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SURGERY SPECTRAL SEQUENCE AND STRATIFIED MANIFOLDS

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ABSTRACT. Cappell and Shaneson pointed out in 1978 interesting properties of Browder - Livesay invariants which are similar to differentials in some spectral sequence. Such spectral sequence was constructed in 1991 by Hambleton and Kharshiladze. This spectral sequence is closely related to a problem of realization of elements of Wall groups by normal maps of closed manifolds. The main step of construction of the spectral sequence is an infinite filtration of spectra in which only the first two, as is well-known, have clear geometric sense. The first one is a spectrum $\mathbb{L}(\pi_1(X))$ for surgery obstruction groups of a manifold X and the second $\mathbb{L}P_*(F)$ is a spectrum for surgery on a Browder-Livesay manifold pair $Y \subset X$. The geometric sense of the third term of filtration was explained by Muranov, Repovš, and Spaggiari in 2002. In the present paper we give a geometric interpretation of all spectra of filtration in construction of Hambleton and Kharshiladze. We introduce groups of obstructions to surgery on a system of embedded manifolds and prove that spectra which realize these groups coincide with spectra in the filtration of Hambleton and Kharshiladze. We describe algebraic and geometric properties of introduced obstruction groups and their relations to the classical surgery theory. We prove isomorphism between introduced groups and Browder-Quinn L -groups of stratified manifolds. We give an application of our results to closed manifold surgery problem and iterated Browder-Livesay invariant.

1. Introduction.

Let X^n be a closed n -dimensional CAT ($CAT = TOP, PL, DIFF$) manifold with the fundamental group $\pi = \pi_1(X)$ which is given with a homomorphism of the orientation $w : \pi_1(X) \rightarrow \{\pm 1\}$. In the sequel we shall assume that all groups are given with an orientation homomorphism and shall not specify this in notations without necessity.

A fundamental problem of geometric topology is to describe all closed n -dimensional CAT -manifolds which are homotopy (simple homotopy) equivalent to X . More precisely, let $f : M^n \rightarrow X^n$ be a simple homotopy equivalence of CAT -manifolds.

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The structure set $\mathcal{S}^{CAT}(X)$ is the set of s -cobordism classes of equivalence of CAT -manifolds which are simple homotopy equivalent to X^n (see [34],[29] and [30, p. 542]). The elements of $\mathcal{S}^{CAT}(X)$ are called s -triangulations of the manifold X .

The Sullivan-Novikov-Wall surgery exact sequence

$$(1.1) \quad \cdots \rightarrow L_{n+1}(\pi) \rightarrow \mathcal{S}^{CAT}(X) \rightarrow [X, G/CAT] \xrightarrow{\sigma} L_n(\pi) \cdots$$

is the main tool for describing the structure set $\mathcal{S}^{CAT}(X)$ (see [34] and [30]).

Hereafter we shall consider only topological manifolds ($CAT = TOP$) and groups $L_*(\pi) = L_*^s(\pi)$ which give obstructions to simple homotopy equivalence (see [34, §10] and [30]). To describe the structure set $\mathcal{S}^{TOP}(X)$ we must compute the set of normal invariants $[X, G/TOP]$, the surgery obstruction groups $L_n(\pi)$ and the map σ in (1.1). To describe the map σ we must know what elements of the group $L_n(\pi)$ are realized by normal maps of closed manifolds.

The algebraic surgery exact sequence of Ranicki (see [29] and [30])

$$(1.2) \quad \cdots \rightarrow L_{m+1}(\pi_1(X)) \rightarrow \mathcal{S}_{m+1}(X) \rightarrow H_m(X; \mathbf{L}_\bullet) \xrightarrow{\sigma} L_m(\pi_1(X)) \rightarrow \cdots$$

is defined for any topological space X . In particular, it defines an assembly map

$$(1.3) \quad H_n(K(\pi, 1); \mathbf{L}_\bullet) \xrightarrow{A} L_n(\pi)$$

and $Image(A) \subset L_n(\pi)$ is the subgroup consisting of the elements which can be realized by normal maps of closed manifolds (see, for example, [34, §13]).

If the space X is simple homotopy equivalent to a topological n -dimensional manifold $n \geq 5$, then the exact sequence (1.1) is isomorphic to corresponding part of (1.2). Exact sequence (1.2) is realized on the spectra level by a map of spectra

$$(1.4) \quad X_+ \wedge \mathbf{L}_\bullet \rightarrow \mathbb{L}(\pi_1(X))$$

where $\mathbb{L}(\pi_1(X))$ is the surgery L -spectrum of the fundamental group $\pi_1(X)$ with

$$\pi_n(\mathbb{L}(\pi_1(X))) \cong L_n(\pi_1(X))$$

and \mathbf{L}_\bullet is the 1-connected cover of the Ω -spectrum $\mathbb{L}(\mathbb{Z})$ such that $\mathbf{L}_{\bullet 0} \simeq G/TOP$.

In particular, for manifold X we have isomorphisms $\mathcal{S}_{n+1}(X) \cong \mathcal{S}^{TOP}(X)$ and $H_n(X; \mathbf{L}_\bullet) \cong [X, G/TOP]$ (see [29] and [30]).

Approaches to computation of the structure set $\mathcal{S}^{TOP}(X)$ are very different for the cases finite and infinite group π . The case of infinite group is closely related to the Novikov Conjecture (see, for example [9]). In the case of finite groups the solution of the problem for the special case of decorations (the case of intermediate groups L') was given in [12]. The fundamental results of [12] are based on analysis of assembly map and methods of [5] and [10]. The methods developed in [4], [5], [10], and [16] make it possible to prove the nonrealizability of elements of the Wall group $L_n(\pi)$ which do not lie in the image of the natural map $L_n(1) \rightarrow L_n(\pi)$ for arbitrary case. In particular, Hambleton solved in [10] the corresponding problem for projective Novikov groups L_*^p . These methods are mostly algebraic and are based on the algebraic theory of splitting homotopy equivalence along submanifolds.

Let $Y \subset X$ be a submanifold of codimension q in a closed topological manifold X of dimension n . A simple homotopy equivalence $f : M \rightarrow X$ splits along the submanifold Y if it is homotopy equivalent to a map g which is transversal to Y with a submanifold $N = g^{-1}(Y) \subset M$ and the restrictions

$$(1.5) \quad g|_N : N \rightarrow Y, \quad g|_{(M \setminus N)} : M \setminus N \rightarrow X \setminus Y$$

are simple homotopy equivalences. Let U be a tubular neighborhood of the submanifold Y in X with boundary ∂U . Denote by

$$(1.6) \quad F = \begin{pmatrix} \pi_1(\partial U) & \rightarrow & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(U) & \rightarrow & \pi_1(X) \end{pmatrix}$$

a push-out square of fundamental groups with orientations. There exists a group $LS_{n-q}(F)$ of obstructions to splitting (see [34] and [30]) which depends only on $n - q \bmod 4$ and the square F .

Let $(f, b) : M \rightarrow X$ be a normal map with $b : \nu_M \rightarrow \xi$ a map of fibrations covering f where ξ is a topological reduction of the Spivak normal fibration over X ([29] and [30]). In this case an obstruction for existence in the normal bordism class of the map (f, b) a map (g, b') with properties (1.5) lies in the group $LP_{n-q}(F)$ of obstructions to surgery on manifold pairs (see [34] and [30]). Also this group depends only on $n - q \bmod 4$ and the square F of fundamental groups.

The main relation between LS_* - and LP_* -groups and algebraic surgery exact sequence (1.2) is given by the following commutative diagram [34, §11]

$$(1.7) \quad \begin{array}{ccccccc} \cdots & \rightarrow & \mathcal{S}_{n+1}(X) & \rightarrow & H_n(X; \mathbf{L}_\bullet) & \xrightarrow{\sigma} & L_n(\pi_1(X)) & \rightarrow \cdots \\ & & \downarrow & & \downarrow \sigma_1 & & \downarrow = & \\ \cdots & \rightarrow & LS_{n-q}(F) & \rightarrow & LP_{n-q}(F) & \xrightarrow{s} & L_n(\pi_1(X)) & \xrightarrow{\partial} \cdots \end{array}$$

where the rows are exact sequences. It follows from (1.7) that the image of the map σ lies in the kernel of the map

$$\partial : L_n(\pi_1(X)) \rightarrow LS_{n-q-1}(F)$$

which has a clear geometric description ([5] and [16]).

The bottom row of diagram (1.7) fits the following braid of exact sequences (see [34, p. 264] and [30, §7.2])

$$(1.8) \quad \begin{array}{ccccccc} \rightarrow & L_{n+1}(C) & \longrightarrow & L_{n+1}(D) & \xrightarrow{\partial} & LS_{n-q}(F) & \rightarrow \\ & \nearrow & & \searrow & & \nearrow & \searrow \\ & & LP_{n-q+1}(F) & & L_{n+1}(C \rightarrow D) & & \\ & \searrow & & \nearrow & & \searrow & \nearrow \\ \rightarrow & LS_{n-q+1}(F) & \longrightarrow & L_{n-q+1}(B) & \longrightarrow & L_n(C) & \rightarrow \end{array}$$

where $A = \pi_1(\partial U)$, $B = \pi_1(Y)$, $C = \pi_1(X \setminus Y)$, and $D = \pi_1(X)$.

Now let a pair of manifolds (X, Y) be a Browder-Livesay pair ([2], [5], [10], [16], and [22]). This means that Y is a one-sided submanifold of codimension 1 of the manifold X and the natural embedding $Y \rightarrow X$ induces an isomorphism of the

fundamental groups. In this case the square F of fundamental groups (1.6) has the following form

$$(1.9) \quad F = \begin{pmatrix} \pi_1(\partial U) & \rightarrow & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \rightarrow & \pi_1(X) \end{pmatrix} = \begin{pmatrix} A & \xrightarrow{\cong} & A \\ \downarrow i_- & & \downarrow i_+ \\ B^- & \xrightarrow{\cong} & B^+ \end{pmatrix}.$$

The orientation of the group B^- in (1.9) differs from the orientation of the group B^+ outside the images of the vertical maps (which are inclusions of index 2). All maps in the square (1.9), except the lower horizontal map, preserve the orientation. The lower isomorphism preserves the orientation on the image of i_- , and reverses the orientation outside this image. In this case, we have an isomorphism

$$LP_n(F) \cong L_{n+1}(i_-^*),$$

where $i^* : L_{n+1}(B^-) \rightarrow L_{n+1}(A)$ is the transfer map. The group $LS_*(F)$ is denoted by $LN_*(A \rightarrow B^+) = LN_*(A \rightarrow B)$ [34], and is called the Browder-Livesay group.

Cappell and Shaneson proved [9] that for a Browder-Livesay pair (X, Y) the elements which do not lie in the kernel of the map

$$\partial : L_n(\pi_1(X)) \rightarrow LN_{n-2}(\pi_1(X \setminus Y) \rightarrow \pi_1(X))$$

cannot be realized by a normal map of closed manifolds.

The diagram (1.8) for Browder-Livesay pairs has an algebraic description (see [10] and [31]). This diagram was investigated from algebraic and geometric point of view in several papers (see [10], [12], [13], [16], [20], [21], [22],[23], [24], and [31]).

Subsequently a spectral sequence in surgery theory was constructed in [13] using realization of commutative diagram (1.8) for a Browder-Livesay pair on the spectra level. Consider the filtration of spectra from [13]

$$(1.10) \quad \cdots \rightarrow \mathbb{X}_{3,0} \rightarrow \mathbb{X}_{2,0} \rightarrow \mathbb{X}_{1,0} \rightarrow \mathbb{X}_{0,0} \rightarrow \mathbb{X}_{-1,0} \rightarrow \cdots$$

where $\mathbb{X}_{0,0} = \mathbb{L}(\pi_1(X))$ is a surgery spectrum with $\pi_n(\mathbb{L}(\pi_1(X))) = L_n(\pi_1(X))$ and $\mathbb{X}_{1,0}$ is a spectrum for surgery obstruction groups on the manifold pair (X, Y)

$$\Sigma \mathbb{L}P(F) = \mathbb{L}(i_-^*).$$

The map s in commutative diagrams (1.7) and (1.8) is induced by the map of spectra $\mathbb{X}_{1,0} \rightarrow \mathbb{X}_{0,0}$ from filtration (1.10). Another spectra of filtration is defined inductively using the pullback construction and as is well-known they have no geometric meaning. It follows from [13] that the surgery spectral sequence is closely related to the iterated Browder-Livesay invariants and to the oozing problem. Other versions of surgery spectral exact sequence were obtained in papers [6], [7], [15], and [20].

Let $Z \subset Y \subset X$ be a triple of closed topological manifolds such that n is the dimension of X , q is the codimension of Y in X , and q' is the codimension of Z in Y . Groups of obstructions to surgery $LT_{n-q-q'}(X, Y, Z)$ on manifold triples were introduced in [26]. These groups are realized on the spectra level by a spectrum $\mathbb{L}T(X, Y, Z)$ and they are a natural generalization of surgery obstruction groups

LP_* for manifold pairs. The natural forgetful map $t : LT_n(X, Y, Z) \rightarrow LP_{n+1}$ which is realized on the spectra level is well-defined.

