{F}}$ (acting on the set of fibres $\mathcal{F}$) is permutation isomorphic to $\mathcal{L}(\Omega) = \mathrm{Aut}(\Phi)$. ■

4 Self-complementary two-graphs

In this section we shall investigate self-complementary two-graphs and their relationship with almost self-complementary double covers.

The *complement* of a two-graph $\Phi\colon \Omega^{(3)} \to \mathbb{Z}_2$ is the two-graph $\Phi^{\mathrm{C}}\colon \Omega^{(3)} \to \mathbb{Z}_2$ defined by

$$\Phi^{\mathrm{C}}(C) = 1 + \Phi(C) \qquad \text{for every } C \in \Omega^{(3)}, \tag{5}$$

where the addition is performed modulo 2. Recall that $r_\Phi(v)$ and $\lambda_\Phi(e)$ denote the number of coherent triples of Φ containing $v \in \Omega$ and $e \in \Omega^{(2)}$, respectively. Note that the set $\mathcal{T}_{\Phi^{\mathrm{C}}}$ of coherent triples of Φ^{C} is the complement of the set $\mathcal{T}_\Phi$ in $\Omega^{(3)}$. Hence, if $|\Omega| = n$, then

$$r_\Phi(v) + r_{\Phi^{\mathrm{C}}}(v) = \frac{1}{2}(n-1)(n-2) \quad \text{and} \quad \lambda_\Phi(e) + \lambda_{\Phi^{\mathrm{C}}}(e) = n-2 \tag{6}$$

for every $v \in \Omega$ and $e \in \Omega^{(2)}$. From the definition (4) of the voltage assignment $\zeta_{\omega,\Phi} : D_{K_\Omega} \to \mathbb{Z}_2$ for any $\omega \in \Omega$ we easily obtain

$$\zeta_{\omega,\Phi^{\mathrm{C}}}(u,v) = \begin{cases} 1 + \zeta_{\omega,\Phi}(u,v) & \text{if } \omega \notin \{u,v\} \\ \zeta_{\omega,\Phi}(u,v) & \text{if } \omega \in \{u,v\}. \end{cases} \tag{7}$$

A two-graph Φ is said to be *self-complementary* if there exists an isomorphism $\varphi\colon \Phi \to \Phi^{\mathrm{C}}$ called an *antimorphism* of Φ. Lemma 16 and Proposition 17 below will be used in the proof of Theorem 24 to determine which doubly transitive two-graphs are self-complementary.

Lemma 16 *If Φ is a regular self-complementary two-graph of order n, then $n \equiv 2 \pmod 4$ and $\lambda_\Phi = \frac{1}{2}n - 1$.*

Proof. Since Φ is a self-complementary two-graph on an n-element set Ω, we have that $|\mathcal{T}_\Phi| = |\mathcal{T}_{\Phi^{\mathrm{C}}}|$ and that $|\Omega^{(3)}| = \frac{1}{6}n(n-1)(n-2)$ is an even number, whence n is congruent to 0, 1 or 2 modulo 4. Since, in addition, Φ is regular, we also have $\lambda_\Phi = \lambda_{\Phi^{\mathrm{C}}}$ and $r_\Phi = r_{\Phi^{\mathrm{C}}}$, whence by (6), $\lambda_\Phi = \frac{1}{2}(n-2)$ and $r_\Phi = \frac{1}{4}(n-1)(n-2)$. The conclusion follows easily. ■

Proposition 17 *If Φ is a self-complementary two-graph on a set Ω, then the index of* $\mathrm{Aut}(\Phi)$ *in its normalizer in* $\mathrm{Sym}(\Omega)$ *is even.*

Proof. If Φ is a self-complementary two-graph on a set Ω, then clearly $\varphi^2 \in \mathrm{Aut}(\Phi)$ and $\varphi \mathrm{Aut}(\Phi) \varphi^{-1} = \mathrm{Aut}(\Phi)$ for every antimorphism φ of Φ. Hence $\mathrm{Aut}(\Phi)$ is a normal subgroup of index 2 in $\langle \mathrm{Aut}(\Phi), \varphi \rangle$, and the statement of the proposition follows. ■

A partial converse to Proposition 17 holds for regular block-transitive two-graphs.

