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ON SPLITTING PROBLEM FOR MANIFOLD PAIRS WITH BOUNDARIES

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ABSTRACT. The problem of splitting a homotopy equivalence along a submanifold closely relates to surgery exact sequence and to the problem of surgery of manifold pairs. In the classical surgery theory there exist two approaches to surgery in the category of manifolds with boundaries. In the *rel* ∂ case the surgery on a manifold pair is considered with the given fixed manifold structure on the boundary. In the relative case the surgery on the manifold with boundary is considered without fixing maps on the boundary. Consider a normal map to a manifold pair $(Y, \partial Y) \subset (X, \partial X)$ with boundaries which is a simple homotopy equivalence on the boundary ∂X . This map defines a mixed structure on the manifold with the boundary in the sense of Wall. We introduce and study groups of obstructions to splitting of such mixed structures along submanifold with boundary $(Y, \partial Y)$. We describe relations of these groups to classical surgery and splitting obstruction groups. Then we consider several geometrical examples.

1. Introduction.

Let $(X^n, \partial X)$ be a closed topological manifold of dimension n with boundary. The set $\mathcal{S}^{CAT}(X, \partial X)$ of *CAT*-manifold structures ($CAT = TOP, PL, DIFF$) on $(X, \partial X)$ consists of the classes of concordance of simple homotopy equivalences of pairs $f: (M, \partial M) \rightarrow (X, \partial X)$, where $(M, \partial M)$ is a compact *CAT*-manifold pair of dimension n with boundary (see [7], [8], and [11]). If ∂X already has a *CAT*-manifold structure then the set of manifold structures on X which are fixed on the boundary is denoted by $\mathcal{S}_\partial^{CAT}(X, \partial X)$.

Let $\mathcal{T}^{CAT}(X, \partial X)$ be the set of classes of normal bordisms of normal maps to the pair $(X, \partial X)$ and $\mathcal{T}_\partial^{CAT}(X, \partial X)$ the set of *rel* ∂ classes of normal bordisms of normal maps (see [7], [8] and [11]).

Let $Y \subset X$ be a submanifold of codimension q in a closed manifold X^n . For a normal map $(f, b): M^n \rightarrow X^n$ there exists a problem to find in the class of normal bordism $[(f, b)]$ a simple homotopy equivalence $g: M_1 \rightarrow X$ which is transversal to Y and $N = g^{-1}(Y)$ is a submanifold of M_1 with the restrictions

$$(1.1) \quad g|_N: N \rightarrow Y, \quad g|_{M_1 \setminus N}: M_1 \setminus N \rightarrow X \setminus Y$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

are simple homotopy equivalences. The obstruction group to do such surgery is denoted by $LP_{n-q}(F)$ (see [11] and [8]) where

$$(1.2) \quad F = \left(\begin{array}{ccc} \pi_1(S(\xi)) & \longrightarrow & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(D(\xi)) & \longrightarrow & \pi_1(X) \end{array} \right),$$

is a pushout square of fundamental groups with orientations and $S(\xi)$ is a boundary of tubular neighborhood $D(\xi)$ of Y in X .

If $f : M \rightarrow X$ is a simple homotopy equivalence then an obstruction to find a map which is transversal to Y with properties (1.1) in the homotopy class of the map f lies in a splitting obstruction group $LS_{n-q}(F)$ (see [11, §11] and [8, §7.2]).

Let $(Y, \partial Y) \subset (X, \partial X)$ be a codimension q manifold pair with boundaries. In *rel* ∂ case the set $\mathcal{T}_\partial^{CAT}(X, \partial X)$ of tangential structures consists of classes of concordance of normal maps

$$(f, \partial f) : (M, \partial M) \rightarrow (X, \partial X)$$

with fixed map

$$\partial f : \partial M \rightarrow \partial X$$

which is already split on the boundary. We have a map

$$(1.3) \quad \mathcal{T}_\partial^{CAT}(X, \partial X) \rightarrow LP_{n-q}(F)$$

which is given by mapping an obstruction to surgery to the normal map of manifold pairs relative to the boundary.