If the triple (X, Y, Z) consists of the Browder-Livesay's pairs (X, Y) and (Y, Z) , then the spectrum $\Sigma^2 \mathbb{L}T(X, Y, Z)$ coincides with the spectrum $\mathbb{X}_{2,0}$ of filtration (1.10). The map $\mathbb{X}_{2,0} \rightarrow \mathbb{X}_{1,0}$ of filtration (1.10) coincides with the map t on the spectra level [26].

Now let

$$(1.11) \quad X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X$$

be a filtration \mathcal{X} of a closed topological manifold X by locally flat embedded submanifolds. Denote by l_j the dimension of the submanifold X_j and by q_j the codimension of X_j in X_{j-1} for $1 \leq j \leq k$. We shall suppose that every pair of manifolds from (1.11) is a topological manifold pair in the sense of Ranicki [30, §7.2] and that dimension $l_k \geq 5$.

For every nonempty subset $B \subset \{k, k-1, \dots, 2, 1, 0\}$ filtration (1.11) defines the restricted filtration \mathcal{X}_B which is obtained by forgetting the submanifolds X_j from filtration (1.10) with $j \in \{k, k-1, \dots, 2, 1, 0\} \setminus B$. In particular a restricted filtration

$$(1.12) \quad X_j \subset X_{j-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X$$

where $B = \{j, j-1, \dots, 2, 1, 0\}$ for every $0 \leq j \leq k$ is well-defined. We shall denote the restricted filtration (1.12) by \mathcal{X}_j .

For a simple homotopy equivalence $f : M \rightarrow X$ we define a concept of an s -triangulation of the filtration (1.11) in section 2 and prove several technical results. In particular, we prove that for the manifold triple $Z \subset Y \subset X$ the surgery obstruction groups $LT_*(X, Y, Z)$ from [26] coincide with the Browder-Quinn groups L^{BQ} (see [3] and [35]) of the stratified manifold $Z \subset Y \subset X$.

We then construct in Section 3 the groups of obstructions to s -triangulation of a filtration \mathcal{X} (1.11) of embedded manifolds and study their properties. We introduce obstruction groups $LM_i^j(\mathcal{X})$ ($0 \leq j \leq k$) which have period 4 for subscript i and which are realized on the spectra level by spectra $\mathbb{L}M^j(\mathcal{X})$ with $\pi_i(\mathbb{L}M^j(\mathcal{X})) = LM_i^j(\mathcal{X})$. The groups $LM_*^k(\mathcal{X})$ coincide with Browder-Quinn stratified L -groups $L_*^{BQ}(\mathcal{X})$ (see [3] and [35]) up to a shift of dimension $*$. The spectrum $\mathbb{L}M^0$ coincides with the spectrum $\mathbb{L}(\pi_1(X))$, the spectrum $\mathbb{L}M^1$ coincides with the spectrum $\mathbb{L}P(F)$ for the pair (X_0, X_1) (see [30] and [34]), and the spectrum $\mathbb{L}M^2$ coincides with the spectrum $\mathbb{L}T$ for the triple (X_0, X_1, X_2) (see [26] and [28]).

Let $(f, b) : (M \rightarrow X)$ be a normal map to the manifold X with the filtration (1.11). For groups introduced above and $0 \leq j \leq k$ an obstruction $\Theta_j(f) \in LM_{l_j}^j$ is defined. It is proved in Theorem 3.9 that this obstruction is trivial if and only if the map f is normally bordant to an s -triangulation of the restricted filtration \mathcal{X}_j (1.12).

In section 3 we define the natural forgetful maps

$$(1.13) \quad LM_{l_k}^k \rightarrow LM_{l_{k-1}}^{k-1} \rightarrow \cdots \rightarrow LM_{l_1}^1 \rightarrow LM_{l_0}^0$$

which are realized on the spectra level by maps of spectra

$$(1.14) \quad \Sigma^{n-l_k} \mathbb{L}M^k \rightarrow \Sigma^{n-l_{k-1}} \mathbb{L}M^{k-1} \rightarrow \cdots \rightarrow \Sigma^{n-l_1} \mathbb{L}M^1 \rightarrow \mathbb{L}M^0.$$

Ranicki introduced in [30] a set $\mathcal{S}_{n+1}(X, Y, \xi)$ of homotopy triangulations of a pair of manifolds (X, Y) , where ξ denotes the normal bundle of Y in X . This set consists of concordance classes of maps $f : (M, N) \rightarrow (X, Y)$ which are split along Y . This structure set is a natural generalization of the structure set $\mathcal{S}_{n+1}(X)$ from exact sequence (1.2) and fits into the exact sequence (see [30, §7.2])

$$(1.15) \quad \cdots \rightarrow \mathcal{S}_{n+1}(X, Y, \xi) \rightarrow H_n(X, \mathbf{L}\bullet) \rightarrow LP_{n-q}(F) \rightarrow \cdots$$

which is a natural generalization of (1.2) to the case of manifold pairs.

In Section 3 we introduce structure sets for the filtration (1.11) which generalize structure sets $\mathcal{S}_{n+1}(X, Y, \xi)$ and $\mathcal{S}_{n+1}(X)$ and we study their properties. Some results for the case of manifold triples were obtained in [26], [27], and [28].

Let all pairs $X_{i+1} \subset X_i$ in (1.11) be Browder-Livesay pairs for $0 \leq i \leq k-1$. In Section 4 we apply our results to an investigation of iterated Browder-Livesay invariants and we describe relations of introduced groups to surgery spectral sequence. It is proved in Theorem 4.1 that in this case filtration (1.14) coincides with the left part starting with $\mathbb{X}_{0,0}$ of filtration (1.10) for spectral sequence of Hambleton and Kharshiladze. Furthermore in Section 4 we investigate relations of groups LM_*^i to the realization of elements of Wall groups by normal maps of closed manifolds.

2. Preliminaries and technical results.

In this section we recall some preliminary results about surgery on topological manifolds and use of surgery L -spectra (see [1], [8], [11], [26], [29], [30], and [33]). We shall give the necessary definitions and prove several technical results.

We shall consider a case of topological manifolds and follow notations from [30, §7.2]. Let (X, Y, ξ) be a codimension q manifold pair in the sense of Ranicki (see [30, §7.2]), i. e. a locally flat closed submanifold $Y \subset X$ given with a normal fibration

$$\xi = \xi_{Y \subset X} : Y \rightarrow \widetilde{BTOP}(q)$$

with the associated (D^q, S^{q-1}) fibration

$$(2.1) \quad (D^q, S^{q-1}) \rightarrow (E(\xi), S(\xi)) \rightarrow Y$$

and we have a decomposition of the closed manifold

$$X = E(\xi) \cup_{S(\xi)} \overline{X \setminus E(\xi)}.$$

A topological normal map [30, §7.2]

$$((f, b), (g, c)) : (M, N) \rightarrow (X, Y)$$

to the manifold pair (X, Y, ξ) is represented by a normal map (f, b) to the manifold X which is transversal to Y with $N = f^{-1}(Y)$, and (M, N) is a topological manifold pair with a normal fibration

$$\nu : N \xrightarrow{f|_N} Y \xrightarrow{\xi} \widetilde{BTOP}(q).$$

Additionally, the following conditions are satisfied:

(i) the restriction

$$(f, b)|_N = (g, c) : N \rightarrow Y$$

is a normal map;

(ii) the restriction

$$(f, b)|_P = (h, d) : (P, S(\nu)) \rightarrow (Z, S(\xi))$$

is a normal map to the pair $(Z, S(\xi))$, where

$$P = \overline{M \setminus E(\nu)}, \quad Z = \overline{X \setminus E(\xi)};$$

(iii) the restriction

$$(h, d)|_{S(\nu)} : S(\nu) \rightarrow S(\xi)$$

coincides with the induced map

$$(g, c)^! : S(\nu) \rightarrow S(\xi),$$

and $(f, b) = (g, c)^! \cup (h, d)$.

The normal maps to (X, Y, ξ) are called t -triangulations of the manifold pair (X, Y) and the set of concordance classes of t -triangulations of the pair (X, Y, ξ) coincides with the set of t -triangulations of the manifold X [30, Proposition 7.2.3].

An s -triangulation of a manifold pair (X, Y, ξ) in topological category [30, p. 571] is a t -triangulation of this pair for which the maps

$$(2.2) \quad f : M \rightarrow X, \quad g : N \rightarrow Y, \quad \text{and} \quad (P, S(\nu)) \rightarrow (Z, S(\xi))$$

are simple homotopy equivalences (s -triangulations).

A simple homotopy equivalence $f : M \rightarrow X$ splits along a submanifold Y if it is homotopy equivalent to a map g which is s -triangulation of (X, Y, ξ) i.e. it satisfies conditions (2.2). In this case f represents an element of $\mathcal{S}_{n+1}(X, Y, \xi)$. It follows from the definition of s -triangulation of the pair (X, Y, ξ) that the forgetful maps

$$\mathcal{S}_{n+1}(X, Y, \xi) \rightarrow \mathcal{S}_{n+1}(X), \quad (f, g) \rightarrow f;$$

$$\mathcal{S}_{n+1}(X, Y, \xi) \rightarrow \mathcal{S}_{n-q+1}(Y), \quad (f, g) \rightarrow g$$

are well-defined. In the general case the map $\mathcal{S}_{n+1}(X, Y, \xi) \rightarrow \mathcal{S}_{n+1}(X)$ is not an epimorphism or a monomorphism [30, p. 571].

Consider a triple $Z^{n-q-q'} \subset Y^{n-q} \subset X^n$ of closed topological manifolds. We shall assume that every submanifold is locally flat in the ambient manifold and that it is equipped by the structure of the normal topological bundle (see [30, pages 562–563] and [26]). Every pair of manifolds defines the following topological normal bundles which we denote in the following way: ξ for the submanifold Y in X , η for the submanifold Z in Y , and ν for the submanifold Z in X . We denote the spaces with boundaries of associated fibrations (2.1) by $(E(\xi), S(\xi))$, $(E(\eta), S(\eta))$, and $(E(\nu), S(\nu))$, respectively. Let $\xi|_{E(\eta)}$ be a restriction of the bundle ξ on a space $E(\eta)$ of normal bundle η with a restriction of fibration (2.1)

$$(D^q, S^{q-1}) \rightarrow (E'(\xi), S'(\xi)) \rightarrow E(\eta)$$

and $\xi|_{S(\eta)}$ be a restriction with a restriction of fibration

$$(D^q, S^{q-1}) \rightarrow (E''(\xi), S''(\xi)) \rightarrow S(\eta)$$

We assume that the space $E(\nu)$ of the normal bundle ν is identified with the space $E'(\xi)$ of the restriction $\xi|_{E(\eta)}$ in such a way that the following condition on the boundary is satisfied

$$(2.3) \quad S(\nu) = E''(\xi) \cup S'(\xi).$$

Remark 2.1. *The existence of normal bundles of the submanifolds for the manifold triple $Z^{n-q-q'} \subset Y^{n-q} \subset X^n$ with the associated fibrations with conditions (2.3) implies that the triple $Z \subset Y \subset X$ is a \mathcal{C} -stratified set in the sense of Browder and Quinn [3].*

Denote by \mathcal{X} a filtration of a closed manifold X^n by a system of submanifolds (1.11). All pairs are given together with normal bundles and corresponding (D^*, S^{*-1}) fibrations (2.1). We shall suppose that for every triple of manifolds $X_j \subset X_l \subset X_m$ with $k \geq j > l > m \geq 0$ the conditions on the normal bundles similar to (2.3) for the triple $Z \subset Y \subset X$ are satisfied.

Remark 2.2. *Under the assumptions above the filtration (1.11) \mathcal{X} gives a \mathcal{C} -stratified set in the sense of Browder and Quinn [3] — this follows from Remark 2.1 and Definition 4.2 from [3].*

A codimension q manifold pair with boundaries $(Y, \partial Y) \subset (X, \partial X)$ is defined in [30, p. 585]. We have a normal fibration $(\xi, \partial\xi)$ over the pair $(Y, \partial Y)$ and a decomposition

$$(2.4) \quad (X, \partial X) = (E(\xi) \cup_{S(\xi)} Z, E(\partial\xi) \cup_{S(\partial\xi)} \partial_+ Z)$$

where $(Z; \partial_+ Z, S(\xi); S(\partial\xi))$ is a manifold triad. Note that here $\partial_+ Z = \overline{\partial X \setminus E(\partial\xi)}$.

A topological normal map of manifold pairs with boundaries

$$(f, \partial f) : (M, \partial M) \rightarrow (X, \partial X)$$

provides a normal fibration $(\nu, \partial\nu)$ over the pair $(N, \partial N)$ (see [30, p. 570]) where

$$(N, \partial N) = (f^{-1}(Y), (\partial f)^{-1}(\partial Y)).$$

We have the following decomposition

$$(2.5) \quad (M, \partial M) = (E(\nu) \cup_{S(\nu)} P, E(\partial\nu) \cup_{S(\partial\nu)} \partial_+ P)$$

where $(P; \partial_+ P, S(\nu); S(\partial\nu))$ is a manifold triad.

Now we define filtration $(\mathcal{X}, \partial\mathcal{X})$ for the case of manifolds with boundaries as filtration

$$(2.6) \quad (X_k, \partial X_k) \subset (X_{k-1}, \partial X_{k-1}) \subset \cdots \subset (X_0, \partial X_0) = (X, \partial X)$$

where all constituent pairs of manifolds with boundaries satisfy properties which are similar to (2.4). We also assume that normal bundles of manifolds of filtration and of boundaries satisfy properties similar to (2.3).