Proposition 18 *Let Φ be a regular block-transitive two-graph on a set Ω with index $\lambda_\Phi = \frac{1}{2}|\Omega| - 1$. Then Φ is self-complementary if and only if* $\mathrm{Aut}(\Phi)$ *is a proper subgroup of its normalizer in* $\mathrm{Sym}(\Omega)$.

Proof. Let $|\Omega| = n$. Observe that since the index of Φ is $\lambda_\Phi = \frac{1}{2}n - 1$, we have $|\mathcal{T}_\Phi| = \frac{1}{12}n(n-1)(n-2)$.

Let $M = \mathrm{Aut}(\Phi)$ and let N be the normalizer of M in $\mathrm{Sym}(\Omega)$. If Φ is self-complementary, then by Proposition 17 the index of M in N is even. Hence, M is a proper subgroup of N. Conversely, if M is a proper subgroup of N, then there exists an element $\varphi \in N \setminus M$. Since $\mathcal{T}_\Phi$ is an orbit of the natural action of M on $\Omega^{(3)}$, either $\varphi(\mathcal{T}_\Phi) = \mathcal{T}_\Phi$ or $\varphi(\mathcal{T}_\Phi) \cap \mathcal{T}_\Phi = \emptyset$. Clearly, the first possibility would imply $\varphi \in M$, a contradiction. Hence, $\varphi(\mathcal{T}_\Phi) \cap \mathcal{T}_\Phi = \emptyset$. But then $|\varphi(\mathcal{T}_\Phi) \cup \mathcal{T}_\Phi| = \frac{1}{6}n(n-1)(n-2) = \binom{n}{3}$ and thus $\varphi(\mathcal{T}_\Phi) = \Omega^{(3)} \setminus \mathcal{T}_\Phi$. It follows that φ is an isomorphism from Φ to Φ^{C}. ■

4.1 Self-complementary two-graphs and almost self-complementary double covers over complete graphs

The next lemma will be crucial in showing that the mapping $\underline{\mathrm{F}}$ from Theorem 15 induces a bijection between the isomorphism classes of self-complementary two-graphs and isomorphism classes of almost self-complementary double covers over complete graphs.

Lemma 19 *Let Φ be a two-graph on a set Ω and $\omega \in \Omega$ a fixed vertex. Furthermore, let ζ and ζ^{C} be the voltage assignments $D_{K_\Omega} \to \mathbb{Z}_2$ induced by Φ and Φ^{C}, respectively; that is, $\zeta = \zeta_{\omega,\Phi}$ and $\zeta^{\mathrm{C}} = \zeta_{\omega,\Phi^{\mathrm{C}}}$. Finally, let $\mathcal{F}$ be the set of fibres of the covering projection $\wp_\zeta$. Then the following statements are equivalent:*

(i) *Φ is a self-complementary two-graph;*

(ii) *$\wp_\zeta$ and $\wp_{\zeta^{\mathrm{C}}}$ are isomorphic covering projections;*

(iii) $\wp_\zeta$ *and* $\wp_{1+\zeta}$ *are isomorphic covering projections;*

(iv) $\mathrm{Cov}(K_\Omega;\zeta)$ *is an almost self-complementary double cover, that is, almost self-complementary with respect to* $\mathcal{F}$*.*

Proof. (i) $\Leftrightarrow$ (ii): If (i) holds, then there exists an isomorphism α between Φ and Φ^{C}. By Lemma 13, α lifts along $(\wp_\zeta, \wp_{\zeta^{\mathrm{C}}})$, whence (ii) follows. Conversely, if (ii) holds, then there exists $\alpha \in \mathrm{Sym}(\Omega)$ that lifts along $(\wp_\zeta, \wp_{\zeta^{\mathrm{C}}})$, and hence by Lemma 13, α is an isomorphism between Φ and Φ^{C}.