In a similar way we have a map

$$(1.4) \quad \mathcal{S}_\partial^{CAT}(X, \partial X) \rightarrow LS_{n-q}(F)$$

to the splitting obstruction group.

It follows from [11, §11, page 136] (see also [12]) that for the relative case we have maps

$$(1.5) \quad \mathcal{T}^{CAT}(X, \partial X) \rightarrow LP_{n-q}(F_\partial \rightarrow F)$$

similarly to (1.3) and (1.4) and

$$(1.6) \quad \mathcal{S}^{CAT}(X, \partial X) \rightarrow LS_{n-q}(F_\partial \rightarrow F)$$

to relative obstruction groups where F_∂ is a pushout square for a splitting problem of the pair $\partial Y \subset \partial X$.

In accordance with Wall [11, §10, p. 116] (see also [3]) it is possible to introduce a mixed type of structures on a manifold with boundary $(X, \partial X)$. Consider a normal map

$$(f, \partial f) : (M, \partial M) \rightarrow (X, \partial X)$$

for which the map

$$\partial f : \partial M \rightarrow \partial X$$

is a simple homotopy equivalence. Two such maps are concordant if they are normally bordant by a bordism for which a restriction to bordism between the boundaries is an equivalence in $S^{CAT}(\partial X)$. Denote the set of concordance classes by $\mathcal{TS}^{CAT}(X, \partial X)$. The elements of $\mathcal{TS}^{CAT}(X, \partial X)$ are called mixed structures on $(X, \partial X)$.

In the present paper we shall work in *TOP*-category and simple surgery obstruction groups (see [7] and [8]).

We think of all surgery and splitting obstruction groups are equipped by decoration "s" although we do not write it.

We introduce groups $LPS_*(F_\partial \rightarrow F)$ and define a map

$$\psi : \mathcal{TS}(X, \partial X) \rightarrow LPS_*(F_\partial \rightarrow F)$$

which gives an obstruction to find in the class of concordance a map which is split along the submanifold pair $(Y, \partial Y) \subset (X, \partial X)$.

We study properties of the introduced groups and their relations to surgery and splitting obstruction groups. The main results are given by braids of exact sequences. Then we consider several geometrical examples in which we compute the introduced groups and natural maps.

In Section 2 we give explicit definitions of different structure sets and recall the necessary technical results about algebraic surgery exact sequences of Ranicki and surgery *L*-spectra.

In Section 3 we recall main properties of splitting obstruction groups and introduce LPS_* -groups. These groups are realized as homotopy groups of a spectrum. We describe algebraic properties of these groups and relations of these groups to surgery and splitting obstruction groups and to surgery exact sequence.

In Section 4 we consider geometric examples in which we compute LPS_* -groups and natural maps between introduced groups and classical obstruction groups which arise naturally in the considered problem.

2. Structure sets and surgery exact sequence.

In definitions of structure sets we shall follow Ranicki [8]. Let X^n be a closed topological manifold. A *t*-triangulation of X is a topological normal map (see [8, §1.2])

$$(f, b) : M \rightarrow X,$$

where M is n -dimensional closed topological manifold. Two *t*-triangulations

$$(f_i, b_i) : M_i \rightarrow X, \quad i = 0, 1$$

are concordant [8, §7.1] if there exists a topological normal map of triads

$$((g, c); (f_0, b_0), (f_1, b_1)) : (W; M_0, M_1) \rightarrow (X \times I; X \times \{0\}, X \times \{1\})$$

where $I = [0, 1]$ and W is a compact $(n + 1)$ -dimensional manifold with a boundary $\partial W = M_0 \cup M_1$. The set of concordance classes of *t*-triangulations of X is denoted by $\mathcal{T}^{TOP}(X)$. Note that we shall consider the case of a manifold X and hence the set $\mathcal{T}^{TOP}(X)$ will be non empty.