Remark 2.3. *Under the assumptions above the filtration (1.11) \mathcal{X} yields a filtration of manifolds with boundaries*

$$(X_{k-1} \setminus X_k, \partial(X_{k-1} \setminus X_k)) \subset (X_{k-2} \setminus X_k, \partial(X_{k-2} \setminus X_k)) \subset \cdots \subset (X \setminus X_k, \partial(X \setminus X_k))$$

This filtration is a \mathcal{C} -stratified manifold with boundary in the sense of [3] and [35]. We shall denote this filtration by $\overline{\mathcal{X}}_k = \overline{\mathcal{X}}$. In a similar way we can construct a filtration $\overline{\mathcal{X}}_j$ using restricted filtration (1.12).

Definition. *A topological normal map to the filtration \mathcal{X} (1.11) (t -triangulation of the filtration \mathcal{X}) is a topological normal map $(f, b) : M \rightarrow X$ which is topologically transversal to every submanifold of filtration with transversal preimages $M_0 = M, M_i = f^{-1}(X_i)$ for $0 \leq i \leq k$. We shall additionally assume that restriction on every pair of submanifolds (M_j, M_l) , ($j \geq l$) is topological normal map to the manifold pair (M_j, M_l) . In a natural way we can define the bordism of such maps, and the bordism classes are denoted by $\mathcal{T}(\mathcal{X})$ (see [3] and [35]).*

It is clear that a t -triangulation of the filtration \mathcal{X} gives a t -triangulation of a restricted filtration \mathcal{X}_B for every nonempty subset $B \subset \{k, k-1, \dots, 2, 1, 0\}$. In particular for every submanifold X_j from the given filtration we have a forgetful map of $\mathcal{T}(\mathcal{X})$ to the set $[X_j, G/TOP]$ of normal maps to the manifold X_j .

Proposition 2.4. ([3] and [30]) *The natural forgetful map $\mathcal{T}(\mathcal{X}) \rightarrow [X, G/TOP]$ is an isomorphism.*

Proof. Topological transversality (see [3], [30, Proposition 7.2.3], and [35]) and induction on the number of elements of filtration. \square

Definition. *A t -triangulation $(f, b) : M \rightarrow X$ of the filtration \mathcal{X} (1.11) is an s -triangulation of the filtration \mathcal{X} if the constituent normal maps of pairs*

$$(M_j, M_l) \rightarrow (X_j, X_l), 0 \leq j < l \leq k$$

are s -triangulations i.e. they satisfy the properties which are similar to properties (2.2) for the manifold pair (X, Y) .

Proposition 2.5. *Let t -triangulation $(f, b) : \mathcal{M} \rightarrow \mathcal{X}$ define an s -triangulation $f_k : \mathcal{M}_k \rightarrow \mathcal{X}_k$ where filtration \mathcal{X}_k is obtained from \mathcal{X} by forgetting the submanifold X_k and similarly for the \mathcal{M}_k . Suppose that the restriction $f|_{\mathcal{M}_k}$ is an s -triangulation of the pair (X_{k-1}, X_k) . Then (f, b) is an s -triangulation of \mathcal{X} .*

Proof. It suffices to prove that for every submanifold $X_j \subset X$, $0 \leq j \leq k-2$ the restricted map

$$f|_{M_j \setminus M_k} : (M_j \setminus M_k) \rightarrow X_j \setminus X_k$$

is a simple homotopy equivalence. However for the triple $X_k \subset X_{k-1} \subset X_j$ the conditions on the boundaries of tubular neighborhoods (2.3) are satisfied. For such triple the result was proved in [28, Proposition 2.1] using properties of simple homotopy equivalences on triads from [8]. \square

The groups $LT_*(X, Y, Z)$ and the map

$$\Theta_*(f, b) : [X, G/TOP] \rightarrow LT_{n-q-q'}(X, Y, Z)$$

were defined in [26] so that the normal map (f, b) is normally bordant to the s -triangulation of the triple (X, Y, Z) if and only if $\Theta_*(f, b) = 0$ (for $n - q - q' \geq 5$).

These groups were defined on the spectra level. First we recall necessary facts about application spectra to L -theory.

A spectrum \mathbb{E} consists of a collection of CW -complexes $\{(E_n, *)\}$, $n \in \mathbb{Z}$, with a collection of cellular maps $\{\epsilon_n : SE_n \rightarrow E_{n+1}\}$, where SE_n is the suspension of the space E_n [33]. The adjoint maps $\epsilon'_n : E_n \rightarrow \Omega E_{n+1}$ (see [33]) are defined and the E is Ω -spectrum if all adjoint maps are homotopy equivalences. Let $\Sigma\mathbb{E}$ be a spectrum with $\{\Sigma\mathbb{E}\}_n = \mathbb{E}_{n+1}$ and $\{\Sigma\epsilon\}_n = \epsilon_{n+1}$. The functor Σ has an inverse functor Σ^{-1} and iterated functors Σ^k , $k \in \mathbb{Z}$ on the category of spectra are defined. For any spectrum E we have an isomorphism

$$\pi_n(\mathbb{E}) = \pi_{n+k}(\Sigma^k\mathbb{E})$$

of homotopy groups. Recall now that in homotopy theory of spectra there is an equivalence between pullback and pushout squares. A homotopy commutative square of spectra

$$(2.7) \quad \begin{array}{ccc} \mathbb{G} & \rightarrow & \mathbb{H} \\ \downarrow & & \downarrow \\ \mathbb{E} & \rightarrow & \mathbb{F} \end{array}$$

is a pullback if the fibers of horizontal or vertical maps are naturally homotopy equivalent [33]. Square (2.7) is a pushout if the cofibres of vertical or horizontal maps are naturally homotopy equivalent.

Such natural maps of L -groups as transfer and induced map are realized on the spectra level. A homomorphism of oriented groups $f : \pi \rightarrow \pi'$ induces a cofibration of Ω -spectra (see [11])

$$(2.8) \quad \mathbb{L}(\pi) \longrightarrow \mathbb{L}(\pi') \longrightarrow \mathbb{L}(f)$$

where $\pi_n(\mathbb{L}(\pi)) = L_n(\pi)$ and similarly for the other spectra. The homotopy long exact sequence of cofibration (2.8) gives the relative exact sequence of L -groups

$$\dots \rightarrow L_n(\pi) \rightarrow L_n(\pi') \rightarrow L_n(f) \rightarrow L_{n-1}(\pi) \rightarrow \dots$$

For a fibration $p : E^{m+n} \rightarrow X^n$ over a closed topological manifold X^n the transfer map

$$p^* : L_n(\pi_1(X)) \rightarrow L_{n+m}(\pi_1(E))$$

is defined (see [18], [19], [34], and [35]) which is realized on the spectra level by a map of Ω -spectra

$$(2.9) \quad p^! : \mathbb{L}(\pi_1(X)) \rightarrow \Sigma^{-m}\mathbb{L}(\pi_1(E)).$$

For a manifold pair (X, Y) we have the following homotopy commutative diagram of spectra

$$(2.10) \quad \begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \xrightarrow{p^!} & \Sigma^{-q}\mathbb{L}(\pi_1(\partial U) \rightarrow \pi_1(U)) & \xrightarrow{\alpha} & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) \\ & p^! \searrow & \downarrow \delta & & \downarrow \delta_1 \\ & & \Sigma^{1-q}\mathbb{L}(\pi_1(\partial U)) & \xrightarrow{\beta} & \Sigma^{1-q}\mathbb{L}(\pi_1(X \setminus Y)), \end{array}$$

where the left maps are transfer maps and the right horizontal maps are induced by the horizontal maps of the square F (1.6). The two right vertical maps in (2.10) obtained from extending cofibration sequences (2.8) for vertical maps of the square F (1.6). The spectrum $\mathbb{L}S(F)$ is a homotopy cofiber of the map

$$(2.11) \quad \Sigma^{-1}(\alpha p_1^!) : \Sigma \mathbb{L}(\pi_1(Y)) \rightarrow \Sigma^{-q-1} \mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X))$$

and the spectrum $\mathbb{L}P(F)$ is a homotopy cofiber of the map

$$(2.12) \quad \Sigma^{-1}(\beta p^!) : \Sigma^{-1} \mathbb{L}(\pi_1(Y)) \rightarrow \Sigma^{-q} \mathbb{L}(\pi_1(X \setminus Y))$$

We have isomorphisms (see [1], [21], [24], and [26])

$$(2.13) \quad \pi_n(\mathbb{L}S(F)) \cong LS_n(F), \quad \pi_n(\mathbb{L}P(F)) \cong LP_n(F).$$

Denote by $\mathcal{S}_{n+1}(X, Y, \xi)$ the set of concordance classes of s -triangulations of the manifold pair (X, Y, ξ) (see [30]).

For the triple $Z \subset Y \subset X$ of closed topological manifolds consider the square of fundamental groups with orientations for the splitting problem for the manifold pair $Z \subset Y$:

$$(2.14) \quad \Psi = \begin{pmatrix} \pi_1(\partial V) & \rightarrow & \pi_1(Y \setminus Z) \\ \downarrow & & \downarrow \\ \pi_1(Z) & \rightarrow & \pi_1(Y) \end{pmatrix}.$$

Consider the commutative diagram (see [30] and [34])

$$(2.15) \quad \begin{array}{ccccccc} \cdots & \rightarrow & \mathcal{S}_{n+1}(X, Y, \xi) & \rightarrow & H_n(X; \mathbf{L}_\bullet) & \xrightarrow{\sigma_1} & LP_{n-q}(F) & \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \rightarrow & \mathcal{S}_{n-q+1}(Y) & \rightarrow & H_{n-q}(Y; \mathbf{L}_\bullet) & \rightarrow & L_{n-q}(Y) & \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow = & \\ \cdots & \rightarrow & LS_k(\Psi) & \rightarrow & LP_k(\Psi) & \rightarrow & L_{n-q}(Y) & \rightarrow \cdots \end{array}$$

in which $k = n - q - q'$ is a dimension of Z , and rows are exact sequences. Observe that the bottom two rows represent the diagram (1.7) for the manifold pair (Y, Z) . Diagram (2.15) is realized on spectra level ([1] and [26]).

In particular, the composition

$$LP_{n-q+1}(F) \rightarrow \mathcal{S}_{n+1}(X, Y, \xi) \rightarrow \mathcal{S}_{n-q+1}(Y) \rightarrow LS_{n-q-q'}(\Psi)$$

of maps from diagram (2.15) is realized by a composition \mathbf{v} of maps of spectra

$$\mathbb{L}P(F) \rightarrow \Sigma^{-q} \mathbb{S}(X, Y, \xi) \rightarrow \mathbb{S}(Y) \rightarrow \Sigma^{q'+1} \mathbb{L}S(\Psi)$$

where

$$\pi_n(\mathbb{S}(X, Y, \xi)) = \mathcal{S}_n(X, Y, \xi), \quad \pi_n(\mathbb{S}(Y)) = \mathcal{S}_n(Y).$$

The spectrum $\mathbb{L}T(X, Y, Z)$ is a homotopy cofiber of the map

$$(2.16) \quad \Sigma^{-q'-1} \mathbf{v} : \Sigma^{-q'-1} \mathbb{L}P(F) \rightarrow \mathbb{L}S(\Psi)$$

and by definition $LT_n(X, Y, Z) = \pi_n(\mathbb{L}T(X, Y, Z))$ (see [26]). The homotopy long exact sequence of cofibration (2.16) gives the exact sequence

$$(2.17) \quad \cdots \rightarrow LP_{n-q+1}(F) \rightarrow LS_{n-q-q'}(\Psi) \rightarrow LT_{n-q-q'}(X, Y, Z) \rightarrow \cdots$$

The triple of manifolds $Z \subset Y \subset X$ is a stratified topological manifold (see [3] and [35]) which we shall denote by \mathcal{X} . Hence the stratified L -groups $L^{BQ}(\mathcal{X})$ of Browder-Quinn are defined. These groups are realized on spectra level and we recall an inductive definition of these groups from [35, p. 129] using our notations. By Remark 2.3, the triple $Z \subset Y \subset X$ yields a pair of manifolds with boundaries

$$(2.18) \quad (Y \setminus Z, \partial(Y \setminus Z)) \subset (X \setminus Z, \partial(X \setminus Z))$$

where $\partial(Y \setminus Z) \subset \partial(X \setminus Z)$ is a manifold pair which coincides with natural decomposition of a boundary of a tubular neighborhood of Z in X . Denote by F_Z the square of fundamental groups for splitting problem relative boundary for the manifold pair (2.18), and by F_U the similar square for the closed manifold pair $\partial(Y \setminus Z) \subset \partial(X \setminus Z)$. In fact, the geometric definition of transfer map $p^!$ in (2.9) and (2.10) for the pair $Z \subset X$ (see [18], [19] [30] and [34]) gives a map

$$(2.19) \quad p^\# : L_{n-q-q'}(\pi_1(Z)) \rightarrow LP_{n-q-1}(F_U)$$

which is realized on spectra level (see [35]) by a map of spectra

$$(2.20) \quad \mathbf{p}^\# : \mathbb{L}(\pi_1(Z)) \rightarrow \Sigma^{-q'+1}\mathbb{L}P_{n-q-1}(F_U).$$

Consider the composition of the map $\mathbf{p}^\#$ (2.20) with the map of spectra

$$\mathbf{b} : \Sigma^{-q'+1}\mathbb{L}P_{n-q-1}(F_U) \rightarrow \Sigma^{-q'+1}\mathbb{L}P_{n-q-1}(F_Z)$$

which is induced by inclusion of the boundary in (2.18). We obtain a cofibration of spectra [35]

$$(2.21) \quad \mathbf{p}^\# \mathbf{b} : \mathbb{L}(\pi_1(Z)) \rightarrow \Sigma^{-q'+1}\mathbb{L}P(F_Z) \rightarrow \Sigma^{-q-q'+1}\mathbb{L}^{BQ}(\mathcal{X})$$

with a cofiber $\Sigma^{-q-q'+1}\mathbb{L}^{BQ}(\mathcal{X})$. By definition (see [3] and [35])

$$\pi_n(\mathbb{L}^{BQ}(\mathcal{X})) = L_n^{BQ}(\mathcal{X}).$$

For the groups L_n^{BQ} index n is equal to dimension of the largest manifold of filtration taken mod 4 (see [3] and [35]). For the case of surgery obstruction groups LP_n Wall and Ranicki (see [30] and [34]) used index n which corresponds to dimension of the smallest manifold from the pair. Similarly to Wall and Ranicki, for the surgery obstruction groups on manifold triples LT_n index n is equal to dimension of the bottom manifold of the filtration.