(ii) $\Leftrightarrow$ (iii): Let $\tilde{\beta}$ be the transposition on $\Omega \times \mathbb{Z}_2$ interchanging $(\omega, 0)$ with $(\omega, 1)$ and fixing all other vertices. Routine calculation using Observation (7) shows that $\tilde{\beta}$ is an isomorphism from $\mathrm{Cov}(K_\Omega;\zeta^{\mathrm{C}})$ to $\mathrm{Cov}(K_\Omega; 1+\zeta)$, and clearly $\wp_{1+\zeta} \circ \tilde{\beta} = \wp_{\zeta^{\mathrm{C}}}$, whence $(\mathrm{id}, \tilde{\beta})$ is an equivalence between $\wp_{\zeta^{\mathrm{C}}}$ and $\wp_{1+\zeta}$. Hence, by Lemma 1, $\wp_\zeta$ is isomorphic to $\wp_{\zeta^{\mathrm{C}}}$ if and only if it is isomorphic to $\wp_{1+\zeta}$.

(iii) $\Leftrightarrow$ (iv): Observe that the almost complement of $\mathrm{Cov}(\Omega;\zeta)$ with respect to the perfect matching $\mathcal{F}$ is $\mathrm{Cov}(\Omega; 1+\zeta)$. However, by Lemma 3, $\mathrm{Cov}(\Omega;\zeta)$ and $\mathrm{Cov}(\Omega; 1+\zeta)$ are isomorphic graphs if and only if $\wp_\zeta$ and $\wp_{1+\zeta}$ are isomorphic covering projections. ∎

A part of the above lemma holds also in the case the two-graph Φ is induced by the voltage assignment ζ (in fact, whenever ζ is equivalent to $\zeta_{\omega,\Phi}$ for some vertex ω) rather than the other way around. The following corollary will allow us to construct self-complementary two-graphs from voltage assignments.

Corollary 20 *Let* $\zeta : D_{K_\Omega} \to \mathbb{Z}_2$ *be a voltage assignment and* $\Phi = \Phi_\zeta$ *the corresponding two-graph on the set* Ω*. Then the two-graph* Φ *is self-complementary if and only if the covering projections* $\wp_\zeta$ *and* $\wp_{1+\zeta}$ *are isomorphic .*

Proof. Let ω be a fixed vertex in Ω and let $\zeta' = \zeta_{\omega,\Phi}$. By Lemma 19 it suffices to prove that $\wp_\zeta$ and $\wp_{1+\zeta}$ are isomorphic if and only if $\wp_{\zeta'}$ and $\wp_{1+\zeta'}$ are isomorphic. We shall do this by showing that $(\wp_\zeta, \wp_{\zeta'})$ and $(\wp_{1+\zeta}, \wp_{1+\zeta'})$ are both pairs of equivalent covering projections.

To prove $\wp_\zeta$ and $\wp_{\zeta'}$ are equivalent, by Corollary 8, it suffices to show that $\zeta'^* = \zeta^*$, where ζ^* and ζ'^* are the unique $\mathbb{Z}_2$-linear extensions of ζ and ζ', respectively, to the first homology group $H_1(K_\Omega;\mathbb{Z}_2)$. Since $H_1(K_\Omega;\mathbb{Z}_2)$ has a basis consisting of all cycles of the form $(w,u)+(u,v)+(v,w)$, where $u, v \in \Omega \setminus \{\omega\}$, it suffices to show that

$$\zeta'(w,u) + \zeta'(u,v) + \zeta'(v,w) = \zeta(w,u) + \zeta(u,v) + \zeta(v,w) \tag{8}$$

for all $u, v \in \Omega \setminus \{\omega\}$, $u \neq v$. However, this is easily established as the left-hand side and the right-hand side of (8) both equal to $\Phi(\{w,u,v\})$ by definitions (3) of $\Phi = \Phi_\zeta$ and (4) of $\zeta = \zeta_{\omega,\Phi}$. In exactly the same way it can be proved that $\wp_{1+\zeta}$ and $\wp_{1+\zeta'}$ are equivalent. It then follows from Lemma 1 that $\wp_\zeta$ and $\wp_{1+\zeta}$ are isomorphic if and only if $\wp_{\zeta'}$ and $\wp_{1+\zeta'}$ are isomorphic, and hence, by Lemma 19, if and only if Φ is self-complementary.

