University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1111 Ljubljana, Slovenia

Preprint series, Vol. 42 (2004), 933

A COMPLETE PROOF OF THE NONEXISTENCE OF REGULAR FOUR-DIMENSIONAL TILINGS IN THE LEE METRIC

Simon Špacapan

ISSN 1318-4865

October 22, 2004

Ljubljana, October 22, 2004

A complete proof of the nonexistence of regular four-dimensional tilings in the Lee metric

Simon Špacapan
University of Maribor, FME
Smetanova 17, 2000 Maribor, Slovenia
simon.spacapan@uni-mb.si

August 23, 2004

Abstract

A family of n-dimensional Lee spheres $\mathcal{L}$ is a *tiling* of $\mathbb{R}^n$, if $\cup\mathcal{L} = \mathbb{R}^n$ and for every $L_u, L_v \in \mathcal{L}$, the intersection $L_u \cap L_v$ is contained in the boundary of L_u. If neighboring Lee spheres meet along entire $(n-1)$-dimensional faces, then $\mathcal{L}$ is called a *regular tiling*. We prove nonexistence of a regular tiling of $\mathbb{R}^4$, which in particular confirms the conjecture of Golomb and Welch about nonexistence of perfect Lee codes for $n = 4$.

Key words: Tiling, Lee Metric, Perfect Codes.

AMS subject classification (2000): 52C22, 94B60

1 Introduction

There have been numerous attempts of proving the nonexistence of perfect Lee codes. The beginning dates back to 1968, when Golomb and Welch proved in [6] the existence of perfect codes in Lee metric, for the case $n = 1$ and all e; for $n = 2$ and all e; and for $e = 1$ and all n, where n denotes the length of the codewords and e number of errors this code corrects. They also proved the nonexistence in some special cases and conjectured [6, 7] the nonexistence of perfect Lee codes for all values of e and n, except of those mentioned above.

Since then numerous articles have been published, each partially confirming the conjecture of Golomb and Welch. For example Post proved in 1975 the nonexistence of perfect Lee codes for $3 \leq n \leq 5$ and $e \geq n - 1$; and for $n \geq 6$ and $e \geq n\sqrt{2}/2 - 3\sqrt{2}/4 - 1/2$. There have also been numerous results given by Astola in [1, 2, 3].

Nevertheless no attempt have been made for proving nonexistence of a tiling in the Lee metric, until recently Gravier, Mollard and Payan proved it in [8], for the case $n = 3$. More precisely they proved nonexistence of a tiling of $\mathbb{R}^3$ with Lee spheres of radii at least two (even if the spheres are allowed to have different radii). The approach of Gravier, Mollard and Payan is fundamentally different from previous approaches, as they disregard the fact that for a perfect Lee code to exist, it is necessary to partition the Cartesian product of cycles and therefore the sphere packing condition must be fulfilled. The focus of Gravier et al. goes more on the local structure of the Cartesian product of cycles, which is actually the same as in the product of paths.

In their proof they list all possible positions of neighboring Lee spheres and conclude that in any case there must be some uncovered space left (or otherwise Lee spheres intersect). In [10] they extended this result to the case when the Lee spheres have radii at least one and at least one of the spheres have radius greater then one.

The main goal of this paper is the proof of next theorem.

Theorem 1.1 *There does not exist a regular tiling of* $\mathbb{R}^4$ *with Lee spheres of radii at least two (even with different radii).*

In Section 2 we give all requiring definitions and lemmas, in particular we state Lemma 2.2, the cruical tool for proving Theorem 1.1 and in Section 3 we give a detailed proof of Theorem 1.1.

2 Notation and Preliminaries

We basically follow the terminology from [8]. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}^n$. The *n-cube* centered on x is the set:

$$C(x) = \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \,|\, \text{for every } i, |x_i - y_i| \leq 1/2\}\,.$$

A *facet* F of n-dimensional cube $C(x)$ is $(n-1)$-dimensional cube

$$F = \{(y_1, y_2, \ldots, y_n) \in C(x) \,|\, \text{for some } k, y_k = c\}\ ,$$

where either $c = x_k + 1/2$ or $c = x_k - 1/2$. Facets are also called $(n-1)$*-dimensional faces of C(X)*. The Lee distance between two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ of $\mathbb{R}^n$ is defined by

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|\,.$$

Let r be a nonnegative integer. The *r-Lee sphere* in $\mathbb{R}^n$ centered on $(0, 0, \ldots, 0) \in \mathbb{R}^n$ is defined by

$$C_r(0) = \bigcup \{C(y) \,|\, y \in \mathbb{Z}^n, d(y, 0) \leq r\}\ ,$$

where r is called the radius of Lee sphere $C_r(0)$ and is denoted by $\mathrm{rad}(C_r(0))$. F is a facet of Lee sphere $C_r(0)$, if F is a facet of $C(y)$ for some $y \in \mathbb{Z}^n, d(y, 0) \leq r$ and F is contained in the boundary of the Lee sphere $C_r(0)$. A 3-dimensional 1-Lee sphere and 2-Lee sphere are depicted on Figure 1. Note that on both spheres, there is only one facet of the sphere marked with gray color.

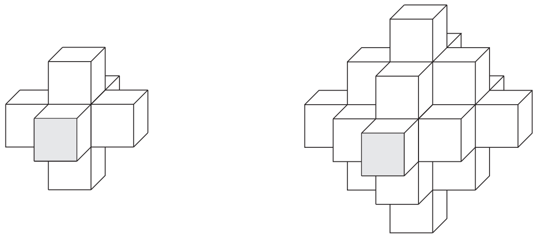

Figure 1: 3-dimensional 1-Lee sphere and 2-Lee sphere

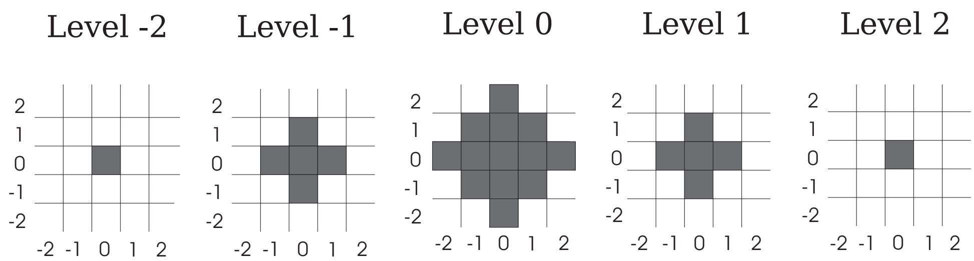

Figure 2: A 2-Lee sphere

Since it is more convenient to present 3-dimensional Lee spheres by their intersections with parallel planes on different levels, we will mainly present Lee spheres with Figures as it is done in Figure 2.

More generally, the r-Lee sphere in $\mathbb{R}^n$ centered on $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is

$$C_r(x) = \bigcup \{C(y) \mid y - x \in \mathbb{Z}^n, d(y,x) \le r\} \ .$$

L is called a *Lee sphere*, if L is an r-Lee sphere for some r. The distance between two Lee spheres is the distance between their centers, and two n-dimensional Lee spheres are said to be neighboring, if their intersection is $(n-1)$-dimensional (that is, the intersection can be embedded in $\mathbb{R}^{n-1}$, but not in $\mathbb{R}^{n-2}$).

If $X \subseteq \mathbb{R}^n$, then we denote the boundary of the set X (in Euclidean topology) by $Bd(X)$. A family of n-dimensional Lee spheres $\mathcal{L}$ is a *tiling* of $\mathbb{R}^n$, if $\cup\mathcal{L} = \mathbb{R}^n$ and for every $L_u, L_v \in \mathcal{L}$, $L_u \cap L_v \subset Bd(L_u)$. If neighboring Lee spheres meet along entire $(n-1)$-dimensional faces, then $\mathcal{L}$ is called a *regular tiling*. Thus a tiling $\mathcal{L}$ is regular, if for every neighboring Lee spheres $L_u, L_v \in \mathcal{L}$ and for any two facets $F_u \subset L_u$ and $F_v \subset L_v$, either $F_u \cap F_v = F_u = F_v$ or $F_u \cap F_v = \emptyset$.

We now state some simple observations about the Lee spheres.

Lemma 2.1 *Let $C_r(u)$ be the r-Lee sphere in $\mathbb{R}^n$ centered on $u = (u_1, \dots, u_n)$. If $z \in \mathbb{Z}$ and $|z| \le r$, then the set*

$$\{(x_1, \dots, x_{n-1}) \mid (x_1, \dots, x_{n-1}, u_n + z) \in C_r(u)\}$$

is $(n-1)$*-dimensional Lee sphere centered on* $(u_1,\dots,u_{n-1})\in\mathbb{R}^{n-1}$*, with radius equal to* $r-|z|$*. More generally, if* $z_i\in\mathbb{Z}, i=k+1,\dots,n$ *and* $\sum_{i=k+1}^{n}|z_i|\le r$*, then the set*

$$\{(x_1,\dots,x_k)\,|\,(x_1,\dots,x_k,u_{k+1}+z_{k+1},\dots,u_n+z_n)\in C_r(u)\}$$

is the k*-dimensional Lee sphere centered on* $(u_1,\dots,u_k)\in\mathbb{R}^k$*, with radius equal to* $r-\sum_{i=k+1}^{n}|z_i|$.

We denote by $\mathcal{E}_c=\{(x_1,x_2,x_3,c)\,|\,x_i\in\mathbb{R}\}$ the 3-dimensional (affine) subspace of $\mathbb{R}^4$, with the last coordinate fixed. If L is 4-dimensional r-Lee sphere, centered on $(\ell_1,\ell_2,\ell_3,\ell_4)$ then

$$\mathcal{E}_{\ell_4-r}\cap L,\mathcal{E}_{\ell_4-r+1}\cap L,\dots,\mathcal{E}_{\ell_4-1}\cap L,\mathcal{E}_{\ell_4}\cap\ L,\mathcal{E}_{\ell_4+1}\cap L,\dots,\mathcal{E}_{\ell_4+r}\cap L$$

are 3-dimensional Lee spheres of radii $1,2,\dots,r-1,r,r-1,\dots,2,1$ in this order. We say that the first $r-1$ Lee spheres (with radii $1,\dots,r-1$) are of *type low*, the r-th Lee sphere is of *type middle* and the last $r-1$ Lee spheres (with radii $r-1,\dots,1$) are of *type high*. Recall that we denote by rad(L) the radius of Lee sphere L. We make the following observation on Lee spheres. Let $z\in\mathbb{Z},|z|\le r, a=(a_1,a_2,a_3,a_4)$ and let

$$P=\{(x_1,x_2,x_3)\,|\,(x_1,x_2,x_3,a_4+z)\in C_r(a)\}$$

be a 3-dimensional Lee sphere. Let

$$P_u=\{(x_1,x_2,x_3)\,|\,(x_1,x_2,x_3,a_4+z+1)\in C_r(a)\},$$
$$P_d=\{(x_1,x_2,x_3)\,|\,(x_1,x_2,x_3,a_4+z-1)\in C_r(a)\}.$$

If $\mathrm{rad}(P_u)<\mathrm{rad}(P)<\mathrm{rad}(P_d)$ (or $\mathrm{rad}(P)<\mathrm{rad}(P_d)$ and $P_u=\emptyset$), then the Lee sphere P is of type high, if $\mathrm{rad}(P_u)>\mathrm{rad}(P)>\mathrm{rad}(P_d)$ (or $\mathrm{rad}(P_u)>\mathrm{rad(P)}$ and $P_d=\emptyset$), then the Lee sphere P is of type low, and if $\mathrm{rad}(P_u)<\mathrm{rad}(P)>\mathrm{rad}(P_d)$ (or $P_d=\emptyset$ and $P_u=\emptyset$), then the Lee sphere P is of type middle. Note that the Lee sphere P is of type middle, if and only if $z=0$ and $\mathrm{rad}(P)=\mathrm{rad}(C_r(a))=r$.

Lemma 2.2 *Let* $a=(a_1,a_2,a_3,a_4)$*,* $b=(b_1,b_2,b_3,b_4)$*,* $c\in\mathbb{R}$ *and*

$$L_1=\{(x_1,x_2,x_3)\,|\,(x_1,x_2,x_3,c)\in C_{r_1}(a)\}$$
$$L_2=\{(x_1,x_2,x_3)\,|\,(x_1,x_2,x_3,c)\in C_{r_2}(b)\}$$

be two 3-dimensional Lee spheres both of type high or both of type low. If $\mathcal{L}$ *is a regular tiling of* $\mathbb{R}^4$ *and* $C_{r_1}(a),C_{r_2}(b)\in\mathcal{L}$*, where* $r_1,r_2\ge 2$*, then we have:*
(i) *If* $\mathrm{rad}(L_1)=\mathrm{rad}(L_2)=0$*, then* $d(L_1,L_2)\ge 5$.
(ii) *If* $\mathrm{rad}(L_1)\ne 0$ *or* $\mathrm{rad}(L_2)\ne 0$*, then* $d(L_1,L_2)\ge 3+r_1+r_2$.

3 Proof of the main result

We divide the proof into 7 steps. Suppose that there exists a regular tiling $\mathcal{L}$ of $\mathbb{R}^4$ with Lee spheres of radii greater or equal 2. We can assume, without loss of generality, that one Lee sphere from $\mathcal{L}$ is $C_r(0,0,0,r)$, and $r\ge 2$. Thus by Lemma 2.1,

$L = \{(x_1, x_2, x_3) \,|\, (x_1, x_2, x_3, 0) \in C_r(0, 0, 0, r)\}$ is a 3-dimensional Lee sphere of type low, centered on $(0, 0, 0)$, with radius equal to 0. Now we will examine all possibilities of tiling the neighborhood of the sphere L, and in all cases we are going to obtain a contradiction (to Lemma 2.2). Clearly, since $\mathcal{L}$ is a regular tiling of $\mathbb{R}^4$, the family of all intersections of the sets from $\mathcal{L}$ with the 3-dimensional subspace $\mathcal{E}_0$, is a regular tiling of $\mathcal{E}_0$, with Lee spheres of radii greater or equal 0. As it will be shown, it is not possible to tile even the space $\mathcal{E}_0$ in such a way, that no two corresponding 4-dimensional Lee spheres intersect (on their interior).

The Lee sphere L has 6 facets and it is impossible for one neighboring 3-dimensional Lee sphere to cover more then three facets of L. Since there are 7 different ways to write 6 as a sum of numbers less or equal 3, there are 7 possible cases. For instance, the case 1+1+1+1+1+1, happens when all the six facets are covered by different Lee spheres. Similarly the case 2+2+2 means, that two facets of L are covered by one sphere, two by second sphere and the rest of two by the third.

We present the proof manly by figures, some comments are added to find correct focus. In all figures the 3-dimensional spheres are labeled with letters A, B, C, and so on. The thinking pattern in figures always goes in the alphabetic order (except of the sphere L, which is given in the beginning), with black dot is usually denoted the cube which cannot be contained in a Lee sphere, such that this sphere doesn't intersect another sphere (on it's interior). In what follows we will frequently say the (a, b, c)-cube, meaning the cube centered on (a, b, c).

3.1 Case 1+1+1+1+1+1

Consider the case when all 6 facets of L are covered by different Lee spheres and recall that the sphere L is of type low, centered on $(0, 0, 0)$, with $r = 0$. Suppose that the $(0, 1, 0)$-cube is contained in the Lee sphere A, $(1, 0, 0)$-cube in B, $(-1, 0, 0)$-cube in D, $(0, -1, 0)$-cube in C, $(0, 0, 1)$-cube in the Lee sphere E and $(0, 0, -1)$-cube in F (see Figure 3). By Lemma 2.2 at most one of the Lee spheres A, B, C, D, E and F is of type high and none of them is of type low (since L is of type low). Without loss of generality assume that A, B, C, D and E are not of type high, thus they are of type middle and therefore have radius greater or equal 2 (c.f. Section 2). Consider the $(0, -1, 1)$-cube and suppose that this cube is contained in the sphere G, with $r = 0$. Thus, by Lemma 2.2 (i) G is of type high, since L is of type low and $d(G, L) = 2$. Thus, by the same lemma, the $(-1, 0, 1)$-cube must be contained in a sphere with $r > 0$, since already L and G are spheres of radii 0. Thus the only possibility is that the $(-1, 0, 1)$-cube is contained in sphere H, with $r = 1$ (for otherwise H would intersect one of the spheres A, E, D, G or L). Since $\mathrm{rad}(H) = 1$, H is of type low or high (c.f. Section 2). But $d(H, L) = 3$, thus H cannot be of typw low and since $d(H, G) = 3$, H cannot be of type high (c.f. Lemma 2.2 (ii)) which is a contradiction. The other possible case is when $(0, -1, 1)$-cube is contained in the sphere G, with $r = 1$. In this case the situation depicted on Figure 4 occur. Here G and H are both of radius 1, and since $d(G, L) = d(H, L) = 3$ neither of them can be of type low, thus G and H are both of type high. Since $d(G, H) = 4$, this is a contradiction to Lemma 2.2.