An *s*-triangulation of a closed topological manifold X^n is a simple homotopy equivalence $f : M \rightarrow X$ where M is a closed topological n -dimensional manifold.

Two s -triangulations

$$(f_i, b_i) : M_i \rightarrow X, \quad i = 0, 1$$

are concordant [8, §7.1] if there exists a simple homotopy equivalence of triads

$$(g; f_0, f_1) : (W; M_0, M_1) \rightarrow (X \times I; X \times \{0\}, X \times \{1\})$$

where W is a compact $(n + 1)$ -dimensional manifold with the boundary $\partial W = M_0 \cup M_1$. The set of concordance classes of s -triangulations of X is denoted by $\mathcal{S}^{TOP}(X)$. This set is called the topological manifold structure set. As before the set $\mathcal{S}^{TOP}(X)$ will be non empty. These sets fits in the surgery exact sequence (see [8, §7] and [11])

$$(2.1) \quad \cdots L_{n+1}(\pi_1(X)) \rightarrow \mathcal{S}^{TOP}(X) \rightarrow \mathcal{T}^{TOP}(X) \rightarrow L_n(\pi_1(X))$$

where $L_*(\pi_1(X))$ are surgery obstruction groups.

Now consider the case of a compact n -dimensional manifold X with the boundary ∂X .

At first, we consider a case of structures which are fixed on the boundary. This is the *rel* ∂ case. A t_∂ -triangulation of $(X, \partial X)$ is a topological normal map of pairs (see [8, §7.1])

$$((f, b), (\partial f, \partial b)) : (M, \partial M) \rightarrow (X, \partial X)$$

with a homeomorphism $\partial f : \partial M \rightarrow \partial X$. Two t_∂ -triangulations

$$((f_i, b_i), (\partial f_i, \partial b_i)) : (M_i, \partial M_i) \rightarrow (X, \partial X), \quad i = 0, 1$$

are concordant if there exists a topological normal map

$$\begin{aligned} ((h, d); (g, c), (f_0, b_0), (f_1, b_1)) : \\ (W; V, M_0, M_1) \end{aligned} \longrightarrow (X \times I; \partial X \times I, \partial X \times \{0\}, \partial X \times \{1\})$$

with

$$V = \partial M_0 \times I, \quad \partial V = \partial M_0 \cup \partial M_1$$

and

$$(g, c) = \partial f_0 \times I : V \rightarrow \partial X \times I.$$

The set of concordance classes is denoted by $\mathcal{T}_\partial^{CAT}(X, \partial X)$ (see [11, §10] and [8, §7.1]).

An s_∂ -triangulation of $(X, \partial X)$ is a simple homotopy equivalence of pairs (see [8, §7.1])

$$(f, \partial f) : (M, \partial M) \rightarrow (X, \partial X)$$

with a homeomorphism $\partial f : \partial M \rightarrow \partial X$. Two s_∂ -triangulations

$$(f_i, \partial f_i) : (M_i, \partial M_i) \rightarrow (X, \partial X), \quad i = 0, 1$$

are concordant if there exists a simple homotopy equivalence of 4-ads

$$(h; g, f_0, f_1) : (W; V, M_0, M_1) \rightarrow (X \times I; \partial X \times I, \partial X \times \{0\}, \partial X \times \{1\})$$

with

$$V = \partial M_0 \times I, \quad \partial V = \partial M_0 \cup \partial M_1$$

and

$$g = \partial f_0 \times I : V \rightarrow \partial X \times I.$$

The set of concordance classes is denoted by $\mathcal{S}_\partial^{CAT}(X, \partial X)$ (see [11, §10] and [8, §7.1]).