∎

Lemma 19 allows us to extend Theorem 15 to the following result. Recall from Corollary 4 that an almost self-complementary double cover over a complete graph with the set of fibres $\mathcal{F}$ is necessarily $\mathcal{F}$-fairly almost self-complementary. Thus, Y is called a *homogeneously almost self-complementary double cover over a complete graph* if it is an almost self-complementary double cover over a complete graph with the set of fibres $\mathcal{F}$ and the group $\mathrm{Aut}_{\mathcal{F}}(Y)$ of $\mathcal{F}$-fair (that is, $\mathcal{F}$-preserving) automorphisms of Y is transitive on the vertex set.

Theorem 21 *The bijection* $\underline{\mathrm{F}}\colon [\mathbf{TGrph}] \to [\mathbf{DCov}]$ *defined by* $\underline{\mathrm{F}}([\Phi]) = [\mathrm{Cov}(K_\Omega; \zeta_{\omega,\Phi})]$ *for a fixed* ω *in the vertex set* Ω *of the two graph* Φ *induces a bijection between:*

(i) *the set of isomorphism classes of self-complementary two-graphs and the set of isomorphism classes of almost self-complementary double covers over complete graphs;*

(ii) *the set of isomorphism classes of vertex-transitive self-complementary two-graphs and the set of isomorphism classes of homogeneously almost self-complementary double covers over complete graphs.*

Proof. Equipped with Theorem 15, all we need to prove is the following two statements:

(i') Φ is a self-complementary two-graph if and only if $\underline{\mathrm{F}}(\Phi)$ is an almost self-complementary double cover.

(ii') Φ is a vertex-transitive self-complementary two-graph if and only if $\underline{\mathrm{F}}(\Phi)$ is a homogeneously almost self-complementary double cover.

(i') and therefore (i) follows directly from Lemma 19. To see (ii'), recall from Theorem 15 that $\mathrm{Aut}(\Phi)$ is permutation isomorphic to $\mathrm{Aut}_{\mathcal{F}}(\underline{\mathrm{F}}(\Phi))^{\mathcal{F}}$, where $\mathcal{F}$ is the set of $p_{\zeta_{\omega,\Phi}}$-fibres of $\underline{\mathrm{F}}(\Phi) = \mathrm{Cov}(K_\Omega; \zeta_{\omega,\Phi})$. Since $p_{\zeta_{\omega,\Phi}}$ is a regular covering projection, $\mathrm{Aut}_{\mathcal{F}}(\underline{\mathrm{F}}(\Phi))^{\mathcal{F}}$ is transitive (on $\mathcal{F}$) if and only if $\mathrm{Aut}_{\mathcal{F}}(\underline{\mathrm{F}}(\Phi))$ is transitive (on the vertex set of $\underline{\mathrm{F}}(\Phi)$). Hence $\mathrm{Aut}(\Phi)$ is transitive if and only if $\mathrm{Aut}_{\mathcal{F}}(\underline{\mathrm{F}}(\Phi))$ is transitive, and so by (i'), Φ is a vertex-transitive self-complementary two-graph if and only if $\underline{\mathrm{F}}(\Phi)$ is a homogeneously almost self-complementary double cover. ■

4.2 Doubly transitive self-complementary two-graphs

A two-graph Φ is called *doubly transitive* if $\mathrm{Aut}(\Phi)$ acts 2-transitively on the vertex set of Φ. An example of a doubly transitive two-graph — and as we shall see shortly, up to isomorphism the only self-complementary doubly transitive two-graph — arises from Construction 9.