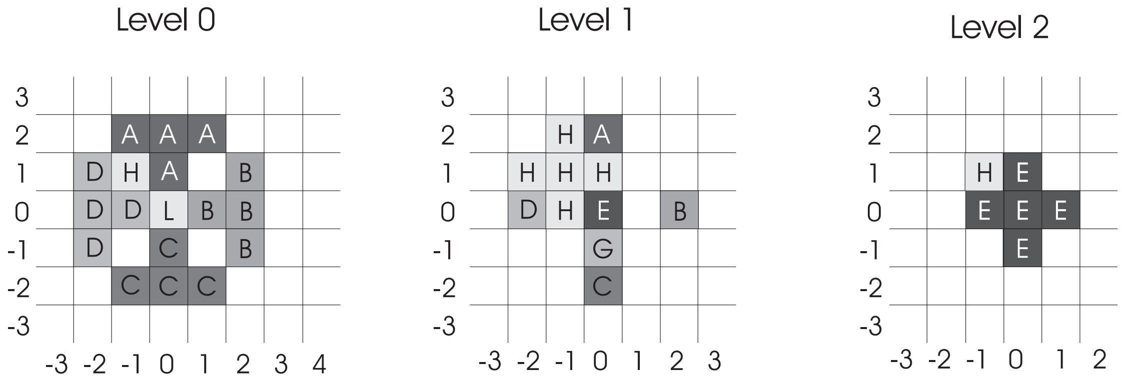

Figure 3: Case 1

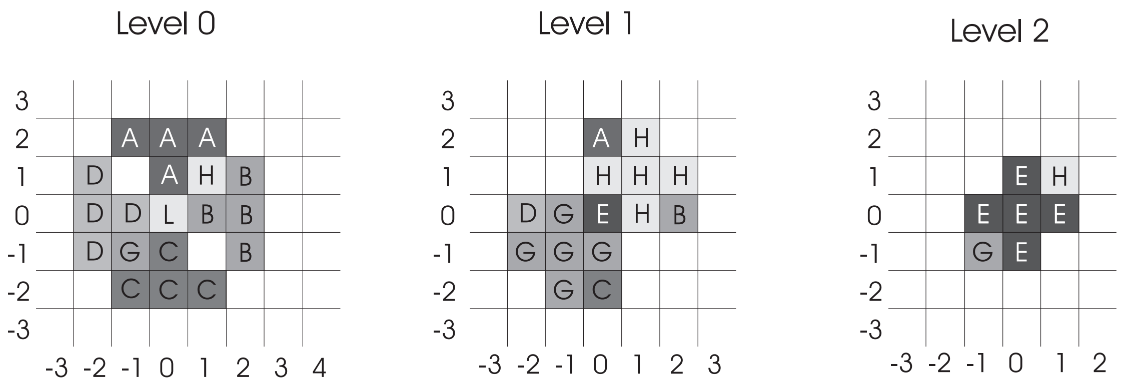

Figure 4: Case 2

3.2 Case 1+1+1+1+2

We have two facets of L covered by a sphere B and spheres A, C, D and E each cover one facet of L. Consider the $(-1,1,0)$-cube and suppose that this cube is contained in a sphere F. Then one of the following three cases appear $\mathrm{rad}(F)=0$ (see Case 1, Figure 5) or $\mathrm{rad}(F)=1$ (see Case 2, Figure 6) or $\mathrm{rad}(F)\geq 2$ (see Case 3, Figure 7).

In Case 1 we have $\mathrm{rad}(F)=0$, thus F is of type high. Then by Lemma 2.2 A and C are of type middle, and $\mathrm{rad}(D),\mathrm{rad}(E)\geq 1$. The $(0,1,1)$-cube is then contained in a sphere G, note that G is also of type middle, since F is of type high and L is of type low. We then have $(-1,0,1)$-cube contained in H, which is by the same argument of type middle (note that G and H 'force' B to be of radius 1). Therefore $(-1,1,1)$-cube must be contained in a sphere with $r=0$, hence this sphere is of type low or high. Since already L is of type low and F is of type high, this is by Lemma 2.2 a contradiction.

Suppose the sphere F (containing the $(-1,1,0)$-cube) is of type high, with $r=1$ (by

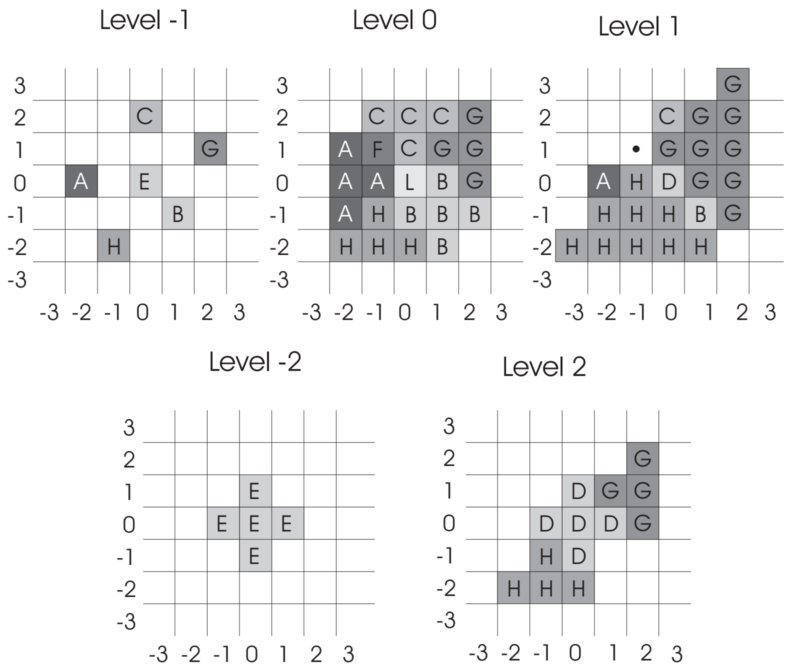

Figure 5: Case 1

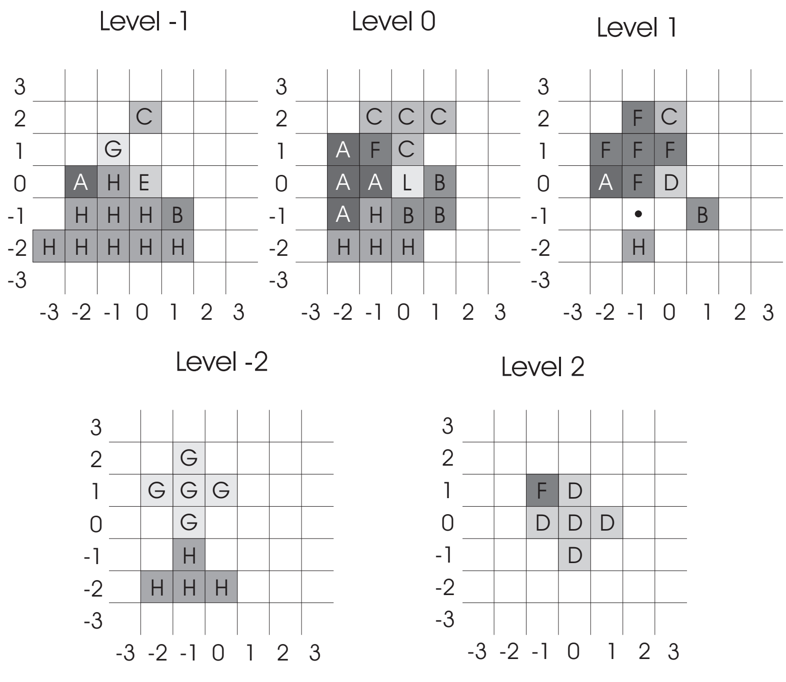

Figure 6: Case 2

the symmetry we can assume that F is centered on $(-1, 1, 1)$). Then D is of type middle and $(-1, 1, -1)$-cube is contained in a sphere G, with $r \geq 1$. Since F (and E) is of type high and L is of type low, the $(-1, 0, -1)$-cube is by Lemma 2.2 contained in a sphere H of type middle and therefore $(-1, -1, 1)$-cube is left uncovered.

If $\text{rad}(F) \geq 2$, then $rad(D) = 0$ and E is of type middle (see Case 3, Figure 7). The $(-1, 1, -1)$-cube is thus contained in a sphere with $r = 0$, which is a contradiction, since L is of type low and D is of type high.

3.3 Case 1+1+2+2

We have two facets of L covered by a sphere A and two by B. One facet is covered by C and one by D. The first case is when the spheres C and D cover opposite facets of L (see Figures 8–11). At most one of the spheres A and B is of type high. Suppose B is of type high and $\text{rad}(B) = 1$ (see Figure 8). Then C and D are both of type middle and either the $(0, -1, 1)$-cube or the $(0, -1, -1)$-cube is contained in a sphere of type middle. Without loss of generality assume the $(0, -1, 1)$-cube is contained in a sphere E of type middle. Thus the $(1, -1, -1)$-cube is contained in a sphere, with $r = 0$, which contradicts the fact that L is of type low and B is of type high.

The other possibility is that A and B are both of type middle, and A and B could be positioned as in Case 2.1.1 or as in Case 2.2 (see Figures 9 and 11 and notice the position of A and B on level 0). In Case 2.1.1 it is easy to see that $(1, -1, 0)$-cube cannot be contained in a sphere E, with $r = 0$, thus $\text{rad}(E) \geq 1$.

Suppose $\text{rad}(E) = 1$ (by the symmetry we can assume that E is centered on $(1, -1, -1)$, otherwise E would be centered on $(1, -1, 1)$), then D is of type middle. Since E is of type high and L is of type low, the $(-1, -1, -2)$-cube is contained in a sphere F, with $r \geq 1$ and the $(-1, 0, -1)$-cube is contained in a sphere G of type middle (see Figure 9).

In the Case 2.1.2 (see Figure 10) we have $\text{rad}(E) \geq 2$, but than $\text{rad}(D) = 0$ and thus D is of type high and C is of type middle.

In the Case 2.2 (see Figure 11) the $(0, 1, -1)$-cube is contained in a sphere E and $(0, -1, -1)$-cube in a sphere F, and at least one of these spheres is of type middle (without loss of generality let E be of type middle). The $(-1, 1, 1)$-cube is then contained in a sphere G and either C is of type middle or $\text{rad}(G) \geq 1$.

Now suppose that the two spheres covering only one facet of L, cover two neighboring facets of L, let this be the spheres B and C (see Figures 12–19). The spheres covering two facets of L are thus A and D. Note that among spheres B and C at most one is of type high. Without loss of generality assume B is not of type high.

The first case is when $\text{rad}(C) = 0$, then D and A are both of type middle. Now sphere D could either have the (two-dimensional) radius 0 on level 0 or the (two-dimensional) radius greater or equal 1. In the Case 3.1.1 (see Figure 12) the radius of D is 0 on level 0, thus $(0, 1, -1)$-cube is contained in a sphere E of type middle and $(1, 1, 1)$-cube in a sphere F, with $r \geq 1$.

In the Case 3.1.2 the (two-dimensional) radius of D on level 0 is greater or equal 1, the $(-1, 1, 0)$-cube is then contained in a sphere E, with $r \geq 1$. Thus the $(-1, -1, 0)$-cube is contained in a sphere F, and the $(1,0,1)$-cube in G, both of them must be of type middle (see Figure 13).

Let now $\text{rad}(C) = 1$, thus C is of type high (see Figure 14). Then again D and A are both of type middle. We have the Case 4.1 (see Figure 14), where the (two-dimensional) radius of A on level 0 is 0, and the Case 4.2.1 and 4.2.2, where the (two-dimensional) radius of A on level 0 is greater or equal 1 (see Figures 15 and 16). Suppose that $(1,1,0)$-cube is contained in a sphere E. Since L is of type low and C is of type high, we infer E is of type middle. The exact size of D is in the Case 4.1 unimportant and in the Case 4.2 the sphere E shapes the sphere D. Note that in the case 4.3 the sphere F containing the $(1,-1,0)$-cube is of type middle.

Suppose $\text{rad}(C) \geq 2$ (see Figures 17, 18 and 19). In the Case 5.1 both A and D have the (two-dimensional) radius 0 on level 0. Thus $(1,1,0)$-cube is contained in a sphere E of type high. Without loss of generality assume E is centered on $(1,1,1)$ (if E is centered on $(1,1,-1)$ we have a symmetric situation). Then $(2,2,0)$-cube is contained in a sphere F, with $r \geq 1$.

In the Case 5.2 one of the spheres A and D have the (two-dimensional) radius greater then 0 on level 0, say the sphere A. It is easy to see that then the $(1,-1,0)$-cube is contained in a sphere with radius greater then 0, thus $\text{rad}(E) = 1$. Then $(1,1,0)$-cube is contained in a sphere F of type middle (see Figure 18).

The last remaining case is the Case 5.3, depicted on Figure 19.