Remark 2.6. For the manifold triple (X, Y, Z) the homotopy long exact sequence of cofibration (2.21) gives the following exact sequence of obstruction groups

$$(2.22) \quad \cdots \rightarrow L_{n-q-q'}(\pi_1(Z)) \rightarrow LP_{n-q-1}(F_Z) \rightarrow L_{n-1}^{BQ}(\mathcal{X}) \rightarrow \cdots$$

where \mathcal{X} denotes the filtration $Z \subset Y \subset X$. \square

For the pair (X, Z) we denote the squares of fundamental groups in the splitting problem by Φ . The groups $LP_*(\Phi)$ fit in the exact sequence (see [30] and [34])

$$(2.23) \quad \cdots \rightarrow L_n(\pi_1(X \setminus Y)) \rightarrow LP_{n-q-q'}(\Phi) \rightarrow L_{n-q-q'}(\pi_1(Z)) \rightarrow \cdots$$

which is realized on the spectra level similar to (2.12) by a cofibration of spectra

$$(2.24) \quad \mathbb{L}P(\Phi) \rightarrow \mathbb{L}(\pi_1(Z)) \rightarrow \Sigma^{-q-q'+1}\mathbb{L}(\pi_1(X \setminus Z)).$$

By [26, Theorem 2] the groups LT_* fit in the commutative diagram of exact sequences

$$(2.25) \quad \begin{array}{ccccccc} & \rightarrow & L_n(C) & \longrightarrow & LP_{n-q}(F) & \longrightarrow & LS_{k-1}(\Psi) \rightarrow \\ & \nearrow & & & \nearrow & & \searrow \\ & & & LT_k(X, Y, Z) & & L_{n-q}(\pi_1(Y)) & \\ & \searrow & & & \searrow & & \nearrow \\ & \rightarrow & LS_k(\Psi) & \longrightarrow & LP_k(\Psi) & \longrightarrow & L_{n-1}(C) \rightarrow, \end{array}$$

where $k = n - q - q'$ and $C = \pi_1(X \setminus Y)$. Diagram (2.25) is realized on spectra level and contains the following exact sequence

$$(2.26) \quad \cdots \rightarrow L_n(\pi_1(X \setminus Y)) \rightarrow LT_{n-q-q'}(X, Y, Z) \rightarrow LP_{n-q-q'}(\Psi) \rightarrow \cdots$$

Exact sequence (2.26) is realized on the spectra level by the cofibration

$$(2.27) \quad \mathbb{L}T(X, Y, Z) \rightarrow \mathbb{L}P(\Psi) \rightarrow \Sigma^{-q-q'+1}\mathbb{L}(\pi_1(X \setminus Z)).$$

Proposition 2.7. There exists the following commutative diagram

$$(2.28) \quad \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \rightarrow & L_n(\pi_1(X \setminus Y)) & \rightarrow & LT_k(X, Y, Z) & \rightarrow & LP_k(\Psi) & \rightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \rightarrow & L_n(\pi_1(X \setminus Z)) & \rightarrow & LP_k(\Phi) & \rightarrow & L_k(\pi_1(Z)) & \rightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \rightarrow & L_n(\pi_1(X \setminus Y) \rightarrow \pi_1(X \setminus Z)) & \rightarrow & LS_{n-q-1}(F_Z) & \rightarrow & L_{n-q-1}(\pi_1(Y \setminus Z)) & \rightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ & \vdots & & \vdots & & \vdots & \end{array}$$

where $k = n - q - q'$. Diagram (2.28) is realized on the spectra level. All the maps in the square

$$(2.29) \quad \begin{array}{ccc} LT_*(X, Y, Z) & \rightarrow & LP_*(\Psi) \\ \downarrow & & \downarrow \\ LP_*(\Phi) & \rightarrow & L_*(\pi_1(Z)) \end{array}$$

of diagram (2.28) are natural forgetful maps. The upper two horizontal rows of the diagram (2.28) coincide with exact sequences (2.26) and (2.23).

Proof. Forgetting the submanifold Y induces the natural maps

$$LT_*(X, Y, Z) \rightarrow LP_*(\Phi) \text{ and } LP_*(\Psi) \rightarrow L_*(\pi_1(Z))$$

which are induced by the maps of spectra from (2.25) and (2.27). Similarly to (2.25) the forgetful map $LP_*(\Psi) \rightarrow L_*(\pi_1(Z))$ is realized on the spectra level. The forgetful map $LT_*(X, Y, Z) \rightarrow LP_*(\Phi)$ is realized on the spectra level by [28, Theorem 3.5]. This map fits in the following exact sequence

$$(2.30) \quad \cdots \rightarrow LT_{n-q-q'}(X, Y, Z) \rightarrow LP_{n-q-q'}(\Phi) \rightarrow LS_{n-q-1}(F_Z) \rightarrow \cdots$$

It follows from this that we have the following homotopy commutative diagram of spectra

$$(2.31) \quad \begin{array}{ccc} \mathbb{L}T(X, Y, Z) & \rightarrow & \mathbb{L}P(\Psi) \\ \downarrow & & \downarrow \\ \mathbb{L}P(\Phi) & \rightarrow & \mathbb{L}(\pi_1(Z)). \end{array}$$

Consider a biinfinite homotopy commutative diagram of spectra

$$(2.32) \quad \begin{array}{ccccc} & \vdots & & \vdots & \\ & \downarrow & & \downarrow & \\ \rightarrow & \mathbb{L}T(X, Y, Z) & \rightarrow & \mathbb{L}P(\Psi) & \rightarrow \Sigma^{-q'-q+1}\mathbb{L}(\pi_1(X \setminus Y)) \rightarrow \\ & \downarrow & & \downarrow & \\ \rightarrow & \mathbb{L}P(\Phi) & \rightarrow & \mathbb{L}(\pi_1(Z)) & \rightarrow \Sigma^{-q'-q+1}\mathbb{L}(\pi_1(X \setminus Z)) \rightarrow \\ & \downarrow & & \downarrow & \\ \rightarrow & \Sigma^{-q'+1}\mathbb{L}S(F_Z) & \rightarrow & \Sigma^{-q'+1}\mathbb{L}(\pi_1(Y \setminus Z)) & \rightarrow \Sigma^{-q'-q+1}\mathbb{L}^{rel} \rightarrow \\ & \downarrow & & \downarrow & \\ & \vdots & & \vdots & \end{array}$$

where $\mathbb{L}^{rel} = \mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X \setminus Z))$. This diagram is obtained from homotopy commutative diagram (2.31) by consideration of cofibrations given by all maps of diagram (2.31) (see [21] and [33]). Application of π_0 to (2.32) gives commutative diagram (2.28). \square

We now recall the following technical result from [21].

Lemma 2.8. *Consider a diagram of spectra*

$$\begin{array}{ccccc} & & \bullet & & \\ & & \downarrow & \searrow & \\ \bullet & \rightarrow & \bullet & \rightarrow & \bullet \\ & \swarrow & \downarrow & & \\ & & \bullet & & \end{array},$$

in which the row and the column are cofibrations. Then the cofibres of the diagonal maps are naturally homotopy equivalent.

Proof. See [21]. \square

Theorem 2.9. *Let \mathcal{X} be a filtration $Z \subset Y \subset X$ of topological manifolds, n the dimension of X , q the codimension of Y in X , and q' the codimension of Z in Y . We have a homotopy equivalence of the spectra*

$$\mathbb{L}T(X, Y, Z) \simeq \Sigma^{-q-q'} \mathbb{L}^{BQ}(\mathcal{X})$$

and hence an isomorphism $LT_{n-q-q'}(X, Y, Z) = L_n^{BQ}(\mathcal{X})$ of surgery obstruction groups for $n = 0, 1, 2, 3 \pmod{4}$.

Proof. It follows by Lemma 2.8 that the cofibres of the diagonal maps of spectra

$$(2.33) \quad \begin{aligned} \Sigma^{-q'} \mathbb{L}(\pi_1(Y \setminus Z)) &\rightarrow \Sigma^{-q'+1} \mathbb{L}(\pi_1(X \setminus Y)), \\ \mathbb{L}T(X, Y, Z) &\rightarrow \mathbb{L}(\pi_1(Z)), \end{aligned}$$

$$\Sigma^{-q'+q} \mathbb{L}(\pi_1(X \setminus Z)) \rightarrow \Sigma^{-q'+1} \mathbb{L}S(F_Z)$$

in diagram (2.32) are naturally homotopy equivalent. The map of spectra

$$(2.34) \quad \Sigma \mathbb{L}(\pi_1(Y \setminus Z)) \rightarrow \Sigma^{-q} \mathbb{L}(\pi_1(X \setminus Y))$$

is a realization on spectra level of the transfer map for manifold pair $(X \setminus Z, Y \setminus Z)$ — this follows from diagram (2.28). Hence a cofiber of the first map in (2.33) coincides with the spectrum $\Sigma^{q'+1} \mathbb{L}P(F_Z)$. Hence the cofiber of the second map in (2.33) coincides with this one. We obtain the following cofibration of spectra

$$(2.35) \quad \mathbb{L}T(X, Y, Z) \rightarrow \mathbb{L}(\pi_1(Z)) \rightarrow \Sigma^{q'+1} \mathbb{L}P(F_Z).$$

Hence (see [33]) the spectrum $\mathbb{L}T(X, Y, Z)$ is defined as homotopical fiber of the transfer map

$$(2.36) \quad \mathbb{L}(\pi_1(Z)) \rightarrow \Sigma^{q'+1} \mathbb{L}P(F_Z).$$

However, by (2.21) a homotopical fiber of this map is a spectrum $\Sigma^{-q-q'} \mathbb{L}^{BQ}(\mathcal{X})$ where \mathcal{X} is the filtration $Z \subset Y \subset X$. Therefore the assertion of the theorem follows. \square

Corollary 2.10. *Under hypothesis of Theorem 2.9 we have the following three braids of exact sequences*

$$(2.37) \quad \begin{array}{ccccccc} \rightarrow & L_{n-q}(D) & \longrightarrow & L_{n-1}(C) & \longrightarrow & LT_{k-1} & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & LP_k(\Psi) & & LP_m(F_Z) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & LT_k & \longrightarrow & L_k(\pi_1(Z)) & \longrightarrow & L_m(D) & \rightarrow, \end{array}$$

$$(2.38) \quad \begin{array}{ccccccc} \rightarrow & LT_k & \longrightarrow & L_k(\pi_1(Z)) & \longrightarrow & L_{n-1}(E) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & LP_k(\Phi) & & LP_m(F_Z) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & L_n(E) & \longrightarrow & LS_m(F_Z) & \longrightarrow & LT_{k-1} & \rightarrow, \end{array}$$

and

$$(2.39) \quad \begin{array}{ccccccc} \rightarrow & L_n(E) & \longrightarrow & LS_m(F_Z) & \rightarrow & L_m(D) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & L_n(C \rightarrow E) & & LP_m(F_Z) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & L_{n-q}(D) & \longrightarrow & L_{n-1}(C) & \longrightarrow & L_{n-1}(E) & \rightarrow, \end{array}$$

where $k = n - q - q'$, $m = n - q - 1$, $D = \pi_1(Y \setminus Z)$, $E = \pi_1(X \setminus Z)$ and $C = \pi_1(X \setminus Y)$. Diagrams (2.37), (2.38), and (2.39) are realized on the spectra level.