Lemma 22 *Let* q *be a prime power congruent to 1 modulo 4 and* $\Omega = GF(q) \cup \{\infty\}$. *Let* $\zeta : D_{K_\Omega} \to \mathbb{Z}_2$ *and* $K^2_{q+1} = \mathrm{Cov}(K_\Omega; \zeta)$ *be the voltage assignment and the corresponding covering graph, respectively, defined in Construction 9. Then* Φ_ζ *(as defined in Section 3.2) is a self-complementary doubly transitive two-graph and* K^2_{q+1} *is a homogeneously almost self-complementary double cover with* $\mathrm{Aut}(K^2_{q+1}) = \mathrm{Aut}_{\mathcal{F}}(K^2_{q+1})$ *acting 2-transitively on the set* $\mathcal{F}$ *of* $\wp_\zeta$*-fibres.*

Proof. A straightforward calculation shows that $\zeta = \zeta_{\infty,\Phi_\zeta}$, so Lemma 19 can be used for Φ_ζ and ζ. Indeed, since the covering projections $\wp_\zeta$ and $\wp_{1+\zeta}$ are isomorphic by Lemma 10, the two-graph Φ_ζ is self-complementary by Lemma 19. Moreover, since by Lemma 10 the 2-transitive group $\mathrm{P\Sigma L}(2,q)$ lifts along ζ, Lemma 13 implies that $\mathrm{P\Sigma L}(2,q) \leq \mathrm{Aut}(\Phi_\zeta)$, and therefore the two-graph Φ_ζ is doubly transitive.

Since $K_{q+1}^2 = \mathrm{Cov}(K_\Omega; \zeta_{\infty,\Phi_\zeta}) = \underline{\mathrm{F}}(\Phi_\zeta)$, by Theorem 21, K_{q+1}^2 is a homogeneously almost self-complementary double cover and by Theorem 15, $\mathrm{Aut}_{\mathcal{F}}(K_{q+1}^2)^{\mathcal{F}}$ is 2-transitive since $\mathrm{Aut}(\Phi_\zeta)$ is. Finally, since K_{q+1}^2 is connected, it follows from Lemma 5 that $\mathrm{Aut}(K_{q+1}^2) = \mathrm{Aut}_{\mathcal{F}}(K_{q+1}^2)$, as asserted. ■

Taylor [16] gives a complete list of doubly transitive two-graphs using the classification of 2-transitive permutation groups (based on the Classification of Finite Simple Groups). We shall now determine which two-graphs on this list are self-complementary. We start with the following simple observation that follows directly from the definition of a 2-transitive permutation group and Lemma 16.

Lemma 23 *Every doubly transitive self-complementary two-graph is regular and its order is congruent to* 2 *modulo* 4.

Theorem 24 *A doubly transitive two-graph* Φ *is self-complementary if and only if it is isomorphic to the two-graph* Φ_ζ *for the voltage assignment* ζ *defined in Construction 9.*

Proof. By [16, Theorem 1], there are only two families of doubly transitive two-graphs of order congruent to 2 modulo 4 — these are the families arising in Subcases (i) and (ii) of Case (A) of [16, Theorem 1]. The family described in Subcase (ii) is associated with the natural 2-transitive action of the group $\mathrm{PSU}(3,q)$, where q is a prime power, on a set of size q^3+1. By [16, Theorem 2], the full automorphism group of a two-graph associated with $\mathrm{PSU}(3,q)$ for $q > 3$ is $\mathrm{P\Gamma U}(3,q)$, and in our case $q > 3$ since $q \equiv 1 \pmod 4$. However, the normalizer of the permutation group $\mathrm{P\Gamma U}(3,q)$ in the full symmetric group is just $\mathrm{P\Gamma U}(3,q)$ itself, whence by Proposition 17 the corresponding two-graphs can not be self-complementary. Therefore, the only candidates for self-complementary doubly transitive two-graphs are the two-graphs of the family arising from Subcase (i) of [16, Theorem 1] associated with the permutation groups $\mathrm{PSL}(2,q)$, where q is an odd prime power congruent to 1 modulo 4. By [16, Theorem 2], this family of two-graphs is unique; it is explicitly described in [15, Example 6.2], and is exactly the family of two-graphs Φ_ζ where ζ is the voltage assignment defined in Construction 9. Lemma 22 concludes the proof. ■

We conclude this section with an analogous result for almost self-complementary graphs.