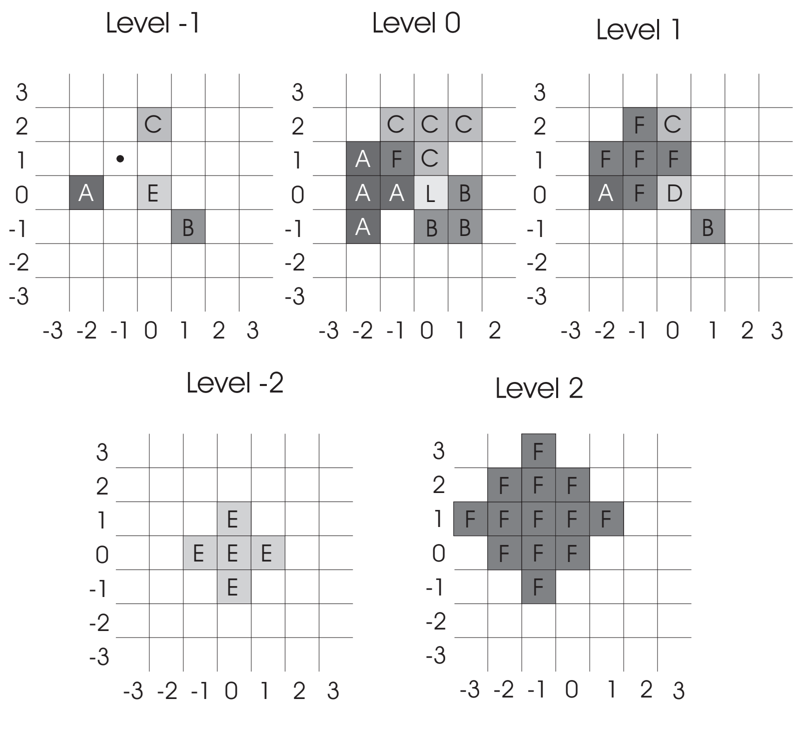

Figure 7: Case 3

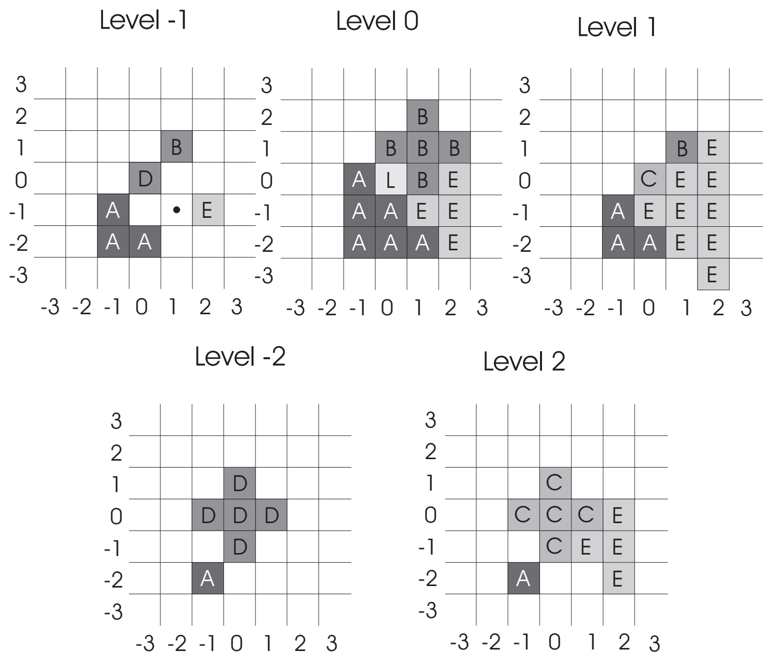

Figure 8: Case 1

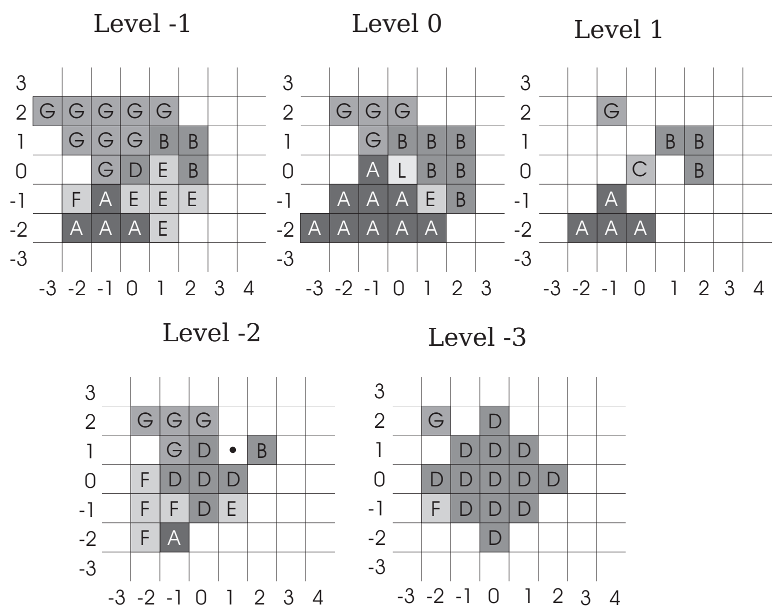

Figure 9: Case 2.1.1

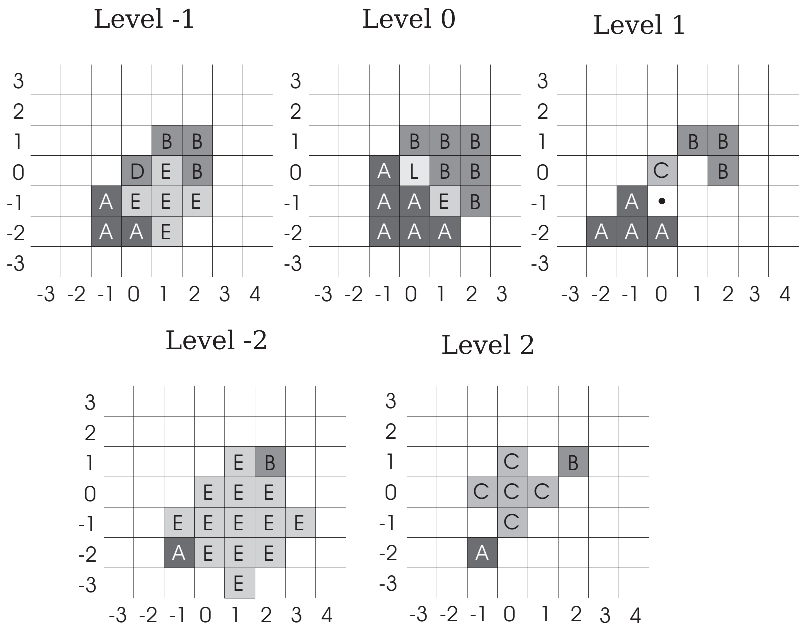

Figure 10: Case 2.1.2

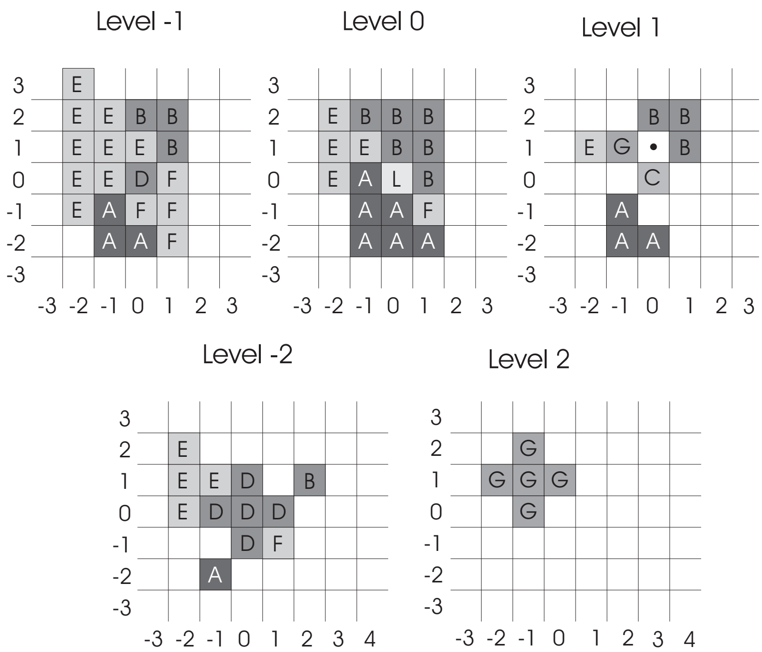

Figure 11: Case 2.2

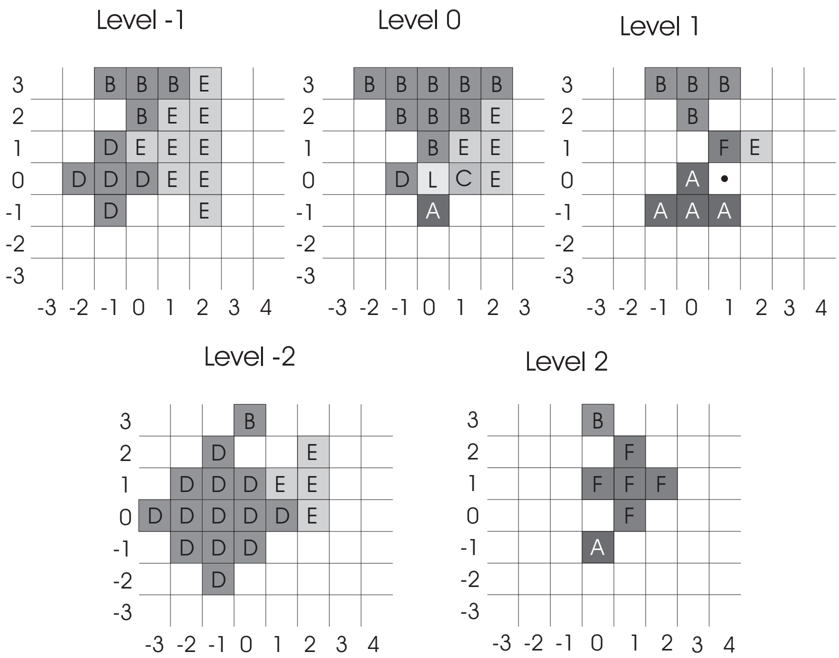

Figure 12: Case 3.1.1

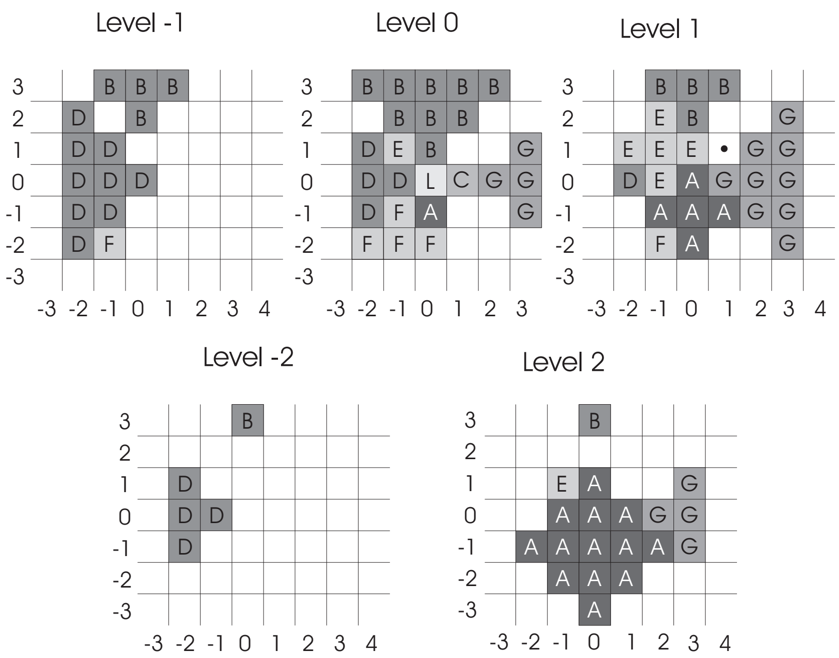

Figure 13: Case 3.1.2

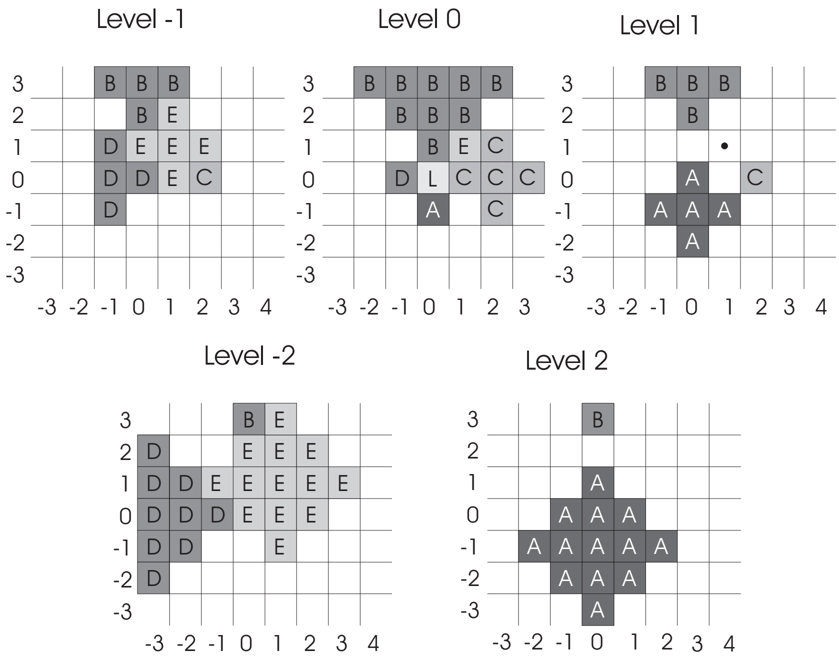

Figure 14: Case 4.1

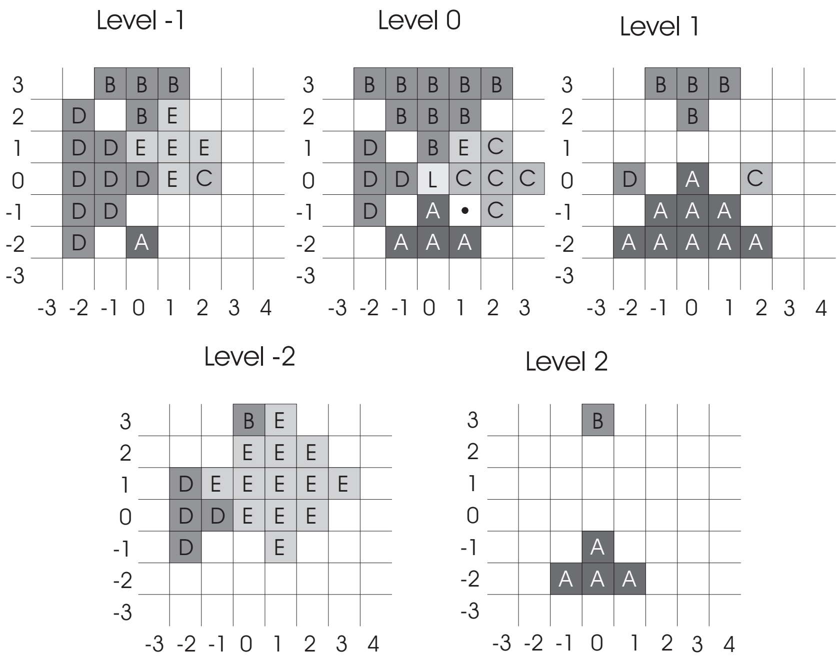

Figure 15: Case 4.2.1

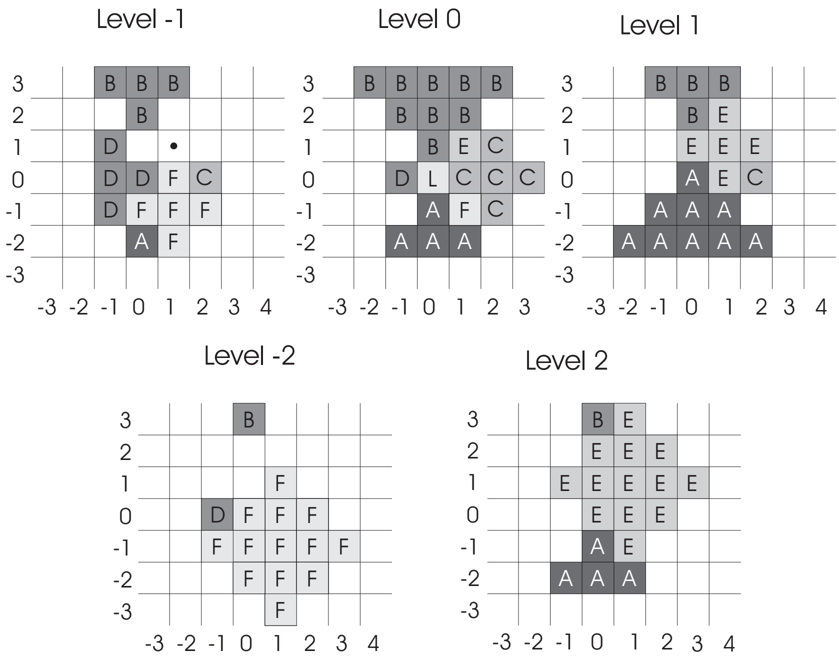

Figure 16: Case 4.2.2

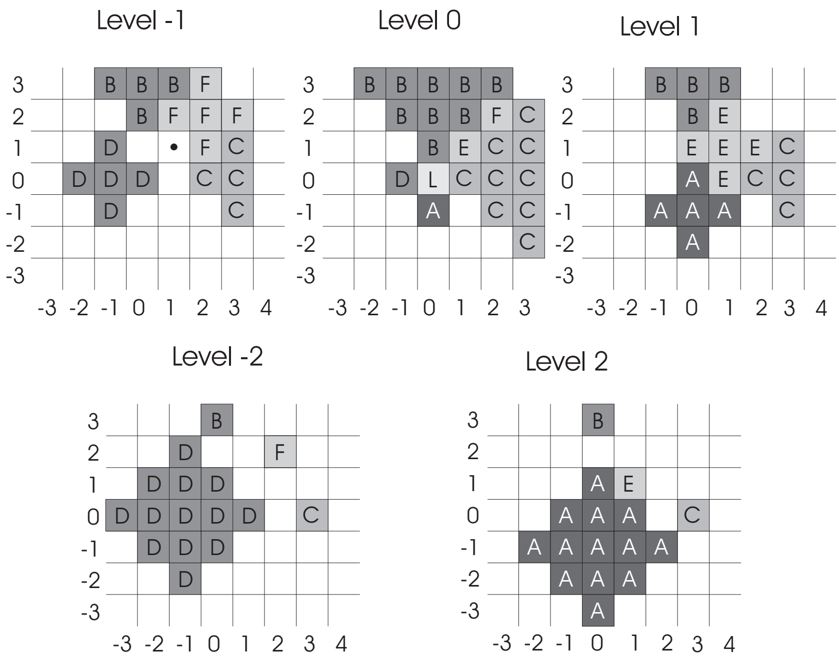

Figure 17: Case 5.1

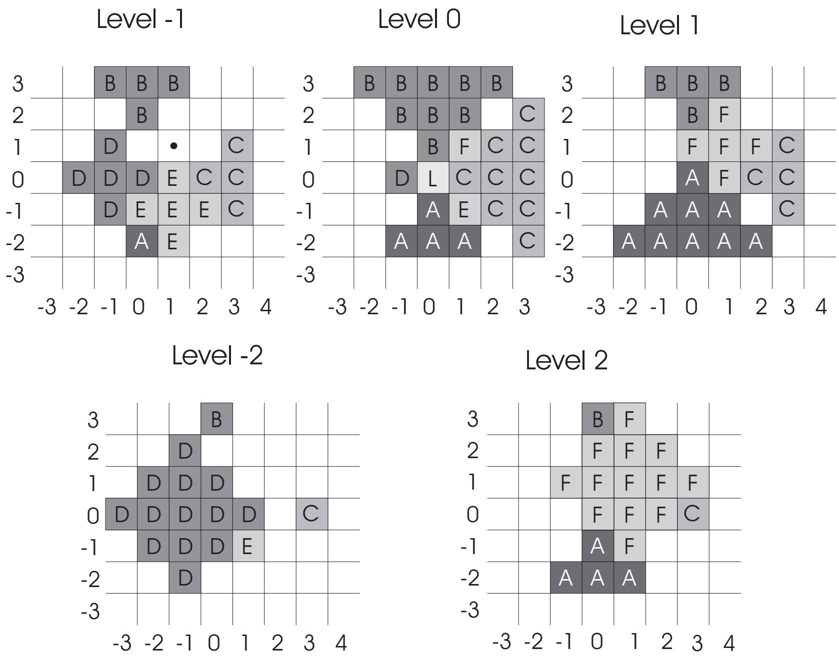

Figure 18: Case 5.2

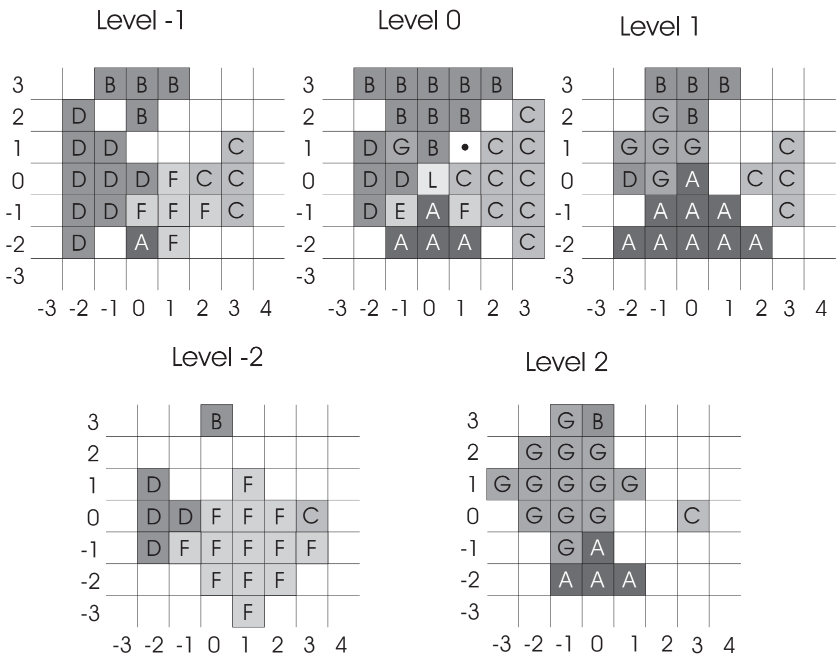

Figure 19: Case 5.3

3.4 Case 6=2+2+2

The given situation is depicted on Figure 20.