Proof. From biinfinite homotopy commutative diagram (2.32) and cofibration (2.35) we obtain the following homotopy commutative diagram of spectra

$$(2.40) \quad \begin{array}{ccccc} \mathbb{L}T(X, Y, Z) & \longrightarrow & \mathbb{L}P(\Psi) & \longrightarrow & \Sigma^{-q'-q+1}\mathbb{L}(\pi_1(X \setminus Y)) \\ = \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}T(X, Y, Z) & \longrightarrow & \mathbb{L}(\pi_1(Z)) & \longrightarrow & \Sigma^{q'+1}\mathbb{L}P(F_Z) \end{array}$$

in which horizontal rows are cofibrations, and right vertical map is induced by two left vertical maps (see [33]). Hence fibers of the two right horizontal maps in (2.40) are naturally homotopy equivalent to the spectrum $\mathbb{L}T(X, Y, Z)$. Hence the right square in (2.40) is a pullback and fibers of vertical map of this square are also naturally homotopy equivalent. Homotopy long exact sequences of this square give commutative diagram (2.37). In a similar way the commutative diagrams (2.38) and (2.39) follow from the other two cofibrations from (2.33) and homotopy commutative diagram (2.32). \square

Remark 2.11. Diagram (2.39) is, in fact, diagram (1.8) constructed for the pair of manifolds with boundaries $(X \setminus Y) \subset (X \setminus Z)$. \square

Remark 2.12. We can consider a manifold pair $Y^{n-q} \subset X^n$ as the stratified manifold \mathcal{X} for which Browder-Quinn groups $L^{BQ}(\mathcal{X})$ are defined. It follows from cofibration (2.12), that the definition of LP_* -groups by Wall and results of Ranicki (see [30] and [34]) give an isomorphism $LP_{n-q}(F) \cong L^{BQ}(\mathcal{X})$. This isomorphism is realized on the spectra level. \square

3. Surgery on a manifold with filtration.

In this section we introduce surgery obstruction groups for the filtration \mathcal{X} (1.11) and describe their main properties. At first we give the motivation of our definition and then we prove Theorem 3.1 and describe relations of introduced groups to L^{BQ} -groups of Browder and Quinn. We shall use the notations of previous sections.

For a manifold pair (X^n, Y^{n-q}) of codimension q realization of the diagram (1.8) on spectra level provides the following homotopy commutative diagram of spectra

$$(3.1) \quad \begin{array}{ccccc} & & \Sigma^q \mathbb{L}(\pi_1(Y)) & & \\ & & \downarrow & \searrow & \\ \mathbb{L}(\pi_1(X)) & \rightarrow & \mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) & \rightarrow & \Sigma \mathbb{L}(\pi_1(X \setminus Y)) \\ & \searrow & \downarrow & & \\ & & \Sigma^{q+1} \mathbb{L}S(F), & & \end{array}$$

in which the vertical column and horizontal row are cofibrations. The cofibres of diagonal maps are naturally homotopy equivalent to the spectrum $\Sigma^{q+1}\mathbb{L}P(F)$ as follows from (1.8) and Lemma 2.8.

Consider now a manifold triple $Z^{n-q-q'} \subset Y^{n-q} \subset X^n$ where q is the codimension of Y in X and q' is the codimension Z in Y . Realization of the diagram (2.25) on spectra level provides the following homotopy commutative diagram of spectra

$$(3.2) \quad \begin{array}{ccccc} & & \Sigma^{q'}\mathbb{L}P(\Psi) & & \\ & & \downarrow & \searrow & \\ \mathbb{L}P(F) & \rightarrow & \mathbb{L}(\pi_1(Y)) & \rightarrow & \Sigma^{-q+1}\mathbb{L}(\pi_1(X \setminus Y)) \\ & \searrow & \downarrow & & \\ & & \Sigma^{q'+1}\mathbb{L}S(\Psi), & & \end{array}$$

in which the vertical column and horizontal row are cofibrations. Recall that F is a square of fundamental groups for splitting problem for the pair (X, Y) , and Ψ is a similar square for the pair (Y, Z) . The cofibres of the diagonal maps are naturally homotopy equivalent to the spectrum $\Sigma^{q'+1}\mathbb{L}T(X, Y, Z)$ as follows from (2.25) and Lemma 2.8.

Now consider filtration \mathcal{X} (1.11) for which the restricted filtrations \mathcal{X}_j for $j = 0, 1, \dots, k$ are defined.

For a pair of submanifolds $X_j \subset X_{j-1}$ of filtration (1.11) we denote the square of fundamental groups for splitting problem by F_j where $1 \leq j \leq k$. We also introduce special notations for the following filtrations. Let \mathcal{Y} be a subfiltration

$$(3.3) \quad X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1$$

of \mathcal{X} and \mathcal{Y}_{j-1} be a restricted subfiltration

$$(3.4) \quad X_j \subset X_{j-1} \subset \cdots \subset X_2 \subset X_1$$

of \mathcal{Y} where $1 \leq j \leq k$. We have $\mathcal{X}_0 = (X_0) = (X)$, $\mathcal{X}_1 = (X_1 \subset X_0)$, and $\mathcal{X}_2 = (X_2 \subset X_1 \subset X_0)$. Denote

$$\mathbb{L}M^0(\mathcal{X}) = \mathbb{L}M^0(X_0) = \mathbb{L}(\pi_1(X_0)),$$

$$\mathbb{L}M^1(\mathcal{X}) = \mathbb{L}M^1(\mathcal{X}_1) = \mathbb{L}M^1(X_1 \subset X_0) = \mathbb{L}P(F_1),$$

and

$$\mathbb{L}M^2(\mathcal{X}) = \mathbb{L}M^2(\mathcal{X}_2) = \mathbb{L}T(X_0, X_1, X_2).$$

For spectra $\mathbb{L}M^i$ defined above with $0 \leq i \leq 2$ and $j \geq i$ we have, by definition, that $\mathbb{L}M^i(\mathcal{X}_j) = \mathbb{L}M^i(\mathcal{X})$. Diagram (3.2) in our notations has the following form

$$(3.5) \quad \begin{array}{ccccc} & & \Sigma^{q_2}\mathbb{L}M^1(\mathcal{Y}) & & \\ & & \downarrow & \searrow & \\ \mathbb{L}M^1(\mathcal{X}) & \rightarrow & \mathbb{L}M^0(\mathcal{Y}) & \rightarrow & \Sigma^{-q_1+1}\mathbb{L}(\pi_1(X_0 \setminus X_1)) \\ & \searrow & \downarrow & & \\ & & \Sigma^{q_2+1}\mathbb{L}S(F_2) & & \end{array}$$

with the cofibres of diagonal maps which are naturally homotopy equivalent to

$$\Sigma^{q_2+1}\mathbb{L}M^2(\mathcal{X}) = \Sigma^{q_2+1}\mathbb{L}M^2(\mathcal{X}_2)$$

The right diagonal map from diagram (3.5) gives a cofibration of spectra

$$(3.6) \quad \mathbb{L}M^2(\mathcal{X}) \rightarrow \mathbb{L}M^1(\mathcal{Y}) \rightarrow \Sigma^{-q_1-q_2+1}\mathbb{L}(\pi_1(X_0 \setminus X_1))$$

where $-q_1 - q_2 = l_2 - n$. The left diagonal map from (3.5) gives a cofibration

$$(3.7) \quad \Sigma^{q_2}\mathbb{L}M^2(\mathcal{X}) \rightarrow \mathbb{L}M^1(\mathcal{X}) \rightarrow \Sigma^{q_2+1}\mathbb{L}S(F_2).$$

For the filtration \mathcal{Y} cofibration (3.7) gives a cofibration

$$(3.8) \quad \Sigma^{q_3}\mathbb{L}M^2(\mathcal{Y}) \rightarrow \mathbb{L}M^1(\mathcal{Y}) \rightarrow \Sigma^{q_3+1}\mathbb{L}S(F_3).$$

We can combine cofibrations (3.6) and (3.8) to obtain the following homotopy commutative diagram

$$(3.9) \quad \begin{array}{ccc} & \Sigma^{q_3}\mathbb{L}M^2(\mathcal{Y}) & \\ & \downarrow & \searrow \\ \mathbb{L}M^2(\mathcal{X}) & \rightarrow & \mathbb{L}M^1(\mathcal{Y}) & \rightarrow & \Sigma^{l_2-n+1}\mathbb{L}(\pi_1(X_0 \setminus X_1)) \\ & \searrow & \downarrow & & \\ & & \Sigma^{q_3+1}\mathbb{L}S(F_3) & & \end{array}$$

in which cofibers of diagonal maps are naturally homotopy equivalent. We shall denote homotopy cofiber of diagonal map in diagram (3.9) by

$$\Sigma^{q_3+1}\mathbb{L}M^3(\mathcal{X}_3) = \Sigma^{q_3+1}\mathbb{L}M^3(\mathcal{X}).$$

It follows from this definition that $\mathbb{L}M^3(\mathcal{X}_j) = \mathbb{L}M_3(\mathcal{X})$ for $3 \leq j \leq k$. We can continue these constructions to give inductive definition of the spectra

$$\mathbb{L}M^i(\mathcal{X}) = \mathbb{L}M^i(\mathcal{X}_i)$$

for $4 \leq i \leq k$.

Let a spectrum $\mathbb{L}M^j(\mathcal{X}) = \mathbb{L}M^j(\mathcal{X}_j)$ be already defined for $k \geq j \geq 2$ in such a way that the spectrum $\Sigma^{q_j+1}\mathbb{L}M^j(\mathcal{X})$, ($j \geq 2$) is a natural homotopy cofiber of diagonal maps in a diagram

$$(3.10) \quad \begin{array}{ccc} & \Sigma^{q_j}\mathbb{L}M^{j-1}(\mathcal{Y}) & \\ & \downarrow & \searrow \\ \mathbb{L}M^{j-1}(\mathcal{X}) & \rightarrow & \mathbb{L}M^{j-2}(\mathcal{Y}) & \rightarrow & \Sigma^{l_{j-1}-n+1}\mathbb{L}(\pi_1(X_0 \setminus X_1)) \\ & \searrow & \downarrow & & \\ & & \Sigma^{q_j+1}\mathbb{L}S(F_j). & & \end{array}$$

The right diagonal map from diagram (3.10) gives a cofibration of the spectra

$$(3.11) \quad \mathbb{L}M^j(\mathcal{X}) \rightarrow \mathbb{L}M^{j-1}(\mathcal{Y}) \rightarrow \Sigma^{l_{j-1}-n-q_j+1}\mathbb{L}(\pi_1(X_0 \setminus X_1))$$

where $l_{j-1} - n - q_j + 1 = l_j - n + 1$. The left diagonal map in (3.10) gives the cofibration

$$(3.12) \quad \Sigma^{q_j}\mathbb{L}M^j(\mathcal{X}) \rightarrow \mathbb{L}M^{j-1}(\mathcal{X}) \rightarrow \Sigma^{q_j+1}\mathbb{L}S(F_j).$$

For the filtrations \mathcal{Y} and \mathcal{Y}_j cofibration (3.12) gives the cofibration

$$(3.13) \quad \Sigma^{q_{j+1}} \mathbb{L}M^j(\mathcal{Y}) \rightarrow \mathbb{L}M^{j-1}(\mathcal{Y}) \rightarrow \Sigma^{q_{j+1}+1} \mathbb{L}S(F_{j+1}).$$

We can combine cofibrations (3.11) and (3.13) to obtain the following homotopy commutative diagram

$$(3.14) \quad \begin{array}{ccccc} & & \Sigma^{q_{j+1}} \mathbb{L}M^j(\mathcal{Y}) & & \\ & & \downarrow & \searrow & \\ \mathbb{L}M^j(\mathcal{X}) & \rightarrow & \mathbb{L}M^{j-1}(\mathcal{Y}) & \rightarrow & \Sigma^{l_j-n+1} \mathbb{L}(\pi_1(X_0 \setminus X_1)) \\ & \searrow & \downarrow & & \\ & & \Sigma^{q_{j+1}+1} \mathbb{L}S(F_{j+1}). & & \end{array}$$

in which cofibers of diagonal maps are naturally homotopy equivalent. We shall denote homotopy cofiber of the diagonal map in diagram (3.14) by

$$\Sigma^{q_{j+1}+1} \mathbb{L}M^{j+1}(\mathcal{X}) = \Sigma^{q_{j+1}+1} \mathbb{L}M^j(\mathcal{X}_j).$$

Thus for $0 \leq j \leq k$ the spectra $\mathbb{L}M^j(\mathcal{X})$ are defined. It follows from the definition that

$$\mathbb{L}M^j(\mathcal{X}) = \mathbb{L}M^j(\mathcal{X}_i), \text{ for } k \geq i \geq j.$$

We define groups $LM_j^i(\mathcal{X})$ as homotopy groups $\pi_j(\mathbb{L}M^i(\mathcal{X}))$. It follows from definition, that j is defined by mod 4.

Proposition 3.1. *Let \mathcal{X} be filtration (1.11). For $0 \leq i \leq k-2$ the groups LM fit in the following braid of exact sequences*

$$(3.15) \quad \begin{array}{ccccccc} \rightarrow & & LS_{l_j}(F_j) & \longrightarrow & LM_{l_j}^{j-1}(\mathcal{Y}) & \rightarrow & L_{n-1}(\pi_1(X_0 \setminus X_1)) \rightarrow \\ & \nearrow & \searrow & & \nearrow & \searrow & \nearrow & \searrow \\ & & & & LM_{l_j}^j(\mathcal{X}) & & LM_{l_{j-1}}^{j-2}(\mathcal{Y}) & \\ & \searrow & \nearrow & & \searrow & \nearrow & \searrow & \nearrow \\ \rightarrow & L_n(\pi_1(X_0 \setminus X_1)) & \rightarrow & LM_{l_{j-1}}^{j-1}(\mathcal{X}) & \rightarrow & LS_{l_{j-1}}(F_j) & \rightarrow, \end{array}$$

where l_j is the dimension of the bottom manifold of filtration. This diagram is realized on the spectra level.