These sets fit in the surgery exact sequence (see [11, §10] and [8, §7])

$$(2.2) \quad \cdots \rightarrow L_{n+1}(\pi_1(X)) \rightarrow \mathcal{S}_\partial^{TOP}(X, \partial X) \rightarrow \mathcal{T}_\partial^{TOP}(X, \partial X) \rightarrow L_n(\pi_1(X)).$$

Now consider the relative case of structures on a manifold with boundary. A t -triangulation of $(X, \partial X)$ is a topological normal map of pairs (see [8, §7.1])

$$((f, b), (\partial f, \partial b)) : (M, \partial M) \rightarrow (X, \partial X).$$

Two t -triangulations

$$((f_i, b_i), (\partial f_i, \partial b_i)) : (M_i, \partial M_i) \rightarrow (X, \partial X), \quad i = 0, 1$$

are concordant if there exists a topological normal map of 4-ads

$$((h, d); (g, c), (f_0, b_0), (f_1, b_1)) : (W; V, M_0, M_1) \rightarrow (X \times I; \partial X \times I, \partial X \times \{0\}, \partial X \times \{1\})$$

with

$$\partial V = \partial M_0 \cup \partial M_1.$$

The set of concordance classes is denoted by $\mathcal{T}^{CAT}(X, \partial X)$ (see [11, §10] and [8, §7.1]).

An s -triangulation of $(X, \partial X)$ is a simple homotopy equivalence of pairs (see [8, §7.1])

$$(f, \partial f) : (M, \partial M) \rightarrow (X, \partial X).$$

Two s -triangulations

$$(f_i, \partial f_i) : (M_i, \partial M_i) \rightarrow (X, \partial X), \quad i = 0, 1$$

are concordant if there exists a simple homotopy equivalence of 4-ads

$$((h; g, f_0, f_1) : (W; V, M_0, M_1) \rightarrow (X \times I; \partial X \times I, \partial X \times \{0\}, \partial X \times \{1\})$$

with

$$\partial V = \partial M_0 \cup \partial M_1.$$

The set of concordance classes is denoted by $\mathcal{S}^{CAT}(X, \partial X)$ (see [11, §10] and [8, §7.1]).

These sets fit in the surgery exact sequence (see [11, §10] and [8, §7])

$$(2.3) \quad \cdots \rightarrow L_{n+1}^{rel} \rightarrow \mathcal{S}^{TOP}(X, \partial X) \rightarrow \mathcal{T}^{TOP}(X, \partial X) \rightarrow L_n^{rel}$$

where $L_*^{rel} = L_*(\pi_1(\partial X) \rightarrow \pi_1(X))$.

Now we define mixed structures on a manifold with boundary (see [11, page 116] and [3]). A t_s -triangulation of $(X, \partial X)$ is a topological normal map of pairs

$$((f, b), (\partial f, \partial b)) : (M, \partial M) \rightarrow (X, \partial X)$$

such that $\partial f : \partial M \rightarrow \partial X$ is an s -triangulation. Two t_s -triangulations

$$((f_i, b_i), (\partial f_i, \partial b_i)) : (M_i, \partial M_i) \rightarrow (X, \partial X), \quad i = 0, 1$$

are concordant if there exists a topological normal map

$$\begin{aligned} ((h, d); (g, c), (f_0, b_0), (f_1, b_1)) : \\ (W; V, M_0, M_1) \longrightarrow (X \times I; \partial X \times I, \partial X \times \{0\}, \partial X \times \{1\}) \end{aligned}$$

with

$$\partial V = \partial M_0 \cup \partial M_1$$

and $g : V \rightarrow \partial X \times I$ is a concordance of s -triangulations ∂f_0 and ∂f_1 . The set of concordance classes is denoted by $\mathcal{TS}^{TOP}(X, \partial X)$ (see [11, page 116]).