There are three possibilities of how the sphere B is positioned (see Figures 21, 22 and 23 and notice the positon of B on level 0).

Suppose that the sphere B is positioned as shown on Figure 21. Then we have few subcases. In the Case 1.1 rad(B) = 1 and the (two-dimensional) radius of A and C on level 0 is 0, this is more precisely depicted on Figure 24. Thus the $(0,-1,1)$-cube is contained in a sphere D and the $(1,0,-1)$-cube is contained in a sphere E (see Figure 25).

In the Case 1.2.1 the (two-dimensional) radius of the sphere C on level 0 is greater or equal 1, so the $(0,-1,1)$-cube is contained in a sphere D of type middle and $(-1,1,0)$-cube is contained in a sphere E. Thus we have either rad(E) ≥ 1 (see Figure 26) or rad(E) $= 0$ (see Figures 27 and 28).

In the Case 1.2.2.1 the $(0,-1,1)$-cube is contained in a sphere D, the $(-1,1,0)$-cube is contained in E, the $(-2,1,0)$ is contained in F and the $(-2,0,0)$-cube in G (see Figure 27). In the Case 1.2.2.2 only the position of G is slightly different from the Case 1.2.2.1 (see Figure 28).

In the Case 1.3.1 the two-dimensional radius of A and C is greater or equal 1. Hence the $(1,1,0)$-cube is contained in a sphere E and the $(2,1,0)$-cube is contained in a sphere F. Note also that the sphere D is of type high (see Figure 29).

In the Case 1.3.2 the $(1,1,0)$ cube is contained in a sphere E, the $(1,0,-1)$-cube is contained in a sphere F, the $(2,0,-1)$-cube is contained in a sphere G and $(-1,0,-1)$-cube in H (see Figure 30).

Suppose now that the sphere B is positioned as shown on Figure 22. Suppose that the two-dimensional radius of A and C on level 0 is 0. (see Case 2.1.1, Figure 31). Since the $(0,1,-1)$-cube is contained in a sphere D of type high, we infer that the $(0,-1,1)$-cube is contained in a sphere E, the $(1,0,1)$-cube is contained in a sphere F and the $(-1,-1,0)$-cube in G. The position of G is either that of the Case 2.1.1 or of the Case 2.1.2 (note that G is in the Case 2.1.2 of type high or low, thus we have a contradiction, since D is of type high and L is of type low).

If the two-dimensional radius of A on level 0 is greater then 0 (and the two-dimensional radius of C on level 0 is 0), then the following two cases occur (see Case 2.2.1 and 2.2.2), in both of them the $(1,0,-1)$-cube is contained in a sphere D of type high. In the Case 2.2.1 the $(1,1,0)$-cube is contained in a sphere E (see Figure 33).

In the case 2.2.2 the $(1,1,0)$-cube is contained in a sphere E (notice the difference in the position of E between the Case 2.2.1 and 2.2.2), the $(0,-1,1)$-cube is contained in a sphere F and the $(-1,-1,0)$-cube in a sphere G, G is of type high or low (see Figure 34).

Suppose that the two-dimensional radius of C on level 0 is greater then 0 and the two-dimensional radius of A on level 0 is 0 (see Case 2.3, Figure 35). Then the $(1,1,0)$-cube is contained in a sphere E, the $(0,-1,1)$ cube is contained in F and the $(-1,-1,0)$-cube is contained in a sphere G, G is of type high or low.

If the two-dimensional radius of A and C on level 0 is greater then 0 (see Case 2.4, Figure 36), then the $(1,1,1)$-cube is contained in a sphere D, the $(0,-1,1)$-cube is contained in a sphere E, the $(-1,1,0)$-cube in F, the $(-2,-1,1)$-cube in G, the $(-1,-1,0)$-cube in H and the $(-2,-1,0)$-cube is contained in a sphere I (I is of type high or low).

Suppose now that the sphere B is positioned as shown on Figure 23. We have three cases that can occur (note the difference of the position of A and C in the three cases).

In the Case 3.1 the $(1,0,-1)$-cube is contained in a sphere D and D is of type high or low (see Figure 37).

In the Case 3.2 the $(0,-1,1)$-cube is contained in a sphere D and the $(1,1,1)$-cube is contained in a sphere E, where D and E are of type high or low (see Figure 38).

In the Case 3.3 the $(-1,1,0)$-cube is contained in a sphere D and the $(-1,-1,0)$-cube is contained in a sphere E (see Figure 39).