Proof. From the definition of LM -groups we get a homotopy commutative square of the spectra

$$(3.16) \quad \begin{array}{ccc} \mathbb{L}M^{j-2}(\mathcal{Y}) & \rightarrow & \Sigma^{l_{j-1}-n+1} \mathbb{L}(\pi_1(X_0 \setminus X_1)) \\ \downarrow & & \downarrow \\ \Sigma^{q_{j+1}} \mathbb{L}S(F_j) & \rightarrow & \Sigma^{q_{j+1}} \mathbb{L}M^j(\mathcal{X}). \end{array}$$

The fibres of parallel maps in (3.16) are naturally homotopy equivalent — this follows from diagram (3.10). Hence square (3.16) is a pullback and consideration of homotopy long exact sequences of the maps from this square completes the proof of the theorem. \square

Corollary 3.2. *For $2 \leq j \leq k$ the spectrum $\mathbb{L}M^j(\mathcal{X})$ fits in the following pullback square of spectra*

$$(3.17) \quad \begin{array}{ccc} \Sigma^{q_j} \mathbb{L}M^j(\mathcal{X}) & \rightarrow & \Sigma^{q_j} \mathbb{L}M^{j-1}(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathbb{L}M^{j-1}(\mathcal{X}) & \rightarrow & \mathbb{L}M^{j-2}(\mathcal{Y}). \end{array}$$

□

We can now define the spectra for the structure sets of filtration \mathcal{X} . In accordance with Ranicki [30] we define a spectrum $\mathbb{S}^0(\mathcal{X}) = \mathbb{S}(X)$ for a manifold X^n as a homotopical fiber of the map (1.4).

For a filtration \mathcal{X} which is given by a manifold pair (X^n, Y^{n-q}) the map

$$H_n(X; \mathbf{L}_\bullet) \rightarrow LP_{n-q}(F)$$

from (1.15) is realized on the spectra level by a map of spectra (see [29], [30], [1], and [26])

$$(3.18) \quad X_+ \wedge \mathbf{L}_\bullet \rightarrow \Sigma^q \mathbb{L}P(F).$$

We denote the cofiber of the map in (3.18) by $\mathbb{S}^1(\mathcal{X}) = \mathbb{S}(X, Y, \xi)$ with homotopy groups

$$\pi_n(\mathbb{S}^1(\mathcal{X})) = S_n(X, Y, \xi)$$

fitting in the exact sequence (1.15).

For a filtration $\mathcal{X} = (Z \subset Y \subset X)$ we define the spectrum $\mathbb{S}^2(\mathcal{X}) = \mathbb{S}(X, Y, Z)$ (see [26]) as a homotopical cofiber of the map

$$(3.19) \quad X_+ \wedge \mathbf{L}_\bullet \rightarrow \Sigma^{q+q'} \mathbb{L}T(X, Y, Z).$$

A t -triangulation of filtration (1.11) \mathcal{X} gives t -triangulations of restricted filtrations \mathcal{X}_k , \mathcal{Y} , and \mathcal{Y}_{k-1} . Thus we obtain the following commutative diagram

$$(3.20) \quad \begin{array}{ccc} \mathcal{T}(\mathcal{X}) & \rightarrow & \mathcal{T}(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathcal{T}(\mathcal{X}_k) & \rightarrow & \mathcal{T}(\mathcal{Y}_k) \end{array}$$

which is realized on the spectra level (see [29], [30], and [26]). By Proposition 2.4 (see [30]), the diagram (3.20) on the spectra level has the following form

$$(3.21) \quad \mathcal{F} = \left(\begin{array}{ccc} (X_0)_+ \wedge \mathbf{L}_\bullet & \rightarrow & \Sigma^{q_1} [(X_1)_+ \wedge \mathbf{L}_\bullet] \\ \downarrow & & \downarrow \\ (X_0)_+ \wedge \mathbf{L}_\bullet & \rightarrow & \Sigma^{q_1} [(X_1)_+ \wedge \mathbf{L}_\bullet] \end{array} \right).$$

It follows from the definition of spectra $\mathbb{L}M^j(\mathcal{X})$ and from (1.4), (3.18) and (3.19) that for $k \geq j \geq 0$ we have the maps

$$(3.22) \quad (X_0)_+ \wedge \mathbf{L}_\bullet \rightarrow \Sigma^{n-l_j} \mathbb{L}M^j(\mathcal{X})$$

with cofiber which we shall denote by $\mathbb{S}^j(\mathcal{X})$. Thus $\mathbb{S}^0(\mathcal{X}) = \mathbb{S}(X_0)$. Using the maps from (3.22) we obtain a map Λ_2 of squares

$$(3.23) \quad \Lambda_2 : \mathcal{F} \longrightarrow \mathcal{G}_2$$

where

$$(3.24) \quad \mathcal{G}_2 = \left(\begin{array}{ccc} \Sigma^{n-l_2} \mathbb{L}M^2(\mathcal{X}) & \rightarrow & \Sigma^{n-l_2} \mathbb{L}M^1(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \Sigma^{n-l_1} \mathbb{L}M^1(\mathcal{X}) & \rightarrow & \Sigma^{n-l_1} \mathbb{L}M^0(\mathcal{Y}). \end{array} \right)$$

which gives a homotopy commutative diagram of spectra in form of a cube. Observe here that square in (3.24) follows from (3.17).

The cofibres of four maps which constitute the map Λ_2 give a pullback square

$$(3.25) \quad \begin{array}{ccc} \mathbb{S}^2(\mathcal{X}) & \rightarrow & \Sigma^{q_1} \mathbb{S}^1(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathbb{S}^1(\mathcal{X}) & \rightarrow & \Sigma^{q_1} \mathbb{S}^0(\mathcal{Y}). \end{array}$$

since squares (3.17) and (3.21) are pullback.

Let \mathcal{G}_i , $k \geq i \geq 2$ be a homotopy commutative square of spectra

$$(3.26) \quad \mathcal{G}_i = \left(\begin{array}{ccc} \Sigma^{n-l_i} \mathbb{L}M^i(\mathcal{X}) & \rightarrow & \Sigma^{n-l_i} \mathbb{L}M^{i-1}(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \Sigma^{n-l_{i-1}} \mathbb{L}M^{i-1}(\mathcal{X}) & \rightarrow & \Sigma^{n-l_{i-1}} \mathbb{L}M^{i-2}(\mathcal{Y}) \end{array} \right)$$

which follows from (3.17).

Proposition 3.3. *For $k \geq i \geq 2$ there exist maps*

$$(3.27) \quad \Lambda_i : \mathcal{F} \longrightarrow \mathcal{G}_i$$

of squares which are given by four maps in such a way that the resulting diagram in form of a cube is homotopy commutative.

Proof. Using induction on i it suffices to define the left upper map in Λ_i when the other three maps are already defined. This is possible since homotopy commutative square (3.26) is a pullback. \square

Definition. *Let \mathcal{X} be a filtration (1.11). For $k \geq j \geq 0$ we shall denote by $\mathbb{S}^j(\mathcal{X})$ a homotopical cofiber of the map*

$$(3.28) \quad (X_0)_+ \wedge \mathbf{L}\bullet \rightarrow \Sigma^{n-l_j} \mathbb{L}M^j(\mathcal{X})$$

which is given by the map Λ_i in (3.27). We shall denote the homotopy groups $\pi_n(\mathbb{S}^j(\mathcal{X}))$ by $\mathcal{S}_n^j(\mathcal{X})$.

The structure sets $\mathcal{S}_n^j(\mathcal{X})$ are natural generalizations of the structure sets of homotopy triangulations $\mathcal{S}_n(X)$ of a manifold X and homotopy triangulations of a manifold pair $\mathcal{S}_n(X, Y, \xi)$. Now we shall describe the main properties of the introduced sets.

Remark 3.4. Let \mathcal{X} be the filtration (1.11). It follows from definition for $k \geq j \geq 0$ that we have the exact sequence

$$(3.29) \quad \cdots \rightarrow \mathcal{S}_{n+1}^j(\mathcal{X}) \rightarrow H_n(X; \mathbf{L}_\bullet) \rightarrow LM_{i_j}^j(\mathcal{X}) \rightarrow \cdots$$

For $j = 0$ the exact sequence (3.29) coincides with (1.2) with $X = X_0$, for $j = 1$ it coincides with (1.15) for the pair $X_1 \subset X_0$, and for $j = 2$ it coincides with homotopy long exact sequence of cofibration (3.19) for the triple $X_2 \subset X_1 \subset X_0$. \square

Proposition 3.5. For $k \geq i \geq 2$ there exist the following homotopy commutative pullback squares of spectra

$$(3.30) \quad \begin{array}{ccc} \mathbb{S}^i(\mathcal{X}) & \rightarrow & \Sigma^{q_1} \mathbb{S}^{i-1}(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathbb{S}^{i-1}(\mathcal{X}) & \rightarrow & \Sigma^{q_1} \mathbb{S}^{i-2}(\mathcal{Y}). \end{array}$$

Proof. The square (3.30) obtained as square of homotopical cofibres of the maps constitute the map Λ_i . Squares (3.21) and (3.27) are pullback. Hence square (3.30) is a pullback. \square

Corollary 3.6. For $k \geq i \geq 2$ there exist the following braids of exact sequences

$$(3.31) \quad \begin{array}{ccccccc} \rightarrow & \mathcal{S}_n(X_0 \setminus X_1) & \longrightarrow & \mathcal{S}_n^{i-1}(\mathcal{X}) & \longrightarrow & LS_{l_i-1}(F_i) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & \mathcal{S}_n^i(\mathcal{X}) & & \mathcal{S}_{n-q_1}^{i-2}(\mathcal{Y}) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & LS_{l_i}(F_i) & \longrightarrow & \mathcal{S}_{n-q_1}^{i-1}(\mathcal{Y}) & \longrightarrow & \mathcal{S}_{n-1}(X_0 \setminus X_1) & \rightarrow \end{array}$$

where $\mathcal{S}_n(X_0 \setminus X_1)$ is the structure set fitting in algebraic surgery exact sequence (1.2) for $X_0 \setminus X_1$.

Proof. The homotopy long exact sequences of the maps from the pullback square (3.30) give the diagram (3.31). \square

Diagram (3.31) for the case of a manifold triple was obtained in [26]. In fact, this diagram is a natural generalization of the diagram [30, Proposition 7.2.6 ii)] which was given there for manifold pairs.

Proposition 3.7. For $k \geq i \geq 1$ there exist the following homotopy commutative pullback squares of spectra

$$(3.32) \quad \begin{array}{ccc} \Sigma^{-1} \mathbb{S}^{i-1}(\mathcal{X}) & \rightarrow & (X_0)_+ \wedge \mathbf{L}_\bullet \\ \downarrow & & \downarrow \\ \Sigma^{n-l_i} \mathbb{L}S(F_i) & \rightarrow & \Sigma^{n-l_i} \mathbb{L}M^i(\mathcal{X}). \end{array}$$

Proof. We have the following homotopy commutative diagram

$$(3.33) \quad \begin{array}{ccccc} \Sigma^{-1} \mathbb{S}^{i-1}(\mathcal{X}) & \rightarrow & (X_0)_+ \wedge \mathbf{L}_\bullet & \rightarrow & \Sigma^{n-l_i-1} \mathbb{L}M^{i-1}(\mathcal{X}) \\ \downarrow & & \downarrow & & \downarrow = \\ \Sigma^{n-l_i} \mathbb{L}S(F_i) & \rightarrow & \Sigma^{n-l_i} \mathbb{L}M^i(\mathcal{X}) & \rightarrow & \Sigma^{n-l_i-1} \mathbb{L}M^{i-1}(\mathcal{X}) \end{array}$$

where the right square follows from Proposition 3.3 and the left map is obtained as a natural map of fibres by [33]. We have such fibres by definition of the spectra $\mathbb{S}^i(\mathcal{X})$ and by Corollary 3.6. Now the cofibres of horizontal maps in the left square of 3.33 are naturally homotopy equivalent and this square is a pushout, and hence it is a pullback. \square

Corollary 3.8. *For $k \geq i \geq 1$ there exist the following braid of exact sequences*

$$(3.34) \quad \begin{array}{ccccccc} \rightarrow & \mathcal{S}_{n+1}^i(\mathcal{X}) & \longrightarrow & H_n(X_0; \mathbf{L}_\bullet) & \rightarrow & LM_{l_{i-1}}^{i-1}(\mathcal{X}) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & \mathcal{S}_{n+1}^{i-1}(\mathcal{X}) & & LM_{l_i}^i(\mathcal{X}) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & LM_{l_{i-1}+1}^{i-1}(\mathcal{X}) & \longrightarrow & LS_{l_i}(F_i) & \longrightarrow & \mathcal{S}_n^i(\mathcal{X}) & \rightarrow . \end{array}$$

Proof. The homotopy long exact sequences of the maps from pullback square (3.32) give commutative diagram of exact sequences (3.31). \square

Diagram (3.34) for the case of a manifold pair was obtained in [30, Proposition 7.2.6 iv)]. For the case of a manifold triple this diagram was obtained in [26, Theorem 4].