It follows from definitions (see also [3]) that the following natural forgetful maps

$$\mathcal{TS}^{TOP}(X, \partial X) \rightarrow \mathcal{T}^{TOP}(X, \partial X),$$

$$(2.4) \quad \mathcal{S}^{TOP}(X, \partial X) \rightarrow \mathcal{TS}^{TOP}(X, \partial X),$$

$$\mathcal{TS}^{TOP}(X, \partial X) \rightarrow \mathcal{S}^{TOP}(\partial X)$$

are correctly defined.

The maps from (2.4) fits in the following exact sequences (see [11, page 116], [8, §7], and [3])

$$(2.5) \quad \cdots \rightarrow L_n(\pi_1(\partial X)) \rightarrow \mathcal{TS}^{TOP}(X, \partial X) \rightarrow \mathcal{T}^{TOP}(X, \partial X) \rightarrow L_{n-1}(\pi_1(\partial X)),$$

$$(2.6) \quad \cdots \rightarrow L_{n+1}(\pi_1(X)) \rightarrow \mathcal{S}^{TOP}(X, \partial X) \rightarrow \mathcal{TS}^{TOP}(X, \partial X) \rightarrow L_n(\pi_1(X)),$$

and

$$(2.7) \quad \cdots \rightarrow \mathcal{T}_\partial^{TOP}(X, \partial X) \rightarrow \mathcal{TS}^{TOP}(X, \partial X) \rightarrow \mathcal{S}^{TOP}(\partial X).$$

Now we recall necessary results about the application of homotopy category of spectra to surgery theory (see [1], [2], [4], [5], [6], and [7]). A spectrum \mathbb{E} is given by a collection of CW -complexes $\{(E_n, *)\}$, $n \in \mathbb{Z}$, together with cellular maps $\{\epsilon_n : SE_n \rightarrow E_{n+1}\}$, where SE_n is the suspension of the space E_n (see [14]). A spectrum \mathbb{E} is a Ω -spectrum if the adjoint maps $\epsilon'_n : E_n \rightarrow \Omega E_{n+1}$, $n \in \mathbb{Z}$ are homotopy equivalences.

In the category of spectra the functor Σ of suspension and iterated functors Σ^k , $k \in \mathbb{Z}$ are defined (see [10]). For every spectrum \mathbb{E} we have an isomorphism of homotopy groups $\pi_n(\mathbb{E}) = \pi_{n+k}(\Sigma^k \mathbb{E})$. Recall that in the homotopy category of spectra the concepts of pull-back and push-out squares are equivalent.

In accordance with [7], [8], and [11] the surgery obstruction groups $L_n(\pi)$ and such natural maps as induced by inclusion and transfer are realized on spectra level. That is, for every group π with an homomorphism of orientation $\omega : \pi \rightarrow \{\pm 1\}$ there exists an Ω -spectrum $\mathbb{L}(\pi, \omega)$ with homotopy groups

$$\pi_n(\mathbb{L}(\pi, \omega)) = L_n(\pi, \omega).$$

In what follows we shall not include homomorphism of orientation in our notations and will suppose that all groups are equipped by such homomorphism and all homomorphisms of groups preserve orientation. Any homomorphism of groups $f : \pi \rightarrow G$ induces a cofibration of spectra

$$(2.8) \quad \mathbb{L}(\pi) \rightarrow \mathbb{L}(G) \rightarrow \mathbb{L}(f)$$

where $\mathbb{L}(f)$ is the spectrum for relative L -groups

$$L_n(f) = L_n(\pi \rightarrow G) = \pi_n(\mathbb{L}(f)).$$

A similar situation we have for the transfer map (see, for example, [11] and [12]).