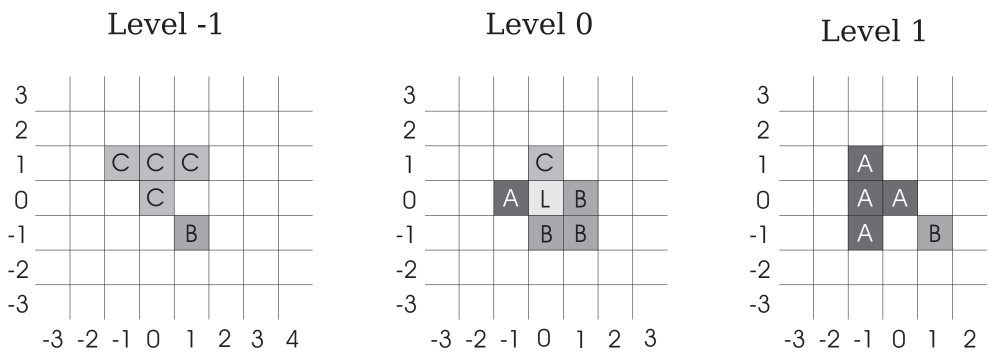

Figure 20: Given situation

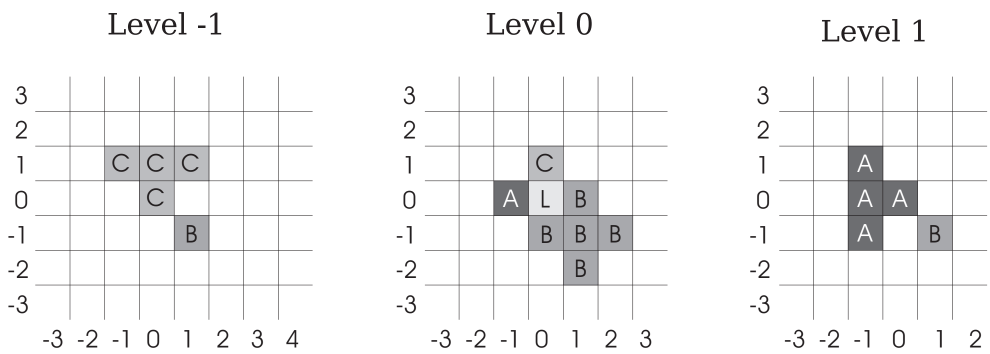

Figure 21: Case 1

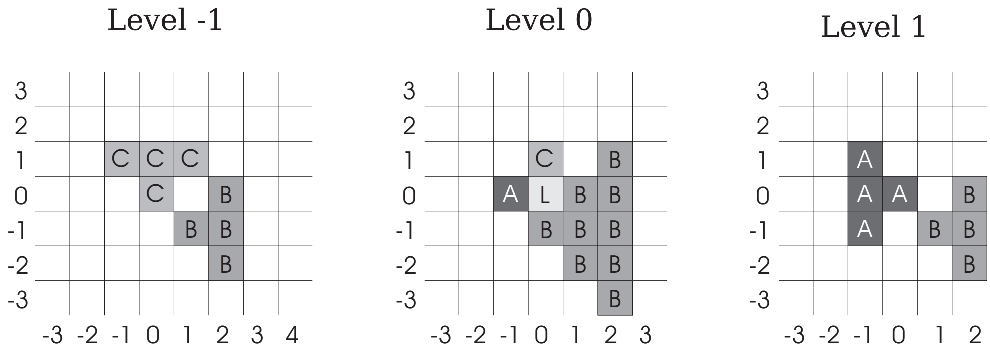

Figure 22: Case 2

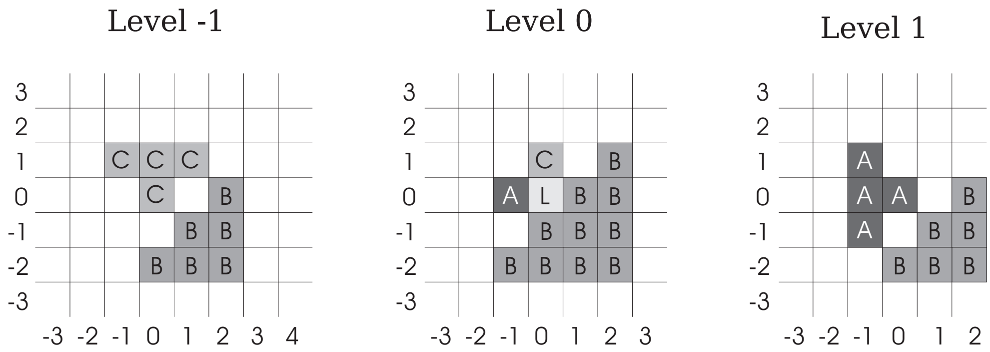

Figure 23: Case 3

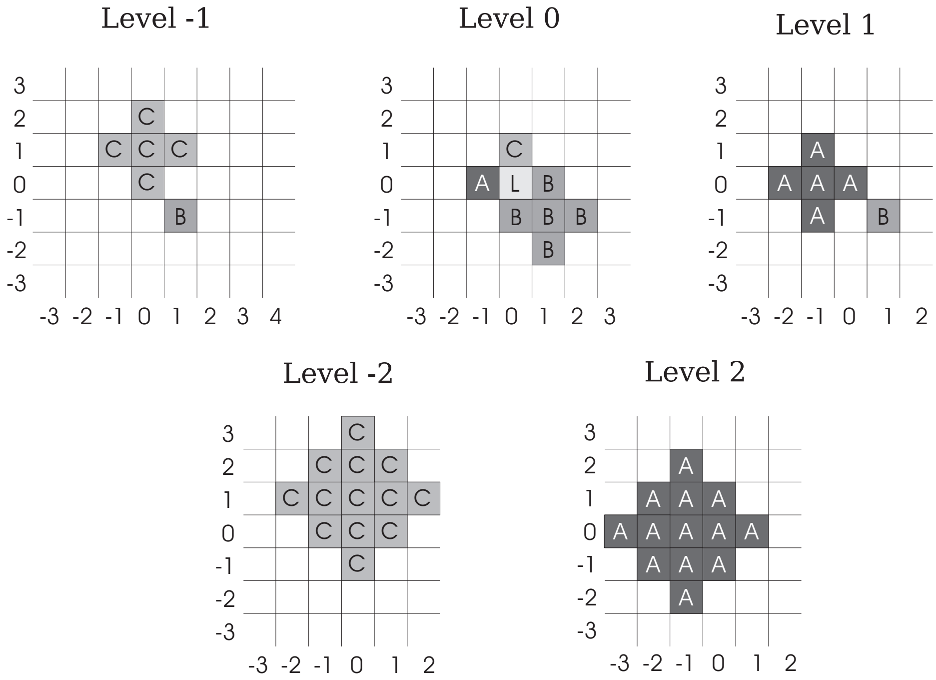

Figure 24: Case 1.1

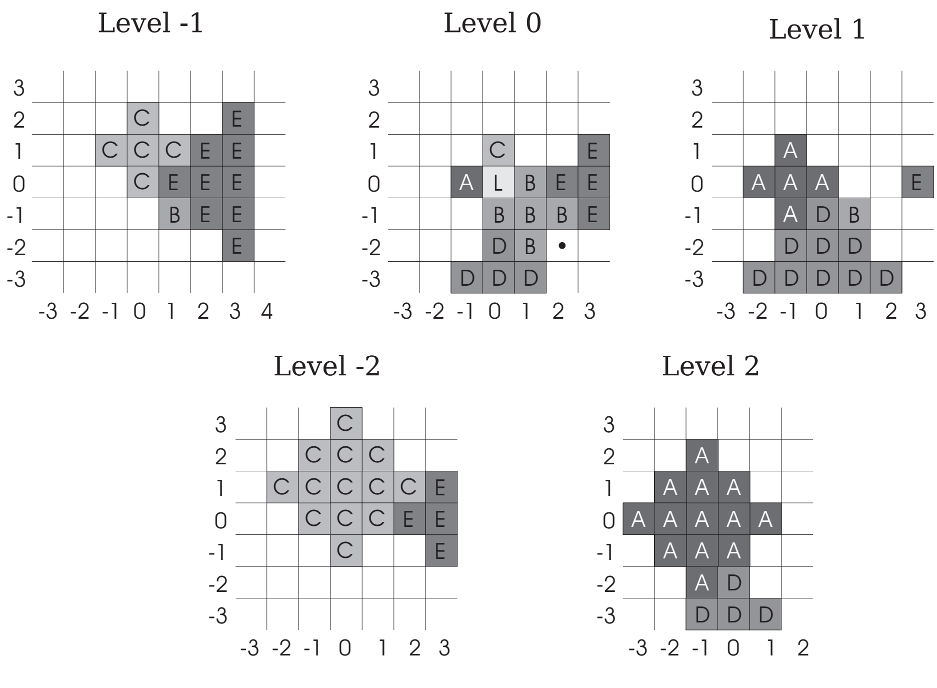

Figure 25: Case 1.1

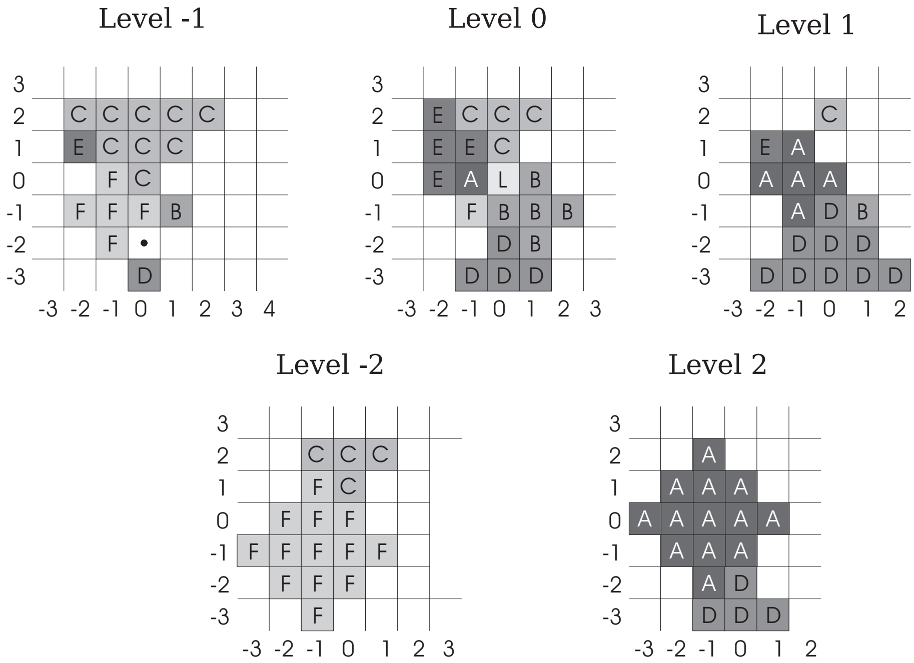

Figure 26: Case 1.2.1

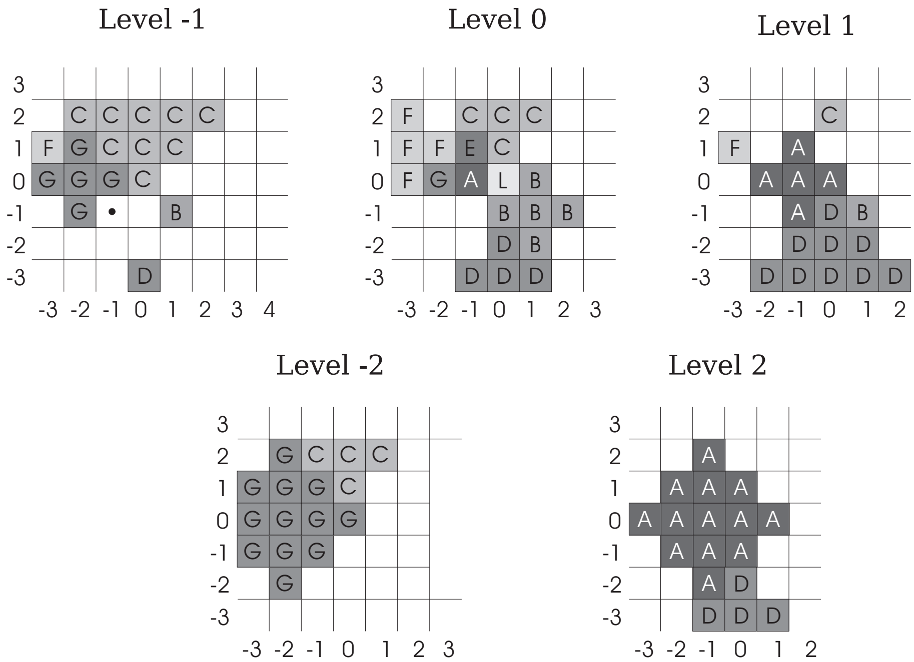

Figure 27: Case 1.2.2.1

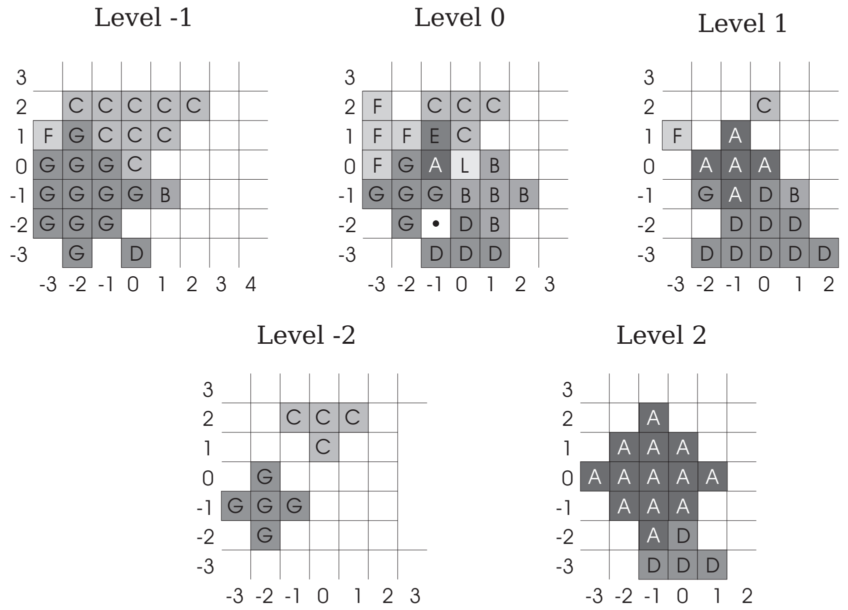

Figure 28: Case 1.2.2.2

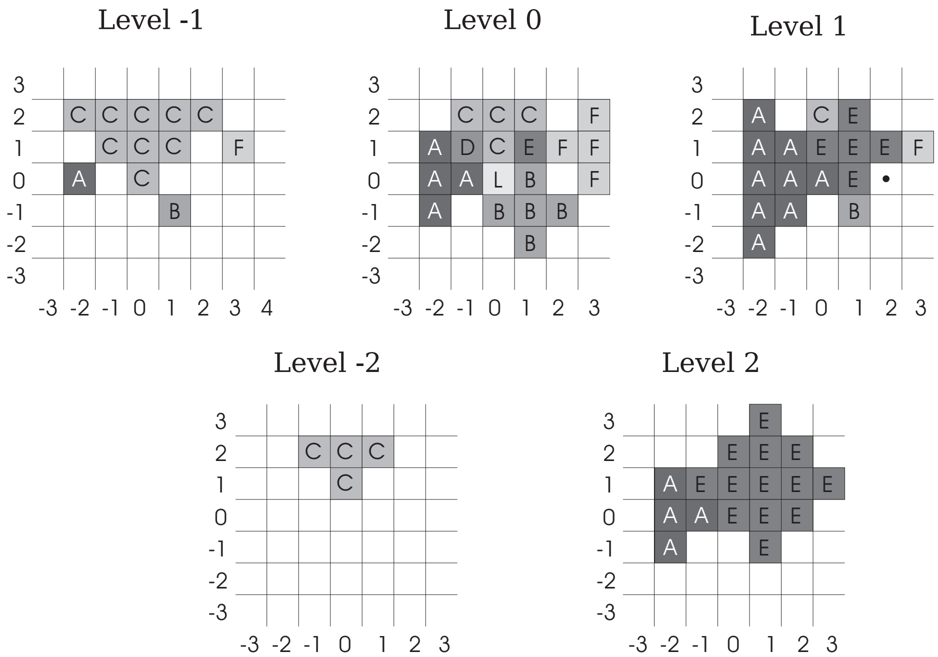

Figure 29: Case 1.3.1

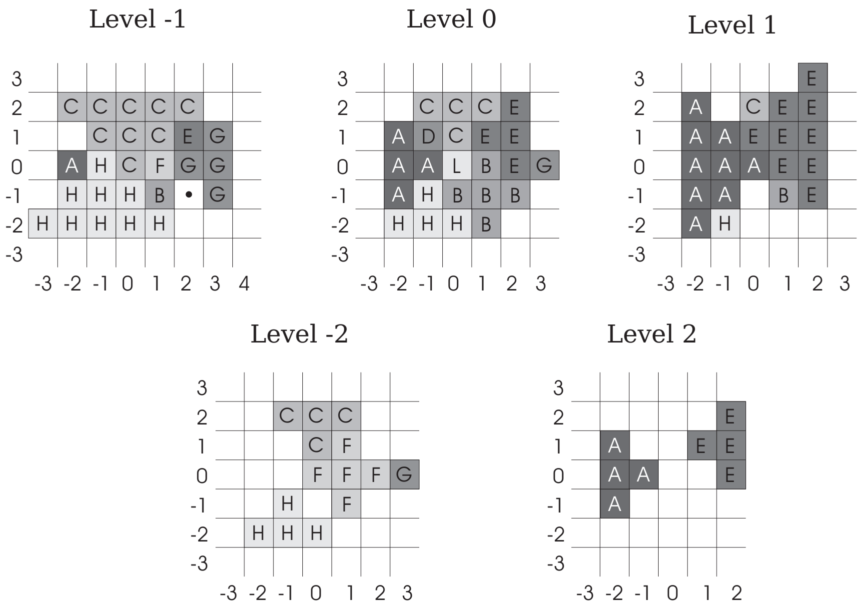

Figure 30: Case 1.3.2

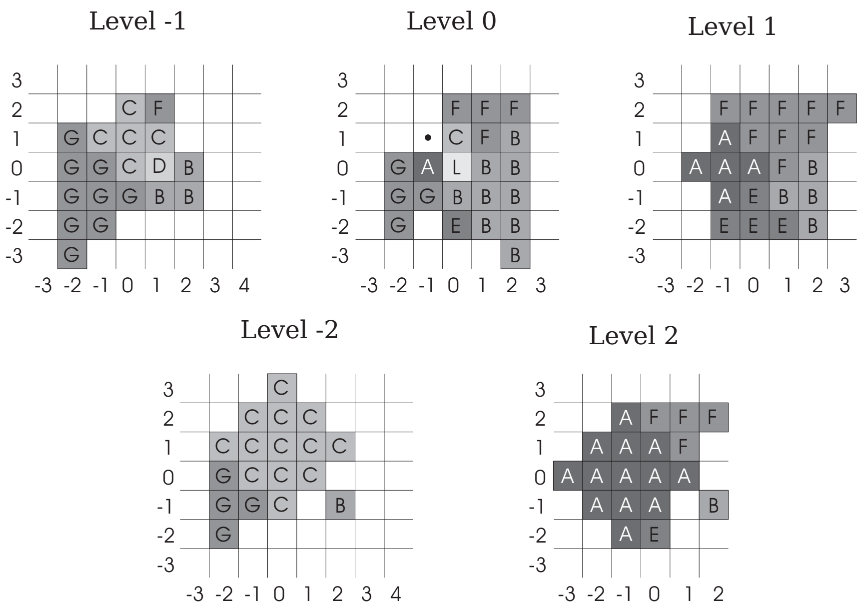

Figure 31: Case 2.1.1

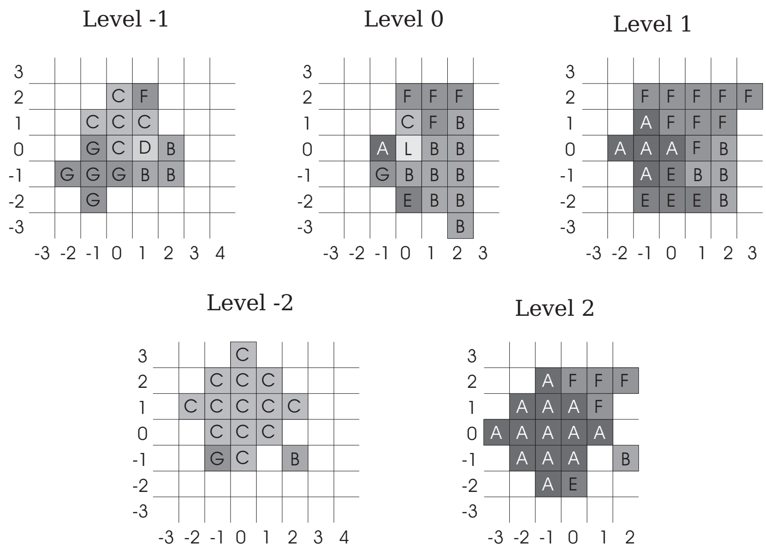

Figure 32: Case 2.1.2

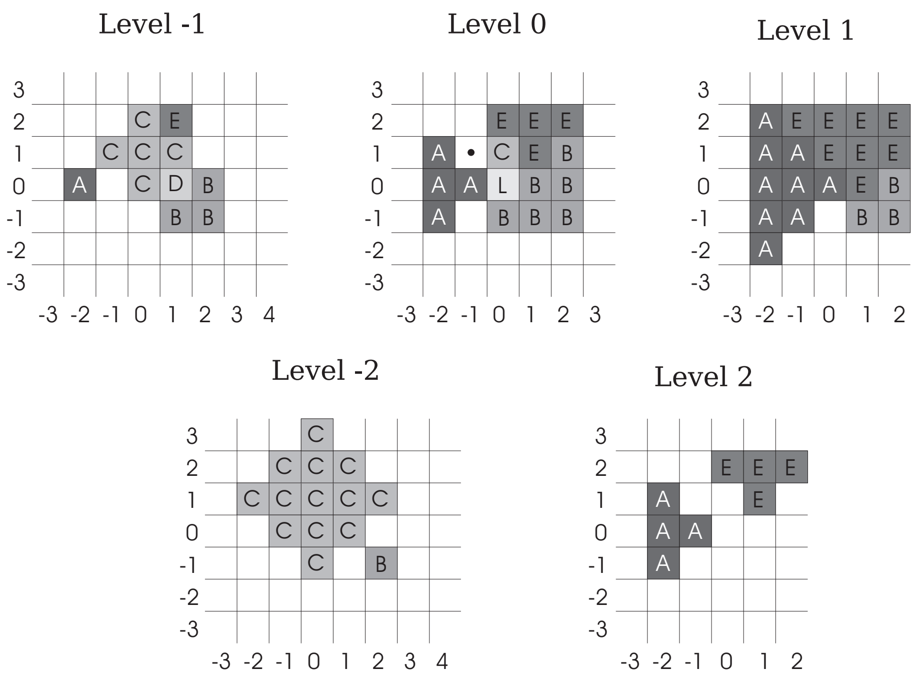

Figure 33: Case 2.2.1

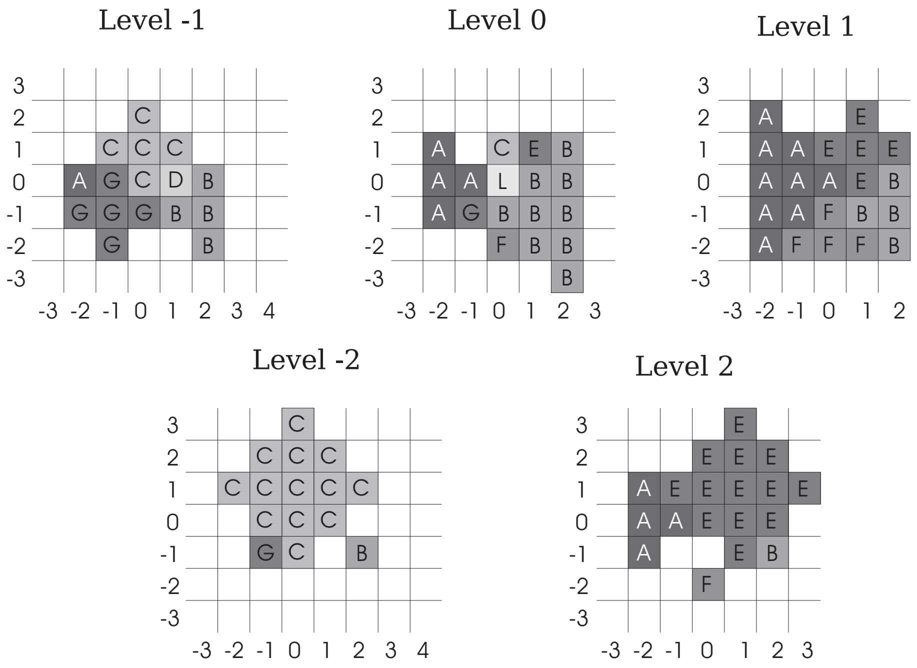

Figure 34: Case 2.2.2

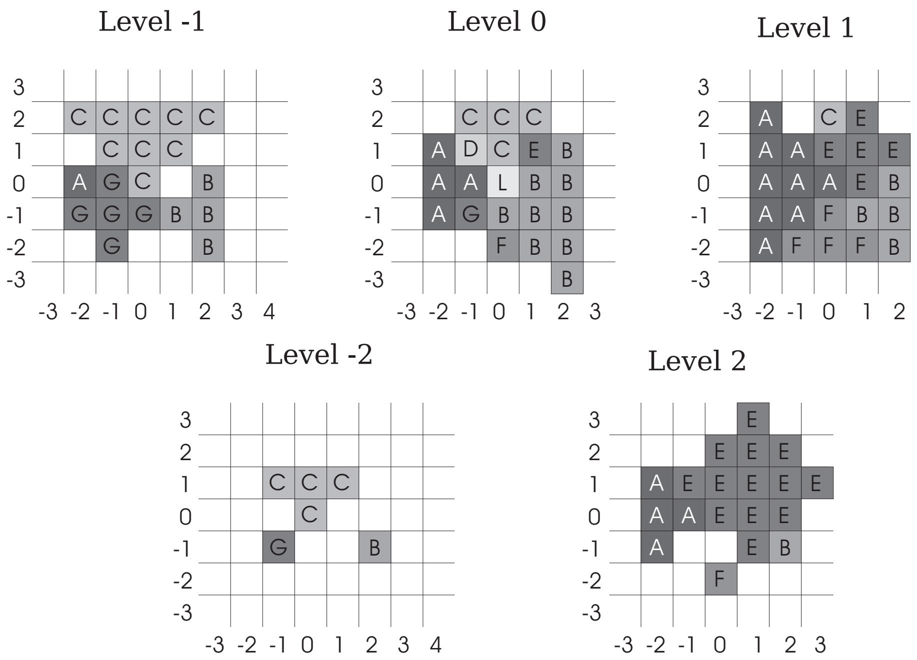

Figure 35: Case 2.3

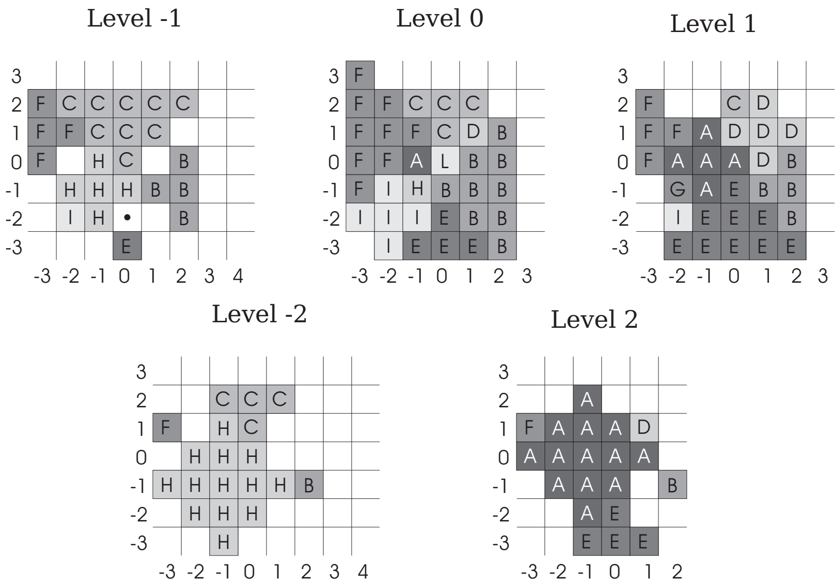

Figure 36: Case 2.4

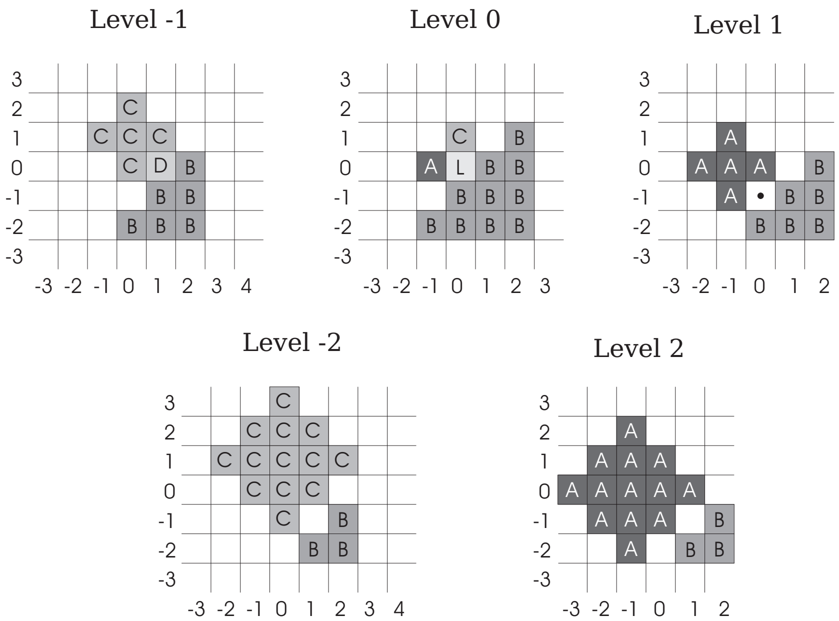

Figure 37: Case 3.1

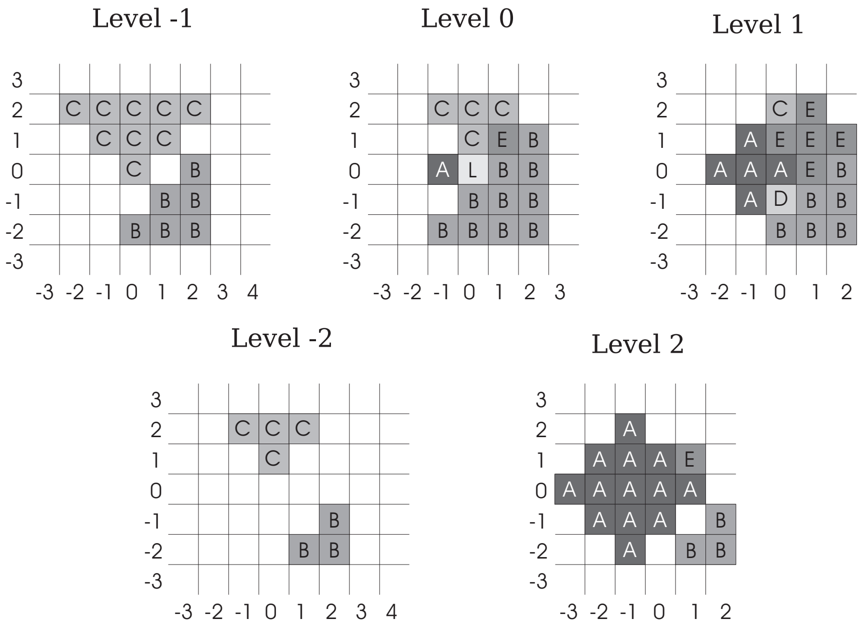

Figure 38: Case 3.2

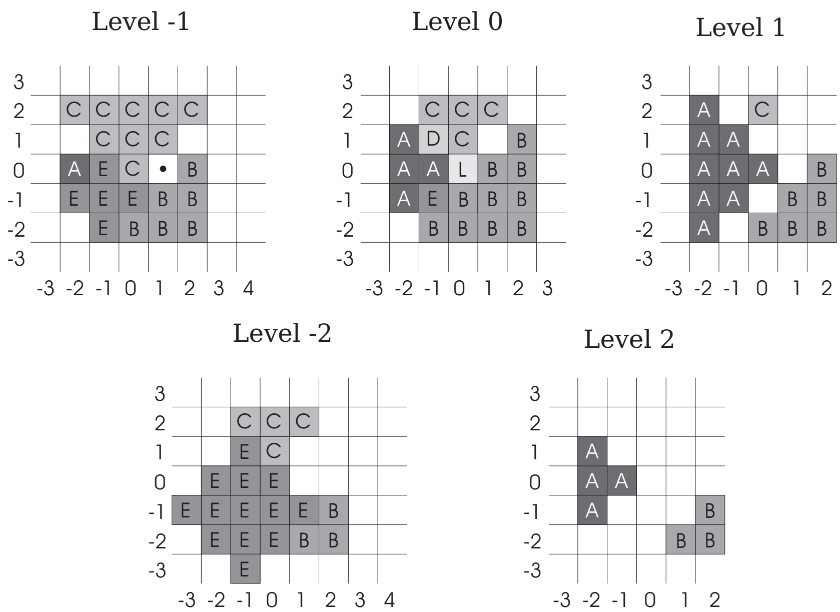

Figure 39: Case 3.3

3.5 Case 6=3+2+1

Three facets of L are covered by a sphere A , two by a sphere B and one facet by a sphere C (see Figure 40). Suppose that the radius of the sphere B is 1, thus B is of type high. Then consider the cubes centered on $(1,0,-1)$ and $(0,1,-1)$, both of them are contained in spheres of type middle, say spheres D and E respectively. Thus we can without loss of generality assume, that we have the situation depicted on Figure 40. In this case the $(1,0,1)$-cube is contained in a sphere F of type middle.

Suppose that the radius of B is greater then 1. Without loss of generality assume that B is positioned as in Figure 41. Consider the cube centered on $(0,1,-1)$, which is contained in a sphere D and the radius of D is either 0,1 or greater then 1.

If $\text{rad}(D) = 1$ (see Figure 42), then the $(1,0,-1)$-cube is contained in a sphere E, which is of type middle, since D is of type high and L is of type low. Thus the $(1,0,-2)$-cube is contained in a sphere F, with $r \geq 1$ (if $r = 1$, then F is of type low), the $(1,-1,-2)$-cube is containd in a sphere G, with $r \geq 1$ and thus the $(2,-1,-2)$-cube is contained in a sphere H of type high, with $r = 0$. Thus the $(0,-1,-3)$-cube is contained in a sphere I with $r \geq 1$, and so the radius of F is equal to 1, hence F is of type low.

Suppose $\text{rad}(D) > 1$ (see Figures 43, 44 and 45) then the cube centered on $(1,0,-1)$ is contained in a sphere E, and the radius of E is either 0 or greater then 0. Suppose $\text{rad}(E) > 0$, then the $(1,0,-2)$-cube is contained in a sphere F of type high, with $r = 0$. Since L is of type low and F is of type high, E is of type middle, thus $\text{rad}(E) > 1$ and $\text{rad}(C) \geq 1$. The $(1,0,1)$-cube is thus contained in a sphere G, with $r \geq 1$ (see Figure 43). If $\text{rad}(E) = 0$ then the cube centered on $(2,1,-1)$ is either contained in the sphere B (see Figure 44) or in sphere F (see Figure 45). In the second case $\text{rad}(F) \geq 1$.

The last case is when $\text{rad}(D) = 0$ (see Figure 46). In this case $\text{rad}(C) \geq 1$ and the $(0,1,-2)$-cube is contained in a sphere E, with $r \geq 1$. Thus the $(0,0,-2)$-cube is contained in a sphere F of type middle, since L is of type low and D is of type high. Thus the $(2,1,0)$-cube is contained in a sphere G, with $r \geq 1$.

3.6 Case 6=3+1+1+1

Suppose that three facets of L are covered by a sphere A one by a sphere B, one by C and one facet by a sphere D (see Figure 47). Among the spheres B and C at most one is of type high, say sphere B.

Let $\text{rad}(B) = 0$, then C and D are both of type middle. Consider the cube centered on $(1,0,1)$. This cube is contained in a sphere E, and E is of type middle. The position of E could be either as in the Case 1.1 (see Figure 48) or in the Case 1.2 (see Figure 49). In the Case 1.1 the $(1,1,-1)$-cube is contained in a sphere F, and F is of type middle. In the Case 1.2 the $(1,1,1)$-cube is contained in a sphere F, with $r \geq 1$ and the $(1,1,0)$-cube is contained in a sphere G, and G is of type middle. We than have the $(0,1,1)$-cube contained in a sphere H and the $(0,2,0)$-cube in a sphere I, also H is of type middle and $\text{rad}(I) \geq 1$.

If $\text{rad}(B) \geq 1$ then consider the cube centered on $(1,1,0)$. Let the $(1,1,0)$-cube be contained in a sphere E and suppose $\mathrm{rad}(E) = 0$ or E is centered in $(1, 1, \alpha)$, where $\alpha \geq 1$ (see Figure 50). Then the $(1, 1, -1)$-cube is contained in a sphere F and the $(0, 1, -1)$-cube in a sphere G and one of this two spheres is of type middle, say sphere F. Thus the other sphere, in this case the sphere G, is of type high, with $r = 0$. Suppose the center of the sphere E is positioned on $(1, 1, -\alpha)$, where $\alpha \geq 1$ (see Figure 51 and 52). Then consider the cubes centered on $(0, 0, 1)$ and $(1, 1, 1)$. Exactly one of them is contained in a sphere of type high, with $r = 0$, and the other is contained in a sphere with $r \geq 1$.