Theorem 3.9. *Let \mathcal{X} be the filtration (1.11) where dimension of the submanifold X_k is equal to $l_k \geq 5$, and let*

$$x = (f, b) \in [X_0, G/TOP] = H_n(X; \mathbf{L}_\bullet)$$

be a t -triangulation of \mathcal{X} with the given map $f : M \rightarrow X = X_0$. Then the map (f, b) is normally bordant to an s -triangulation of the filtration \mathcal{X} if and only if $\Theta_k(x) = 0$. We can identify the set $\mathcal{S}_{n+1}^i(\mathcal{X})$ with the set of concordance classes of s -triangulations of \mathcal{X}_i .

Proof. We use unduction on the number of submanifolds k . For $k = 1, 2$ the result was obtained in [30] and [26], respectively.

Let $x = [(f, b)] \in H_n(X_0; \mathbf{L}_\bullet)$ be an s -triangulation of filtration \mathcal{X}_k . It follows from definition that it is an s -triangulation of the subfiltration \mathcal{X}_{k-1} for which the restriction on X_{k-1} is already split along the submanifold $X_k \subset X_{k-1}$. Hence, by inductive hypothesis, $\Theta_{i-1}(x) = 0$ and it follows from (3.34) that x represents an element $y \in \mathcal{S}_{n+1}^{k-1}(\mathcal{X}_{k-1})$. It follows from diagram (3.31) that the map

$$\sigma : \mathcal{S}_{n+1}^{k-1}(\mathcal{X}) = \mathcal{S}^{k-1}(\mathcal{X}_{k-1}) \rightarrow LS_{l_k}(F_k)$$

in diagram (3.34) is given by the composition

$$(3.35) \quad \mathcal{S}_{n+1}^{k-1}(\mathcal{X}_{k-1}) \rightarrow \mathcal{S}_{l_{k-1}+1}(X_{k-1}) \rightarrow LS_{l_k}(F_k).$$

The last map in (3.35) is the map from diagram (1.7) for the the pair (X_{k-1}, X_k) . Since restriction of x on X_{k-1} splits along X_k by geometric definition of the map $\mathcal{S}_{l_{k-1}+1}(X_{k-1}) \rightarrow LS_{l_k}(F_k)$ we obtain that $\sigma(y) = 0$. Now it follows from commutativity of (3.34) that $\Theta_i(x) = 0$.

We prove now the reverse implication. Let $\Theta_k(x) = 0$. It follows from diagram (3.34) that $\Theta_{k-1}(x) = 0$. Hence there exists an element $y \in \mathcal{S}_{n+1}^{k-1}(\mathcal{X})$ which maps to x . The last set is identified with the classes of concordance of s -triangulations of the filtration \mathcal{X}_{k-1} . Hence the representative y gives an s -triangulation (f', b') of \mathcal{X}_{k-1} . Since $\Theta_k(x) = 0$ it follows by the commutativity (3.34) that y lies in the image of the map

$$\mathcal{S}_{n+1}^k(\mathcal{X}) \rightarrow \mathcal{S}_{n+1}^{k-1}(\mathcal{X})$$

from (3.34). Hence $\sigma(x) = 0$, and by decomposition (3.35) the restriction of the map f' to X_{k-1} splits along the submanifold X_k . We can extend a homotopy to obtain an s -triangulation of \mathcal{X}_{k-1} for which the restriction on X_{k-1} is an s -triangulation of the pair (X_k, X_{k-1}) . Now application of Proposition 2.5 finishes the proof of the theorem. \square

For filtration (1.11) we now describe the relations between surgery obstruction groups $LM_*^i(\mathcal{X}) = LM^i(\mathcal{X}_i)$ introduced above and stratified L -groups $L_*^{BQ}(\mathcal{X}_i)$ of Browder and Quinn (see [3] and [35]).

The Browder-Quinn groups of filtration \mathcal{X} are realized on the spectra level and we recall here an inductive definition of these groups from [35, p. 129] using our notations. In accordance with Theorem 2.9 we have a homotopy equivalence of spectra

$$LM^2(\mathcal{X}) \simeq \Sigma^{l_2-n} \mathbb{L}^{BQ}(\mathcal{X}_2).$$

It is necessary to remark here that in a similar way a homotopy equivalence of spectra immediately follows from (2.10), (2.12), and definition [3, page 129]

$$(3.36) \quad \mathbb{L}M^1(\mathcal{X}) = \mathbb{L}P(F_1) \simeq \Sigma^{l_1-n} \mathbb{L}^{BQ}(\mathcal{X}_1).$$

By Remark 2.3, filtration \mathcal{X} gives filtration $\overline{\mathcal{X}}$ of manifolds with boundaries. The boundaries of the last filtration give a filtration by closed manifolds

$$(3.37) \quad \partial(X_{k-1} \setminus X_k) \subset \partial(X_{k-2} \setminus X_k) \subset \cdots \subset \partial(X_1 \setminus X_k) \subset \partial(X_0 \setminus X_k)$$

which we shall denote by $\partial\overline{\mathcal{X}}$. Note that filtrations $\partial\overline{\mathcal{X}}$ and $\overline{\mathcal{X}}$ contain k spaces, and filtration \mathcal{X} contains $k+1$ spaces.

Consider a homotopy commutative diagram of spectra

$$(3.38) \quad \begin{array}{ccccccc} \mathbb{L}M^j(\overline{\mathcal{X}}) & \rightarrow & \mathbb{L}M^{j-1}(\overline{\mathcal{Y}}) & \rightarrow & \Sigma^{l_j-n+1} \mathbb{L}(C) & \rightarrow & \Sigma \mathbb{L}M^j(\overline{\mathcal{X}}) \\ & & \downarrow & & \downarrow = & & \downarrow \\ \mathbb{L}M^j(\mathcal{X}) & \rightarrow & \mathbb{L}M^{j-1}(\mathcal{Y}) & \rightarrow & \Sigma^{l_j-n+1} \mathbb{L}(C) & \rightarrow & \Sigma \mathbb{L}M^j(\mathcal{X}) \end{array}$$

where $j = k-1 \geq 1$, $C = \pi_1(X_0 \setminus X_1)$, and horizontal rows are cofibrations by (3.14). The vertical maps in (3.38) are induced by a natural inclusion of filtrations $\overline{\mathcal{X}}_k \subset \mathcal{X}_{k-1}$. For $k-1 = j = 1$ the central square of (3.38) fits in a homotopy commutative diagram of spectra

$$(3.39) \quad \begin{array}{ccc} \mathbb{L}M^0(\overline{\mathcal{Y}}) & \rightarrow & \Sigma^{l_1-n+1} \mathbb{L}(C) \\ & \downarrow & \downarrow = \\ \Sigma^{l_1-l_2} \mathbb{L}M^1(\mathcal{Y}) & \rightarrow & \Sigma^{l_1-n+1} \mathbb{L}(C) \\ & \downarrow & \downarrow = \\ \mathbb{L}M^0(\mathcal{Y}) & \rightarrow & \Sigma^{l_1-n+1} \mathbb{L}(C) \end{array}$$

which follows from diagram (3.14) and diagram (1.8) for the pair (X_1, X_2) on spectra level.

Proposition 3.10. *Let \mathcal{X} be filtration (1.11) with $k \geq 2$. There exists the following homotopy commutative diagram of spectra*

$$(3.40) \quad \begin{array}{ccccc} \mathbb{L}M^{k-2}(\overline{\mathcal{Y}}) & \rightarrow & \Sigma^{l_{k-1}-n+1}\mathbb{L}(C) & \rightarrow & \Sigma\mathbb{L}M^{k-1}(\overline{\mathcal{X}}) \\ & & \downarrow = & & \downarrow \\ \Sigma^{q_k}\mathbb{L}M^{k-1}(\mathcal{Y}) & \rightarrow & \Sigma^{l_{k-1}-n+1}\mathbb{L}(C) & \rightarrow & \Sigma^{q_k+1}\mathbb{L}M^k(\mathcal{X}) \\ & & \downarrow = & & \downarrow \\ \mathbb{L}M^{k-2}(\mathcal{Y}) & \rightarrow & \Sigma^{l_{k-1}-n+1}\mathbb{L}(C) & \rightarrow & \Sigma\mathbb{L}M^{k-1}(\mathcal{X}) \end{array}$$

where $l_{k-1} - l_k = q_k$ and $C = \pi_1(X_0 \setminus X_1)$. The right vertical composition coincides with the right vertical map in diagram (3.38) for $j = k - 1$.

Proof. For $k = 2$ the result follows from diagrams (3.39) and (3.38) if we define the right vertical maps in (3.40) as natural maps of homotopical cofibers of the horizontal maps from (3.39) (see [33]). Induction on k now finishes proof of the proposition. \square

Corollary 3.11. *Let \mathcal{X} be filtration (1.11) with $k \geq 1$. Then a homotopical fiber of the map*

$$(3.41) \quad \mathbb{L}M^{k-1}(\overline{\mathcal{X}}) \rightarrow \Sigma^{q_k}\mathbb{L}M^k(\mathcal{X})$$

is naturally homotopy equivalent to $\Sigma^{q_k-1}\mathbb{L}(\pi_1(X_k))$.

Proof. For $k = 1$ the result follows from definition of spectra $\mathbb{L}M^0, \mathbb{L}M^1 = \mathbb{L}P(F_1)$, and cofibration (2.12). For $k \geq 2$ the result follows inductively from a homotopy commutative diagram

$$(3.42) \quad \begin{array}{ccccc} \Sigma^{l_{k-1}-n}\mathbb{L}(C) & \rightarrow & \mathbb{L}M^{k-1}(\overline{\mathcal{X}}) & \rightarrow & \mathbb{L}M^{k-2}(\overline{\mathcal{Y}}) \\ & & \downarrow = & & \downarrow \\ \Sigma^{l_{k-1}-n}\mathbb{L}(C) & \rightarrow & \Sigma^{q_k}\mathbb{L}M^k(\mathcal{X}) & \rightarrow & \Sigma^{q_k}\mathbb{L}M^{k-1}(\mathcal{Y}) \end{array}$$

which follows from diagram (3.40). The right square in (3.42) is pullback since fibers of horizontal maps are naturally homotopy equivalent. Hence fibers of the vertical maps are naturally homotopy equivalent, too. By inductive hypothesis fiber of the right vertical map is naturally homotopy equivalent to $\Sigma^{q_k-1}\mathbb{L}(\pi_1(X_k))$. From this result of the corollary follows. \square

Recall now, that in [35, page 129] an inductive definition of spectra $\mathbb{L}^{BQ}(\mathcal{X})$ is given with homotopy groups

$$\pi_n(\mathbb{L}^{BQ}(\mathcal{X})) = L_n^{BQ}(\mathcal{X})$$

which are Browder-Quinn stratified L -groups [3].

Theorem 3.12. *Let \mathcal{X} be the filtration (1.11) with the smallest manifold of filtration X_k of dimension l_k . We have a naturally homotopy equivalence*

$$(3.43) \quad \mathbb{L}M^k(\mathcal{X}) \simeq \Sigma^{l_k-n}\mathbb{L}^{BQ}(\mathcal{X})$$

Proof. For $k \geq 1$ the cofibration (3.41) yields a cofibration

$$\mathbb{L}(\pi_1(X_k)) \rightarrow \Sigma^{1-q_k}\mathbb{L}M^{k-1}(\overline{\mathcal{X}}) \rightarrow \Sigma^1\mathbb{L}M^k(\mathcal{X})$$

which coincides with cofibration for inductive definition of the spectra $\mathbb{L}^{BQ}(\mathcal{X})$ [35, page 129] up to a shift of dimension of spectra. \square

Proposition 3.13. *Let \mathcal{X} be filtration (1.11) with $k \geq 2$. We have the following braid of exact sequences*

$$(3.44) \quad \begin{array}{ccccccc} \rightarrow & L_{l_k+1}(X_k) & \rightarrow & LM_{l_{k-1}}^{k-2}(\bar{\mathcal{Y}}) & \rightarrow & L_{n-1}(C) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & LM_{l_{k-1}}^{k-1}(\bar{\mathcal{X}}) & & LM_{l_k}^{k-1}(\mathcal{Y}) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & L_n(C) & \rightarrow & LM_{l_k}^k(\mathcal{X}) & \rightarrow & L_{l_k}(X_k) & \rightarrow \end{array}$$

where $C = \pi_1(X_0 \setminus X_1)$.

Proof. The homotopy long exact sequences of the maps from the right pullback square of diagram (3.42) give commutative braid of exact sequences (3.44). \square

4. Application to Browder-Livesay invariants.

We shall call a filtration \mathcal{X} (1.11) a Browder-Livesay filtration if for every $k \geq j \geq 1$ the pair of manifolds $X_j \subset X_{j-1}$ is a Browder-Livesay pair. Note that F_j , $1 \leq j \leq k$, is a square of fundamental groups for splitting problem for the pair of manifolds $X_j \subset X_{j-1}$ from filtration (1.11). For a Browder-Livesay filtration any manifold X_j is an one-sided submanifold of codimension 1 in X_{j-1} , horizontal maps in squares F_j are isomorphisms, and vertical maps are inclusions of index 2.