Let X be a topological space. An algebraic surgery exact sequence of Ranicki (see [7] and [8])

$$(2.9) \quad \cdots \rightarrow L_{n+1}(\pi_1(X)) \rightarrow \mathcal{S}_{n+1}(X) \rightarrow H_n(X, \mathbf{L}_\bullet) \rightarrow L_n(\pi_1(X)) \rightarrow \cdots$$

is defined. Here \mathbf{L}_\bullet is the 1-connected cover of the surgery Ω -spectrum $\mathbb{L}(1)$ with $\{\mathbf{L}_\bullet\}_0 \simeq G/TOP$. The algebraic surgery exact sequence (2.9) is a homotopy long exact sequence of the cofibration

$$(2.10) \quad X_+ \wedge \mathbf{L}_\bullet \rightarrow \mathbb{L}(\pi_1(X)).$$

By definition, we have $\mathcal{S}_i(X) = \pi_i(\mathbb{S}(X))$ for a homotopy cofiber $\mathbb{S}(X)$ of the map in (2.10). For a closed n -dimensional topological manifold X we have

$$(2.11) \quad \pi_{n+1}(\mathbb{S}(X)) = \mathcal{S}_{n+1}(X) \cong \mathcal{S}^{TOP}(X), \quad \mathcal{T}(X) \cong H_n(X; \mathbf{L}_\bullet),$$

and surgery exact sequence (2.1) is isomorphic to the left part of the algebraic surgery exact sequence (2.9) (see [8, Proposition 7.1.4]).

For the case of a compact topological manifold X with the boundary ∂X the algebraic surgery exact sequences for relative case and for *rel* ∂ case fit in the following commutative diagram (see [7] and [8, §7])

$$(2.12) \quad \begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & & \vdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & L_{n+1}(\pi) & \rightarrow & \mathcal{S}_{n+1}^\partial(X, \partial X) & \rightarrow & H_n(X; \mathbf{L}_\bullet) & \rightarrow & L_n(\pi) & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & L_{n+1}^{rel} & \rightarrow & \mathcal{S}_{n+1}(X, \partial X) & \rightarrow & H_n(X, \partial X; \mathbf{L}_\bullet) & \rightarrow & L_n^{rel} & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & L_n(\rho) & \rightarrow & \mathcal{S}_n(\partial X) & \rightarrow & H_{n-1}(\partial X; \mathbf{L}_\bullet) & \rightarrow & L_{n-1}(\rho) & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

where $\pi = \pi_1(X)$, $\rho = \pi_1(\partial X)$, and $L_*^{rel} = L_*(\rho \rightarrow \pi)$. Diagram (2.12) is realized on spectra level (see [8], [1], and [3]). For the topological manifold X the left part of the upper row of diagram (2.12) is isomorphic to the exact sequence (2.2). The left part of middle row of diagram (2.12) is isomorphic to exact sequence (2.3). We shall denote by $\mathbb{S}^\partial(X, \partial X)$ a spectrum with homotopy groups

$$\pi_i(\mathbb{S}^\partial(X, \partial X)) = \mathcal{S}_i^\partial(X, \partial X)$$

and by $\mathbb{S}(X, \partial X)$ a spectrum with homotopy groups

$$\pi_i(\mathbb{S}(X, \partial X)) = \mathcal{S}_i(X, \partial X).$$

In particular, we have isomorphisms

$$(2.13) \quad \mathcal{S}^{TOP}(X, \partial X) \cong S_{n+1}(X, \partial X), \quad \mathcal{S}_\partial^{TOP}(X, \partial X) \cong S_{n+1}^\partial(X, \partial X),$$

and

$$(2.14) \quad \mathcal{T}^{TOP}(X, \partial X) \cong H_n(X, \partial X; \mathbf{L}_\bullet), \quad \mathcal{T}_\partial^{TOP}(X, \partial X) \cong H_n(X; \mathbf{L}_\bullet).$$