Thus in the Case 2.2 we have the $(1, 1, 1)$ -cube contained in a sphere F of type high, with $r = 0$ and therefore the $(0, 0, 1)$-cube is contained in a sphere D of type middle. Note also that the spheres D and E are both of type middle, since L is of type low and F is of type high.

In the Case 2.3 the $(0, 0, 1)$-cube is contained in a sphere D of type high, with $r = 0$ and the $(1, 1, 1)$-cube in a sphere F with $r \geq 1$. Thus the $(0, 1, 1)$-cube is contained in a sphere G (G is of type middle) and the $(1, 0, 1)$-cube in a sphere H (H is also of type middle). Since D is of type high and L is of type low, the $(0, 0, 2)$-cube is contained in a sphere I of type middle.

3.7 Case 6=3+3

Consider the case, when three facets of L are covered by one sphere, say sphere A, and the other three facets of L by a sphere B. Thus we have the situation depicted on Figure 53. Note that the cube centered on $(1, 1, 0)$ is not covered by any of the spheres A and B. There are few possible subcases which can occur.

Case 1: The cube centered on $(1, 1, 0)$ is contained in a sphere C of type middle (with $r \geq 2$). There are three possibilities of how the center of C is positioned, but the Cases 1.1 and 1.3 are symmetric, so we will consider only the case 1.1 (see Figure 54). Note that the third coordinate of the center of C must be 0.

The cube centered on $(0, 1, 1)$ is either contained in a sphere with $r = 0$ or $r \geq 1$. Suppose the $(0, 1, 1)$-cube is contained in a sphere D, with $r = 0$ (see Case 1.1.1, Figure 55), then the $(0, 2, 1)$-cube is contained in a sphere E (by Lemma 2.2 $\mathrm{rad}(E) \geq 1$), the $(-1, 2, 1)$-cube is contained in a sphere F, with $r \geq 1$ and the $(-1, 0, 1)$-cube in G. Since L is of type low and D is of type high G is of type middle. Thus $\mathrm{rad}(E) = 1$ and E is of type low, therefore F is of type middle.

In the Case 1.1.2 the $(0, 1, 1)$-cube is contained in a sphere D (with $r \geq 1$), the $(0, 2, 1)$-cube is then contained in E (with $r = 0$), the $(-1, 2, 1)$-cube in F, and by Lemma 2.2 $\mathrm{rad}(F) \geq 1$ (see Figure 56).

Consider the Case 1.2 (see Figure 54). In this case C is centered on $(\alpha, \beta, 0)$, where $\alpha, \beta \geq 2$. Then consider the cube centered on $(-1, -1, 0)$. If this cube is covered by a Lee sphere D of type high, then we have a symmetric case to *Case 2* or *Case 3* below (see Figure 57). If the $(-1, -1, 0)$-cube is contained in a Lee sphere D with center in $(-1, -\gamma, 0)$ or $(-\gamma, -1, 0)$, $\gamma \geq 3$, then we have a symmetric situation as in the Case 1.1.1 or 1.1.2 above (see Figure 57). Thus the only case left is when the center of D is in $(-\delta, -\varrho, 0)$, where $\delta, \varrho \geq 2$ (see Figure 58). Thus the $(1, 1, 0)$-cube is contained in C, the

$(-1,-1,0)$-cube is contained in D (C and D are both of type middle). Among the $(0,1,1)$-cube and $(-1,0,1)$-cube exactly one is contained in a sphere with $r = 0$, and the other is contained in a sphere of type middle. Without loss of generality assume the $(0,1,1)$-cube is contained in a sphere E (of type middle). Then the $(-1,0,1)$-cube is contained in a Lee sphere F (of type high), with $r = 0$. The cube centered on $(0,2,1)$ is then contained in a Lee sphere G, with $r \geq 1$ and the $(-1,2,1)$-cube in a sphere H, with $r \geq 1$, as shown on Figure 58. Since F is of type high the cube centered on $(-1,2,2)$ is contained in a sphere I of type low, with $r = 0$. Therefore H is of type middle and $\text{rad}(H) \geq 2$ which is a contradiction, since then the $(-2,0,1)$-cube is contained in a sphere with $r = 0$ (this is impossible, since F is of type high and L is of type low).

Case 2: The $(1,1,0)$-cube is contained in a sphere C (of type high), with $r = 1$ centered on $(1,2,0)$. In this case the $(0,1,1)$-cube is covered by a sphere D of type middle, and so the $(0,2,1)$-cube is covered by a Lee sphere E (with $r \geq 1$). Thus the $(-1,2,1)$-cube is covered by a sphere F, with $r \geq 1$, as shown on Figure 59. The $(-1,2,0)$-cube is covered by a sphere of radius 0. Since C is of type high and L is of type low, this is by Lemma 2.2 a contradiction.

Case 3: The $(1,1,0)$-cube is covered by a sphere C, with $r = 0$. Then consider the cube centered on $(-1,-1,0)$. If this cube is covered by a sphere D of radius 0, then we have a contradiction, since the distance between $(-1,-1,0)$ and $(1,1,0)$ is 4 (see Figure 60), and if the $(-1,-1,0)$-cube is not covered by a sphere of radius 0, then we have a case symmetric to *Case 1* or to *Case 2* above, except of in the case when C is positioned as in the Case 1.2 (see Figure 61).

In this case the $(0,1,1)$-cube is contained in a sphere E, and $\text{rad}(E) \geq 1$. Thus the $(-1,0,1)$-cube is contained in sphere F of type middle and the $(-1,2,1)$-cube in a sphere G, with $r \geq 1$. Also the $(-1,0,2)$-cube is covered by a sphere H, with $r \geq 1$, hence $\text{rad}(E) = 1$ and E is of type high.