Theorem 4.1. *Let \mathcal{X} be a Browder-Livesay filtration (1.11) in which all the squares F_j for $1 \leq j \leq k$ are the same. Then filtration of spectra (1.14) has the following form*

$$(4.1) \quad \Sigma^k \mathbb{L}M^k \rightarrow \Sigma^{k-1} \mathbb{L}M^{k-1} \rightarrow \dots \rightarrow \Sigma^1 \mathbb{L}M^1 \rightarrow \mathbb{L}M^0$$

and coincides with the left part beginning with $\mathbb{X}_{0,0}$ of filtration (1.10) for spectral sequence of Hambleton and Kharshiladze.

Proof. It follows from Corollary (3.2) that the spectrum $\mathbb{L}M^j(\mathcal{X})$ is defined inductively. This spectrum is constructed using three others spectra from diagram (3.17) to obtain a pullback square. But the spectrum $\mathbb{X}_{j,0}$ of filtration (1.10) is defined inductively using the same construction (see [13] and [26]). \square

Let $i_+ : A \rightarrow B^+$ be an inclusion of groups of index 2 as in square (1.9). For such inclusion an algebraic version of diagram (1.8) was constructed by Ranicki in [31]. It has the following form

$$(4.2) \quad \begin{array}{ccccccc} \rightarrow & L_{n+1}(A) & \longrightarrow & L_{n+1}(B^+) & \xrightarrow{\partial} & LN_{n-1}(A \rightarrow B^+) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & L_{n+1}(i_-^*) & & L_{n+1}(A \rightarrow B^+) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & LN_n(A \rightarrow B^+) & \longrightarrow & L_n(B^+) & \longrightarrow & L_n(A) & \rightarrow \end{array}$$

For the Browder-Livesay pair $Y \subset X$ with the square (1.9) of fundamental groups diagram (1.8) coincides with diagram (4.2). The map ∂ is called Browder-Livesay invariant. If $\partial(x) \neq 0$ then no element $x \in L_{n+1}(B)$ can be realized by a normal map of closed manifolds (see [9]).

This diagram is realized on the spectra level and we can write down the following pullback square of spectra

$$(4.3) \quad \begin{array}{ccc} & \mathbb{L}(B) & \\ \nearrow & & \searrow \\ \mathbb{L}(i^*) & & \mathbb{L}(A \rightarrow B) \\ \searrow & & \nearrow \\ & \Sigma\mathbb{L}(B^\epsilon) & \end{array}$$

where i^* denotes i_-^* and B^ϵ means that orientation on the bottom group B differs from orientation of the upper group B outside of the image of the map $i : A \rightarrow B$. Consider a sequence \mathcal{A} of inclusions of subgroups of index 2 into a group B with an orientation

$$(4.4) \quad i_1 : A_1 \rightarrow B; i_2 : A_2 \rightarrow B; \dots; i_k : A_k \rightarrow B; \dots$$

Every inclusion in (4.4) gives a pullback square similar to 4.3 and we can write down the following column of pullback squares

$$(4.5) \quad \begin{array}{ccccc} & & \mathbb{L}(B) & & \\ & \nearrow & & \searrow & \\ & \mathbb{L}(i_1^*) & & & \mathbb{L}(A_1 \rightarrow B) \\ & \searrow & & \nearrow & \\ & & \Sigma\mathbb{L}(B^{\epsilon_1}) & & \\ & \nearrow & & \searrow & \\ & \Sigma\mathbb{L}(i_2^*) & & & \Sigma\mathbb{L}(A_2 \rightarrow B) \\ & \searrow & & \nearrow & \\ & & \Sigma^2\mathbb{L}(B^{\epsilon_2}) & & \\ & \nearrow & & \searrow & \\ & \Sigma^2\mathbb{L}(i_3^*) & & & \mathbb{L}(A_3 \rightarrow B) \\ & \searrow & & \nearrow & \\ & & \Sigma^3\mathbb{L}(B^{\epsilon_3}) & & \\ & & \vdots & & \\ & & \Sigma^{k-1}\mathbb{L}(B) & & \\ & \nearrow & & \searrow & \\ & \Sigma^{k-1}\mathbb{L}(i_k^*) & & & \Sigma^{k-1}\mathbb{L}(A_k \rightarrow B) \\ & \searrow & & \nearrow & \\ & & \Sigma^k\mathbb{L}(B^{\epsilon_k}) & & \\ & & \vdots & & \end{array}$$

in which we have the same agreement on orientations as in the square (4.3).

Let $\mathbb{L}M^0(\mathcal{A}) = \mathbb{L}(B)$ and $\Sigma\mathbb{L}M^1(\mathcal{A}) = \mathbb{L}(i_1^*)$. Using the pullback construction we can extend diagram to the left direction similarly to [13]. In particular, we obtain a filtration of spectra

$$(4.6) \quad \dots \rightarrow \Sigma^k\mathbb{L}M^k(\mathcal{A}) \rightarrow \Sigma^{k-1}\mathbb{L}M^{k-1}(\mathcal{A}) \rightarrow \dots \rightarrow \Sigma^1\mathbb{L}M^1(\mathcal{A}) \rightarrow \mathbb{L}M^0(\mathcal{A})$$

which is the left upper diagonal row of the diagram.

We can use filtration (4.6) to construct a surgery spectral sequence

$$E_r^{p,q} = E_r^{p,q}(\mathcal{A})$$

for a sequence of inclusions \mathcal{A} similar to [13]. We define the first term

$$(4.7) \quad E^{p,q} = \pi_{q-p}(\Sigma^p \mathbb{L}M^p(\mathcal{A}), \Sigma^{p+1} \mathbb{L}M^{p+1}(\mathcal{A})) \cong LN_{q-2p-2}(A_p \rightarrow B^{\epsilon_p})$$

and the first differential

$$d_1^{p,q} : E_1^{p,q} \rightarrow E_1^{p+1,q}$$

which coincides with the composition

$$(4.8) \quad LN_{q-2p-2}(A_{p+1} \rightarrow B^{\epsilon_p}) \rightarrow L_{q-2p-2}(B^{\epsilon_{p+1}}) \rightarrow LN_{q-2p}(A_{p+2} \rightarrow B^{\epsilon_{p+1}}).$$

The first map of the composition 4.8 lies in the diagram (4.3) for the inclusion

$$A_{p+1} \rightarrow B^{\epsilon_p},$$

and the second map lies in the same diagram for the inclusion

$$A_{p+2} \rightarrow B^{\epsilon_{p+1}}$$

(see [13]). Note that the obtained spectral sequence is a natural generalization of the spectral sequence constructed in [13]. General results about surgery spectral sequence from [13] are applicable to obtained surgery spectral sequence.

Remark 4.2. *Let in (4.4) all the groups A_i be equal to A . Then the surgery spectral sequence constructed above coincides with the spectral sequence from [13] for the inclusion $A \rightarrow B$ of index 2.*

Note that a finite sequence \mathcal{A} of inclusions i_j , $1 \leq j \leq k$ as (4.4) gives a finite filtration

$$(4.9) \quad \Sigma^k \mathbb{L}M^k(\mathcal{A}) \rightarrow \Sigma^{k-1} \mathbb{L}M^{k-1}(\mathcal{A}) \rightarrow \dots \rightarrow \Sigma^1 \mathbb{L}M^1(\mathcal{A}) \rightarrow \mathbb{L}M^0(\mathcal{A})$$

of spectra. Browder-Livesay filtration \mathcal{X} (1.11) gives a finite sequence of squares

$$(4.10) \quad F_j = \begin{pmatrix} A_j & \xrightarrow{\cong} & A_j \\ \downarrow i_- & & \downarrow i_+ \\ B^{\epsilon_j} & \xrightarrow{\cong} & B^{\epsilon_{j-1}} \end{pmatrix}$$

of fundamental groups for $1 \leq j \leq k$ which are similar to (1.9) Right vertical inclusions from (4.10) give finite sequence $\mathcal{A}(\mathcal{X})$ of inclusions of index 2 into the group B .

Proposition 4.3. *Under the assumptions above we have*

$$\mathbb{L}M^j(\mathcal{X}) = \mathbb{L}M^j(\mathcal{A}(\mathcal{X}))$$

for $1 \leq j \leq k$.

Proof. The same as the proof of Theorem 4.1. \square

For the sequence of inclusions \mathcal{A} (4.4) we can construct filtration of spectra (4.6). We denote the homotopy groups of the spectra from this filtration as follows

$$\pi_n(\mathbb{L}M^k(\mathcal{A})) = LM_n^k(\mathcal{A}).$$

Thus filtration (4.6) gives a tower of groups

$$(4.11) \quad \cdots \rightarrow LM_{n-j}^j(\mathcal{A}) \rightarrow LM_{n-j+1}^{j-1}(\mathcal{A}) \rightarrow \cdots \rightarrow LM_{n-1}^1(\mathcal{A}) \rightarrow LM_n^0(\mathcal{A}) = L_n(B)$$

Denote by ϕ_j the map

$$(4.12) \quad LM_{n-j}^j(\mathcal{A}) \rightarrow LM_n^0(\mathcal{A}) = L_n(B)$$

given by a composition of maps from (4.11). The map ϕ_1 is the map s in diagram (4.2).

Theorem 4.4. *Suppose that an element $x \in L_n(B)$, where n is given by mod 4, does not lie in the the image of ϕ_j for some sequence of inclusions \mathcal{A} and some natural number j . Then x cannot be realized by a normal map of closed manifolds.*

Proof. Let the element $x \in L_n(B)$ be realized by a normal map of closed manifolds $(f, b) : M^n \rightarrow X^n$. In accordance with [34, §9] we can take a product of this surgery problem with the projective complex space $P_2(\mathbb{C})$ of dimension 4 to obtain the surgery problem

$$f \times Id : M^n \times P_2(\mathbb{C}) \rightarrow X^n \times P_2(\mathbb{C})$$

in dimension $n+4$ with surgery obstruction $x \in L_n(B)$. Iterating this construction we can obtain a normal map of closed manifolds

$$g = f \times Id : M \times (P_2(\mathbb{C}))^k \rightarrow X \times (P_2(\mathbb{C}))^k = X_0$$

in dimension $m = n + 4k \geq j + 5$ with surgery obstruction $\Theta_0(g, b') = x \in L_n(B)$. Denote by \mathcal{A} the sequence of inclusions

$$(4.13) \quad i_1 : A_1 \rightarrow B; i_2 : A_2 \rightarrow B; \dots; i_j : A_j \rightarrow B,$$

which defines the map ϕ_j . Consider a map

$$\psi_1 : X_0 \rightarrow P_N(\mathbb{R}),$$

which induces epimorphism of fundamental group with kernel A_1 . Here $P_N(\mathbb{R})$ is a real projective space of high dimension. By changing the map ψ_1 in its homotopy class we can assume that ψ_1 is transversal to $P_{N-1}(\mathbb{R}) \subset P_N(\mathbb{R})$ with $\psi_1^{-1}(P_{N-1}(\mathbb{R})) = X_1$ and that $X_1 \subset X_0$ is a Browder-Livesay pair (see [5], [10], [13], and [17]). Now in a similar way we consider the map

$$\psi_2 : X_1 \rightarrow P_N(\mathbb{R}),$$

which induces epimorphism of the fundamental groups with kernel A_2 with $\psi_2^{-1}(P_{N-1}(\mathbb{R})) = X_2$ and with $X_2 \subset X_1$ is a Browder-Livesay pair. Iterating this construction we obtain a Browder-Livesay filtration \mathcal{X}

$$(4.14) \quad X_j \subset X_{j-1} \subset \cdots \subset X_1 \subset X_0$$

with $\mathcal{A}(\mathcal{X}) = \mathcal{A}$. From Corollary 3.8 we obtain the following commutative diagram

$$(4.15) \quad \begin{array}{ccc} H_m(X_0; \mathbf{L}\bullet) & \xrightarrow{\Theta_0} & L_m(B) \\ & \searrow \Theta_j & \nearrow \phi_j \\ & & LM_{n-j}^j \end{array}$$

It follows from (4.15) that

$$\phi_j \Theta_j(g, b') = \Theta_0(g, b') = x \in L_m(B)$$

and hence the element x lies in the image of ϕ_j . We have obtained a contradiction and thus the theorem is proved. \square

An element of $L_n(B)$ doesn't lie in the image ϕ_1 if and only if it maps nontrivially by Browder-Livesay invariant ∂ from (4.2). An element of $L_n(B)$ doesn't lie in the image ϕ_2 if and only if first or second invariant of Browder-Livesay are nontrivial. The second Browder-Livesay invariant was introduced by Hambleton in [10] (see also [27]) and it is defined only if Browder-Livesay invariant is trivial. The iterated Browder-Livesay invariants were introduced by Kharshiladze (see [16], [17], and [22]). Elements of $L_n(B)$ which don't lie in the image of ϕ_j for some j (and only these elements) detected by iterated Browder-Livesay invariants as follows immediately from [13], [16], and [17]. The nonrealizability of such elements by a normal map of closed manifolds was proved by Kharshiladze (see [16] and [17]) by geometric methods.

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