Theorem 25 *A graph* X *on* $2n$ *vertices is almost self-complementary with respect to a perfect matching* $\mathcal{I}$ *in* X^{c} *such that* $\mathrm{Aut}_{\mathcal{I}}(X)$ *is acting* 2*-transitively on* $\mathcal{I}$ *if and only if* $n = q+1$ *for some prime power* q *congruent to* 1 *modulo* 4 *and* X *is isomorphic to the graph* K_{q+1}^2 *defined in Construction 9.*

Proof. The "if" part is immediate: the graph K_{q+1}^2 has the required property by Lemma 22.

To prove the "only if" part, let X be a graph on $2n$ vertices that is almost self-complementary with respect a perfect matching $\mathcal{I}$ in X^{C}, and suppose $\mathrm{Aut}_{\mathcal{I}}(X)^{\mathcal{I}}$ is 2-transitive. First we observe that all bricks of X (that is, subgraphs of X induced by two elements of $\mathcal{I}$) are pairwise isomorphic, and are "symmetric" in the sense that there exists an automorphism of X swapping the two elements of $\mathcal{I}$ that are the bipartition sets of the brick. Moreover, since X has $n(n-1)$ edges and $\binom{n}{2}$ bricks, each brick must contain two edges and therefore be isomorphic to $2K_2$. Hence X is a double cover over the complete graph K_n and thus isomorphic to $\mathrm{Cov}(K_n;\zeta')$ for some voltage assignment $\zeta' : D_{K_n} \to \mathbb{Z}_2$. Moreover, there exists an isomorphism that maps $\mathcal{I}$ to the set $\mathcal{F}$ of $\wp_{\zeta'}$-fibres, and thus $\mathrm{Cov}(K_n;\zeta')$ is an almost self-complementary double cover.

Let ω be a fixed vertex of K_n and consider the spanning tree (in fact, a star) of K_n containing all edges incident with ω. By Proposition 2 there exists a voltage assignment ζ'' equivalent to ζ' such that $\zeta''(\omega, u) = 0$ for all $u \in V(K_n) \setminus \{\omega\}$. Then $\mathrm{Cov}(K_n;\zeta'')$ is also an almost self-complementary double cover isomorphic to X, and $\mathrm{Aut}_{\mathcal{F}}(\mathrm{Cov}(K_n;\zeta''))$ acts 2-transitively on the set $\mathcal{F}$ of $\wp_{\zeta''}$-fibres.

Let Φ be the two-graph $\Phi_{\zeta''}$. An easy calculation shows that $\zeta'' = \zeta_{\omega,\Phi}$, whence by Lemma 19, the two-graph Φ is almost self-complementary since $\mathrm{Cov}(K_n;\zeta'')$ is an almost self-complementary double cover. Moreover, since $\mathrm{Cov}(K_n;\zeta'') = \underline{\mathrm{F}}(\Phi)$ and $\mathrm{Aut}_{\mathcal{F}}(\mathrm{Cov}(K_n;\zeta''))^{\mathcal{F}}$ is 2-transitive, Φ is doubly transitive by Theorem 15. But then, by Lemma 22, Φ is isomorphic to Φ_ζ for the voltage assignment $\zeta : D_{K_\Omega} \to \mathbb{Z}_2$ defined in Construction 9 for some prime power q congruent to 1 modulo 4 and $\Omega = GF(q) \cup \{\infty\}$. But as $\zeta = \zeta_{\infty,\Phi_\zeta}$, the graphs $K_{q+1}^2 = \mathrm{Cov}(K_\Omega;\zeta)$ and $\mathrm{Cov}(K_n;\zeta'')$ are isomorphic by Theorem 15. Hence $X \cong K_{q+1}^2$, and necessarily $n = q+1$. ■

4.3 Constructions of self-complementary two-graphs

As we have seen in Theorem 24, Construction 9 results in a unique family of doubly transitive self-complementary two-graphs. In this last section we shall give two more constructions of self-complementary two-graphs via voltage assignments. In each case we determine sufficient conditions for the constructed two-graphs to be vertex-transitive.