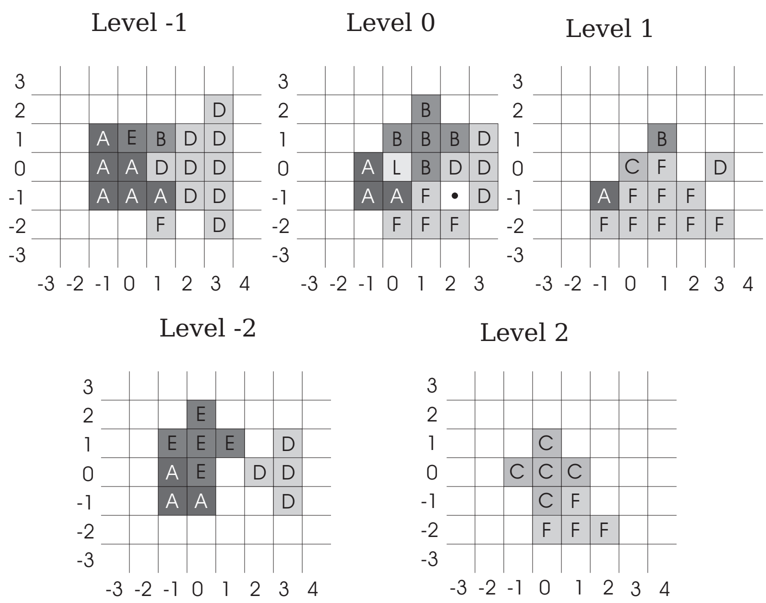

Figure 40: Case 1

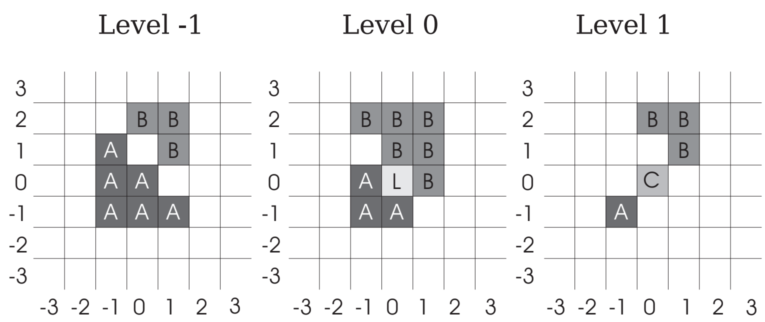

Figure 41: The radius of B is greater then 1

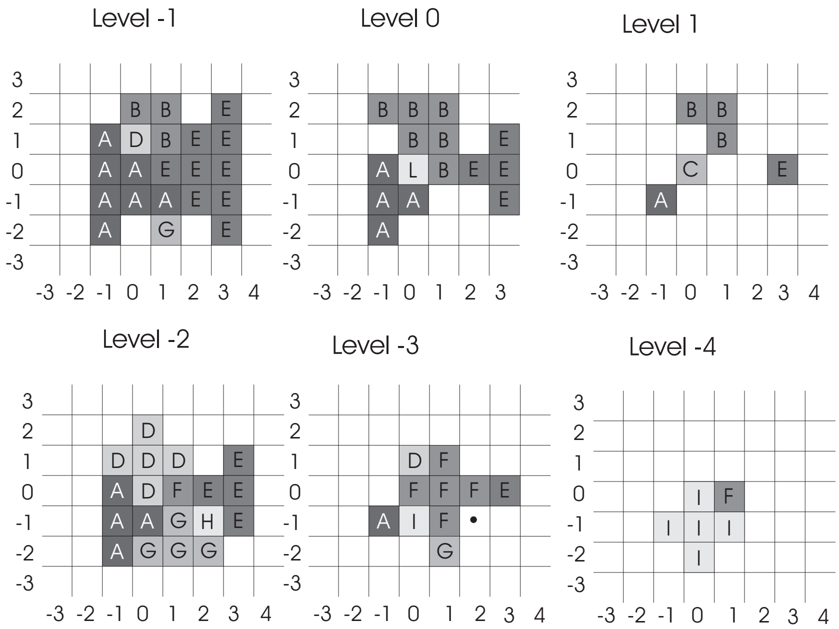

Figure 42: Case 2

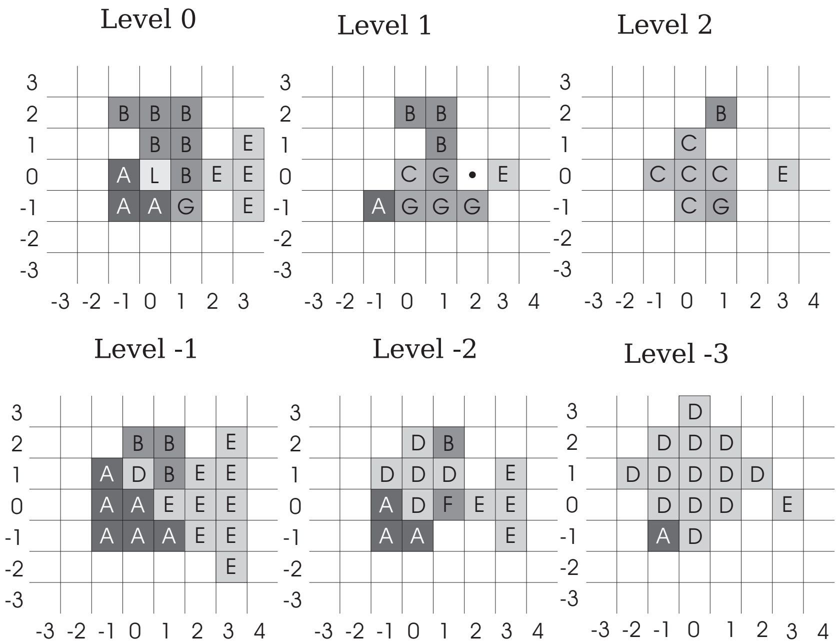

Figure 43: Case 3

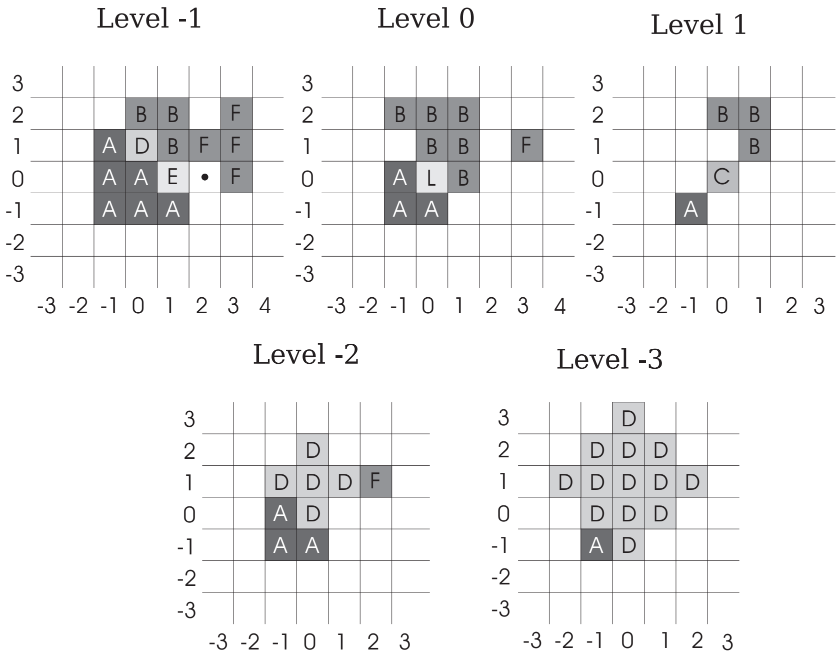

Figure 44: Case 4

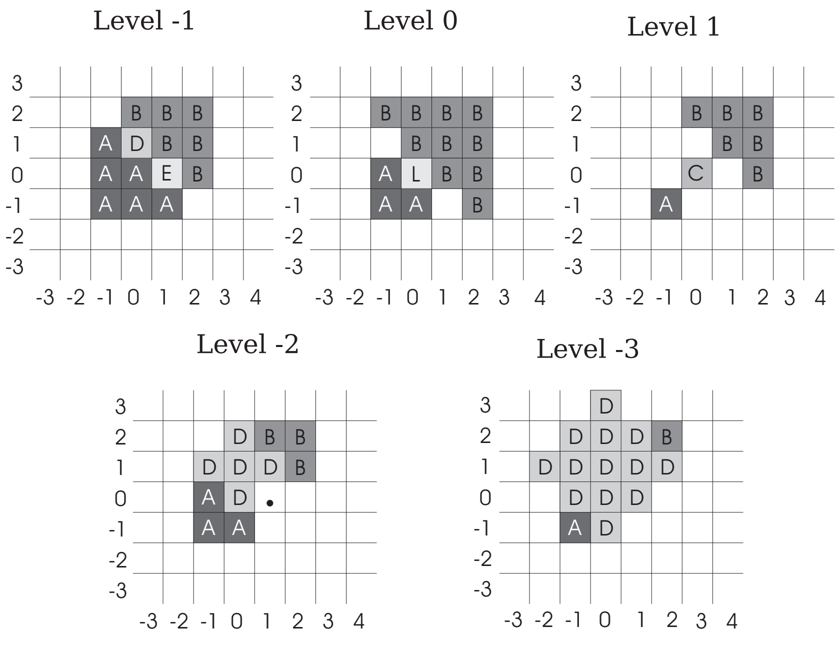

Figure 45: Case 5

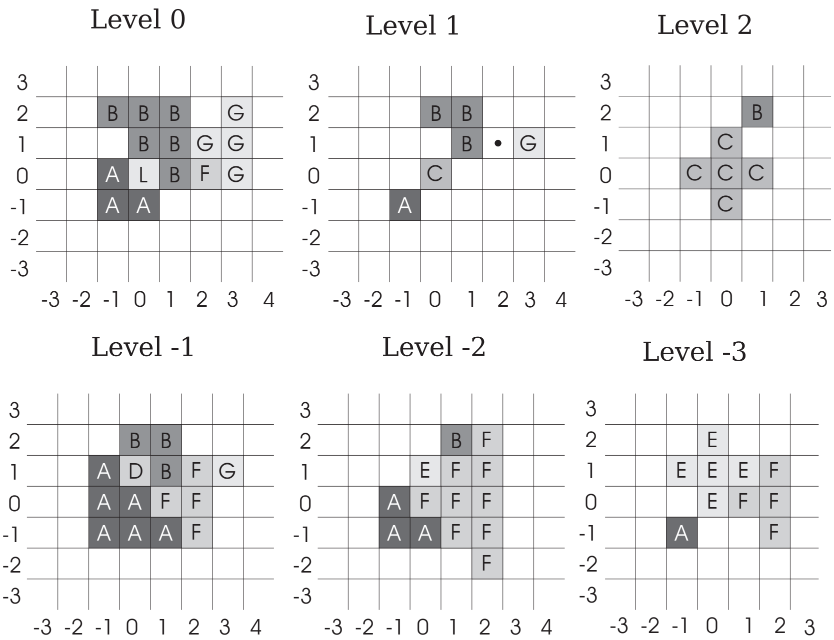

Figure 46: Case 6

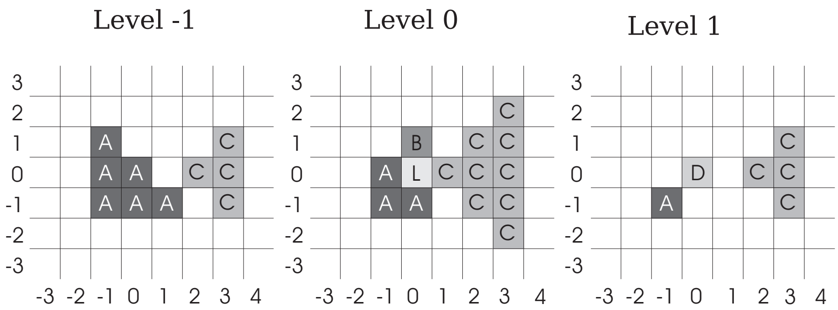

Figure 47: Beggining situation

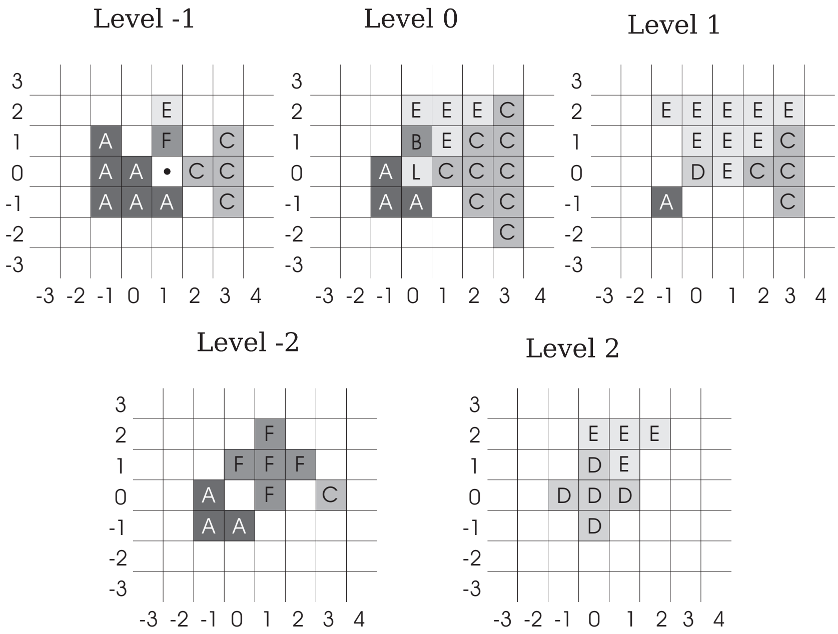

Figure 48: Case 1.1

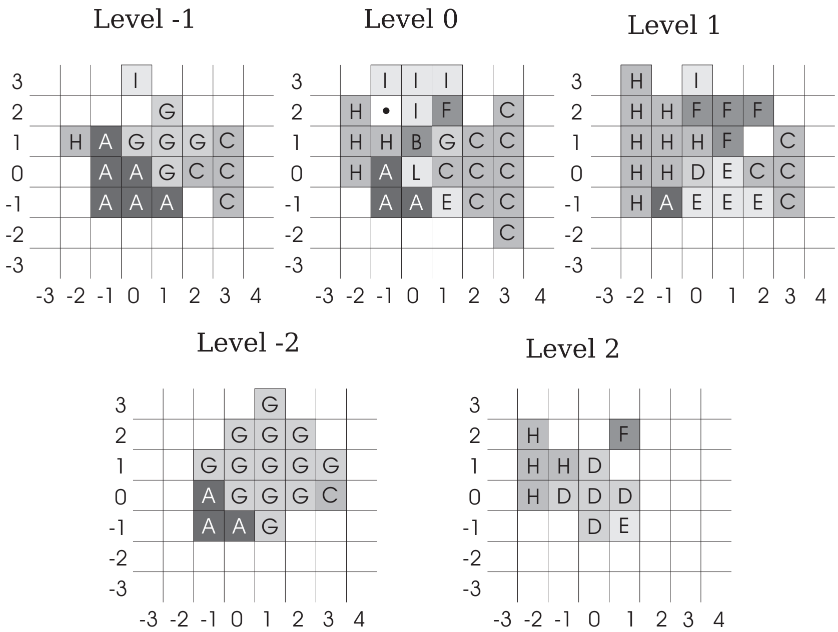

Figure 49: Case 1.2

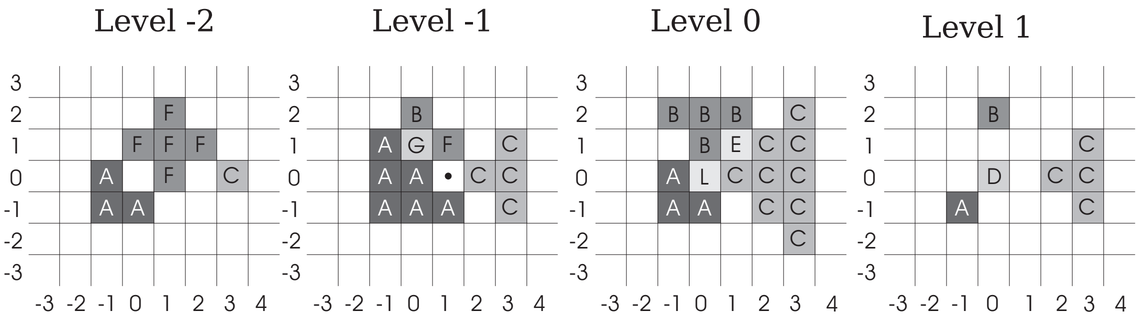

Figure 50: Case 2.1

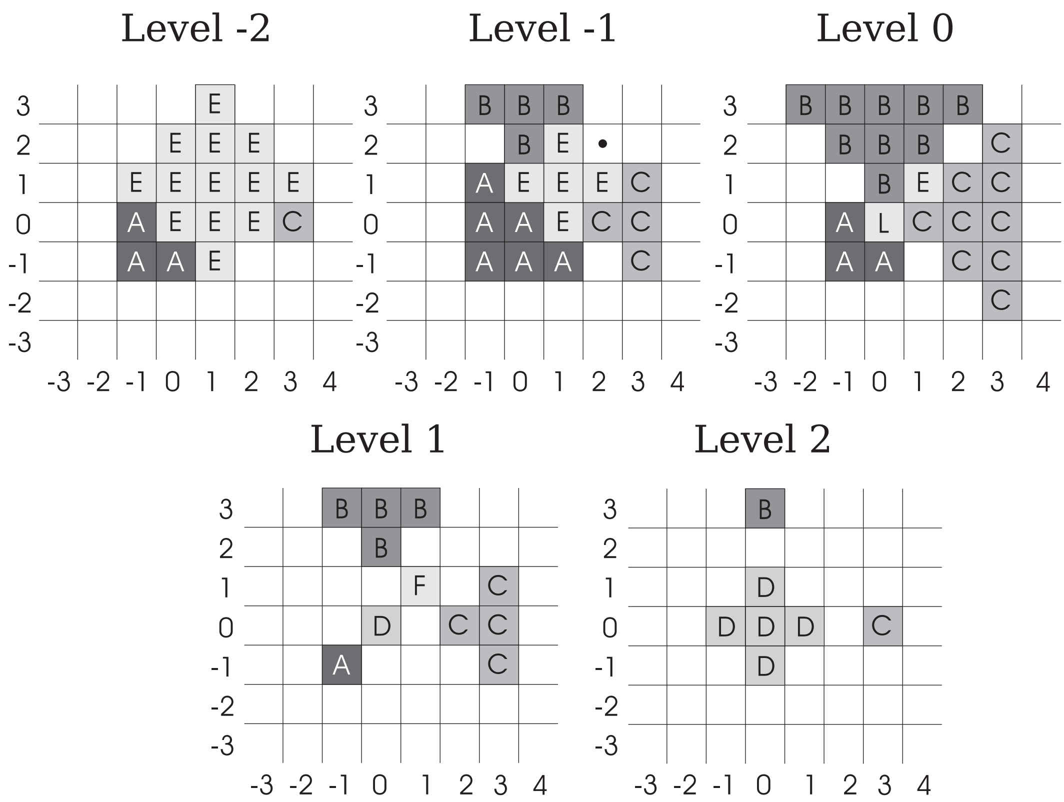

Figure 51: Case 2.2

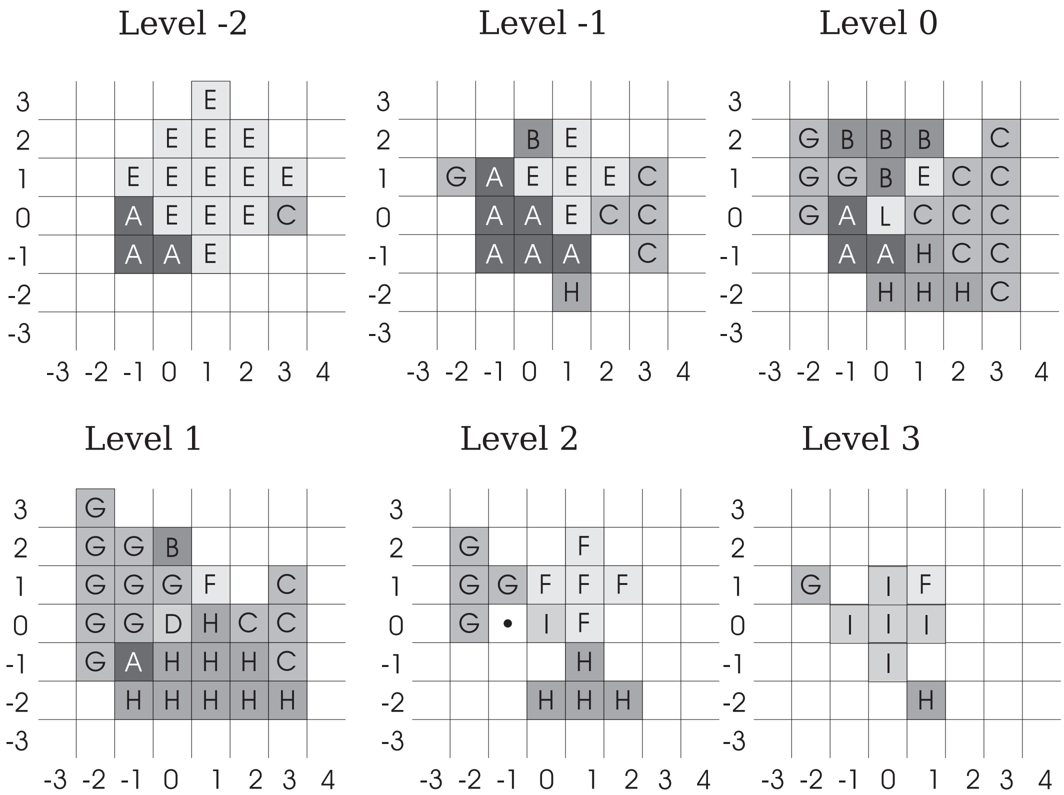

Figure 52: Case 2.3

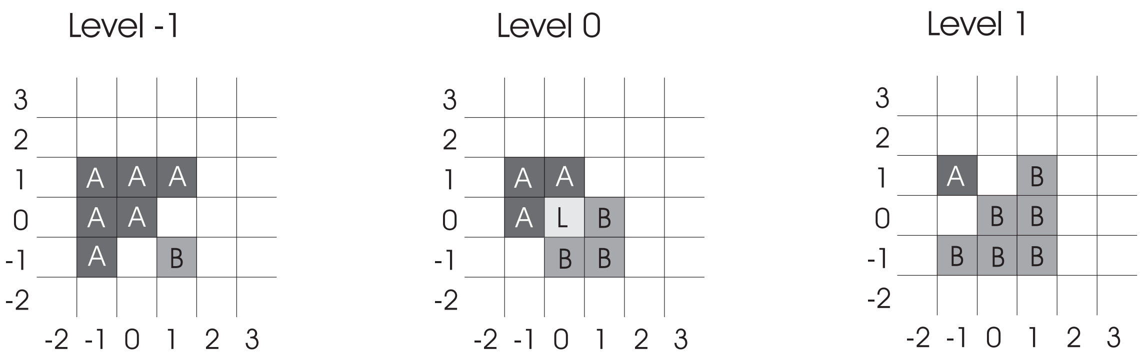

Figure 53: Three faces of L are covered by A and three by B.

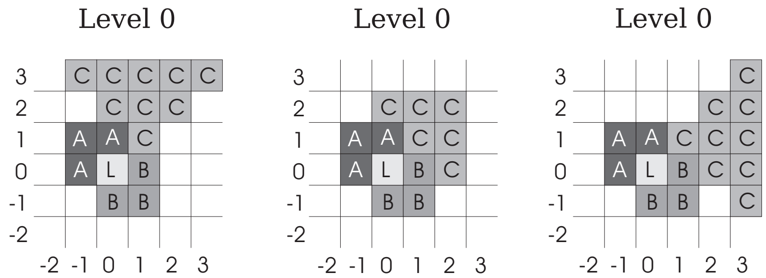

Figure 54: Case 1.1, 1.2 and 1.3

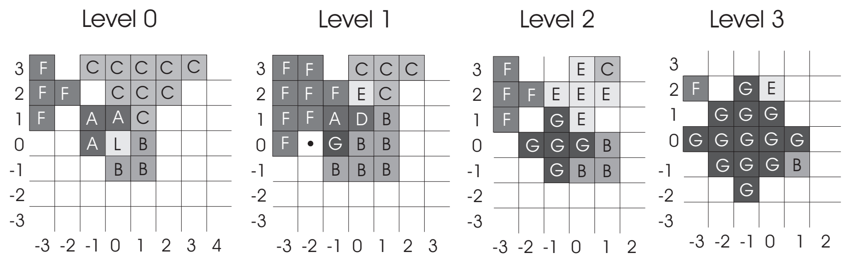

Figure 55: Case 1.1.1

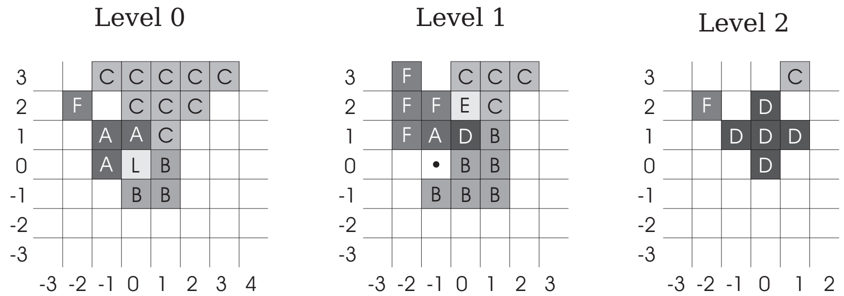

Figure 56: Case 1.1.2

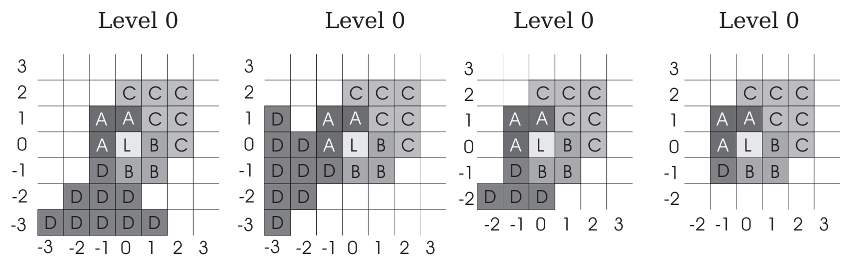

Figure 57: Symmetric cases

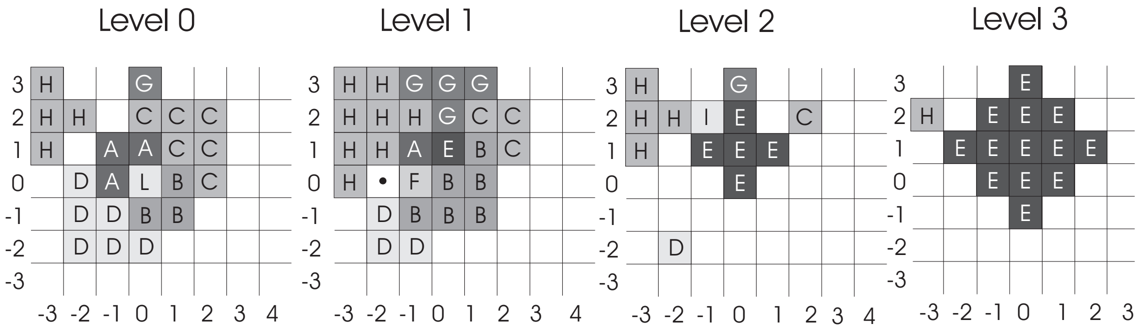

Figure 58: Case 1.2

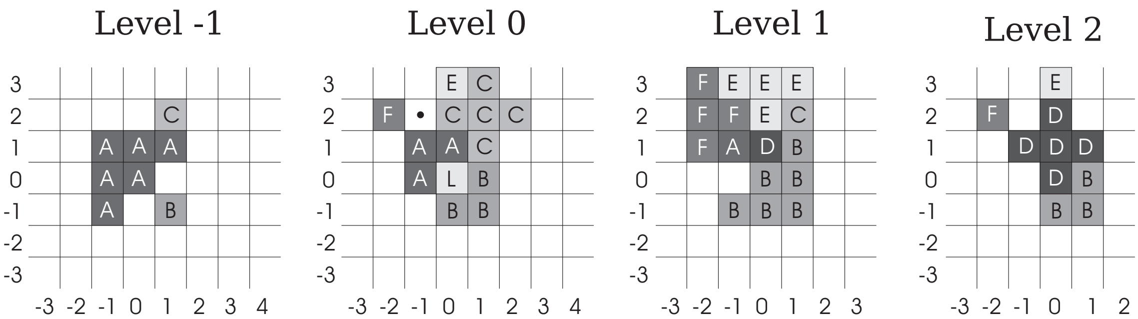

Figure 59: Case 2

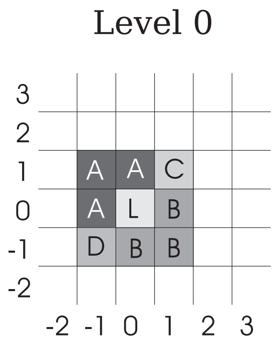

Figure 60: Case 3.1

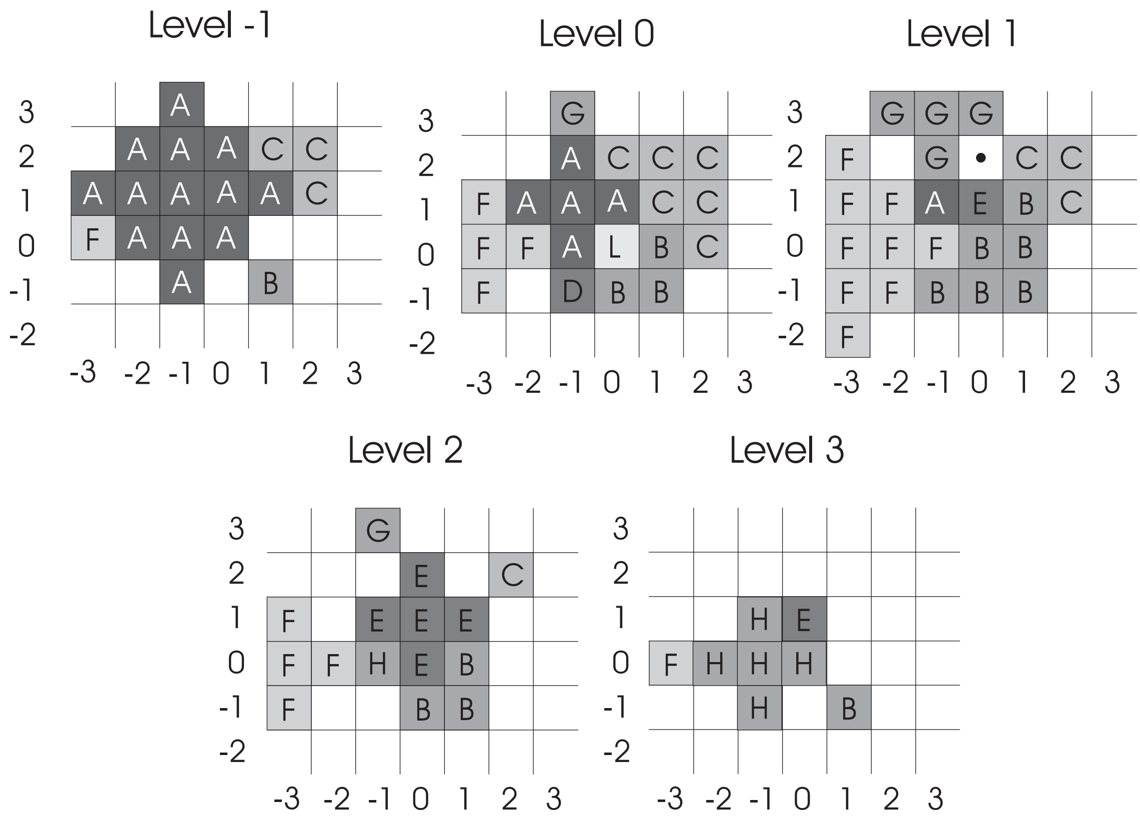

Figure 61: Case 3.2

References

[1] J. T. Astola, On perfect codes in the Lee-metric Ann. Univ. Turku. Ser. A I no. 176 (1978) 56 pp.

[2] J. T. Astola, An Elias-type of bound for Lee codes over large alphabets and its application to perfect codes, IEEE Trans. Inform. Theory 28 (1982) no. 1 111–113.

[3] J. T. Astola, On the asimptotic behaviour of Lee-codes, Discrete Appl. Math. 8 (1984) no. 1 13–23.

[4] N. Biggs, Perfect codes in graphs, J. Combin. Theory Ser. B 15 (1973) 289–296.

[5] H. Everett and D. Hickerson, Packing and covering by translates of certain nonconvex bodies, Proc. Amer. Math. Soc. 75 (1979) no.1 87–91.

[6] S. W. Golomb and L. R. Welch, Algebraic coding and the Lee metric, Proc. Sympos. Math. Res. Center, Madison, Wis. (1968) 175–194 John Wiley, New York.

[7] S. W. Golomb and L. R. Welch, Perfect codes in the Lee metric and the packing of polyominoes, SIAM J. Appl. Math. 18 (1970) 302–317.

[8] S. Gravier, M. Mollard and C. Payan, On the Non-existence of 3-Dimensional Tiling in the Lee Metric, Europ. J. Combin. 19 (1998) 567–572.

[9] S. Gravier, M. Mollard and C. Payan, Variations on tilings in the Manhattan metric, Geom. Dedicata 76 (1999) no.3 265–273.

[10] S. Gravier, M. Mollard and C. Payan, On the nonexistence of three-dimensional tiling in the Lee metric II, Discrete Math. 235 (2001) 151–157.

[11] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, J. Wiley & Sons, New York, 2000.

[12] J. Kratochvíl, *Perfect Codes in General Graphs*, Rozpravy Československé Akad. Věd Řada Mat. Přírod. Věd no. 7, 126 pp. Akademia Praha (1991).

[13] K. A. Post, Noexistance theorems on Perfect Lee codes over large alphabets, Inform. and Control 29 (1975) 369–380.

[14] S. K. Stein and S. Szabo, *Algebra and tiling. Homomorphisms in the service of geometry,* Carus Mathematical Monographs 25. Mathematical Association of America, Washington, DC, 1994.