Construction 26 Let X be a self-complementary graph with vertex set Ω, let $\tau \in \mathrm{Sym}(\Omega)$ be an antimorphism of X, and let $\zeta \colon D_{K_\Omega} \to \mathbb{Z}_2$ be the voltage assignment on K_Ω with $\zeta(u,v) = 1$ if and only if $u \sim_X v$. Then $(1+\zeta)^* = \zeta^* \circ \tau^*$ (where, as usual, * denotes the unique $\mathbb{Z}_2$-linear extension to the first homology group $H_1(K_\Omega;\mathbb{Z}_2)$ of all cycles) and so τ lifts along $(\wp_{1+\zeta}, \wp_\zeta)$ by Corollary 8. Hence p_ζ and $p_{1+\zeta}$ are isomorphic covering projections and by Corollary 20, the two-graph Φ_ζ is self-complementary.

Moreover, if α is an automorphism of X, then $\zeta^* \circ \alpha^* = \zeta^*$, whence α lifts along $\wp_\zeta$. Thus by Lemma 13, any automorphism of X is an automorphism of Φ_ζ, and $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(\Phi_\zeta)$. In particular, if X is a vertex-transitive self-complementary graph, then Φ_ζ is a vertex-transitive self-complementary two-graph.

Construction 27 Let X be a graph and let $\Omega = V_X \times \mathbb{Z}_2$. Define $\zeta\colon D_{K_\Omega} \to \mathbb{Z}_2$ by the rule

$$\zeta((u,i),(v,j)) = \begin{cases} 0 & \text{if } i \neq j \\ 0 & \text{if } i = j = 0 \text{ and } u \not\sim_X v \\ 0 & \text{if } i = j = 1 \text{ and } u \sim_X v \\ 1 & \text{in all other cases} \end{cases} .$$

Observe that $\zeta((u,i),(v,i)) = 1 + \zeta((u,i+1),(v,i+1))$ and $\zeta((u,i),(v,i+1)) = \zeta((u,i+1),(v,i))$ for all $u,v \in V_X$ and $i \in \mathbb{Z}_2$. Since any cycle in K_Ω contains an even number of darts of the form $((u,i),(v,i+1))$, the permutation α swapping the vertices $(u,0)$ and $(u,1)$ for all $u \in V_X$ satisfies the equality $(1+\zeta)^* = \zeta^* \circ \alpha^*$. By Corollary 8, α lifts along the pair of covering projections $(\wp_{1+\zeta}, \wp_\zeta)$, whence $\wp_\zeta$ and $\wp_{1+\zeta}$ are isomorphic. Therefore, Φ_ζ is a self-complementary two-graph by Corollary 20.

Note also that for every automorphism $\alpha \in \mathrm{Aut}(X)$, the permutation $\bar\alpha \in \mathrm{Sym}(\Omega)$ defined by $\bar\alpha(u,i) = (\alpha(u), i)$ lifts along $\wp_\zeta$ since $\zeta^* \circ \bar\alpha^* = \zeta^*$. Hence by Lemma 13, $\bar\alpha$ is an automorphism of the two-graph Φ_ζ. Therefore, if X is vertex-transitive, then $\mathrm{Aut}(\Phi_\zeta)$ has at most two orbits on Ω. Moreover, if X is self-complementary with an antimorphism σ, then the permutation $\bar\sigma \in \mathrm{Sym}(\Omega)$ defined by $\bar\sigma(u,i) = (\sigma(u), i+1)$ lifts along $\wp_\zeta$ and is therefore an automorphism of Φ_ζ. This shows that if X is a vertex-transitive self-complementary graph, then Φ_ζ is a vertex-transitive self-complementary two-graph.

We conclude the paper with the following open problem.

Problem 28 *Find the set of positive integers n such that there exists a vertex-transitive self-complementary two-graph of order n.*

A solution of this problem would be analogous to Muzychuk's determination of all orders of vertex-transitive self-complementary graphs [11]. Note that Constructions 26 and 27 imply existence of vertex-transitive self-complementary graphs of all orders of the form n and $2n$, where n is an odd integer such that the highest power of every prime dividing n is congruent 1 modulo 4.

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