Consider a composition

$$(2.15) \quad L_{n+1}(\pi_1(X)) \rightarrow L_{n+1}(\pi_1(\partial X) \rightarrow \pi_1(X)) \rightarrow \mathcal{S}^{TOP}(X, \partial X)$$

where the first map lies in the relative exact sequence of L -groups for the map $\pi_1(\partial X) \rightarrow \pi_1(X)$ and the second map lies in (2.3). As follows from (2.12) the composition (2.15) is realized by a map of spectra (see also [3])

$$(2.16) \quad \mathbb{L}(\pi_1(X)) \rightarrow \mathbb{S}(X, \partial X)$$

and cofiber of the map in (2.16) is denoted by $\mathbb{TS}(X, \partial X)$. We shall denote

$$\pi_n(\mathbb{TS}(X, \partial X)) = \mathcal{TS}_i(X, \partial X)$$

and we have an isomorphism [3]

$$\mathcal{TS}_{n+1}(X, \partial X) \cong \mathcal{TS}^{TOP}(X, \partial X).$$

Remark: in a similar way (see [3]) it is possible to describe the spectrum $\mathbb{TS}(X, \partial X)$ as homotopical cofiber of any from the following maps

$$(2.17) \quad \mathbb{S}(\partial X) \rightarrow \Sigma(X_+ \wedge \mathbf{L}_\bullet) \text{ and } (X/\partial X)_+ \wedge \mathbf{L}_\bullet \rightarrow \Sigma\mathbb{L}(\pi_1(\partial X)).$$

3. Splitting problem for a manifold with boundary.

Let (X, Y, ξ) be a codimension q manifold pair in the sense of Ranicki (see [8, §7.2]) i. e. a locally flat closed submanifold Y is given with a normal fibration

$$\xi = \xi_{Y \subset X} : Y \rightarrow \widetilde{BTOP}(q)$$

with the associated (D^q, S^{q-1}) fibration

$$(3.1) \quad (D^q, S^{q-1}) \rightarrow (D(\xi), S(\xi)) \rightarrow Y$$

and we have a decomposition of the closed manifold

$$X = D(\xi) \cup_{S(\xi)} \overline{X \setminus D(\xi)}.$$

The normal disk bundle (3.1) provides transfer maps on spectra level (see [1], [6], [11], and [12])

$$(3.2) \quad p^\sharp : \mathbb{L}(\pi_1(Y)) \rightarrow \Omega^q \mathbb{L}(\pi_1(S(\xi)) \rightarrow \pi_1(D(\xi)))$$

and

$$(3.3) \quad p_1^\sharp : \mathbb{L}(\pi_1(Y)) \rightarrow \Omega^{q-1} \mathbb{L}(\pi_1(S(\xi))).$$

Transfer maps (3.2) and (3.3) fit in a homotopy commutative diagram of spectra (3.4)

$$(3.4) \quad \begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \xrightarrow{p^\sharp} & \Omega^q \mathbb{L}(\pi_1(S(\xi)) \rightarrow \pi_1(D(\xi))) & \rightarrow & \Omega^q \mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) \\ & p_1^\sharp \searrow & \downarrow & & \downarrow \\ & & \Omega^{q-1} \mathbb{L}(\pi_1(S(\xi))) & \rightarrow & \Omega^{q-1} \mathbb{L}(\pi_1(X \setminus Y)), \end{array}$$

where the horizontal maps of the right square are induced by the horizontal maps of F and the vertical maps are maps from cofibrations of spectra as in (2.8) for the vertical maps of the square F .

The spectra $\mathbb{L}\mathbb{S}(F)$ for splitting obstruction groups for the manifold pair $Y \subset X$ and $\mathbb{L}\mathbb{P}(F)$ for surgery obstruction groups for the manifold pair fit in the homotopy commutative diagram of spectra

$$(3.5) \quad \begin{array}{ccccc} \Omega \mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q+1} \mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) & \longrightarrow & \mathbb{L}\mathbb{S}(F) \\ = \downarrow & & \downarrow & & \downarrow \\ \Omega \mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^q \mathbb{L}(\pi_1(X \setminus Y)) & \longrightarrow & \mathbb{L}\mathbb{P}(F) \end{array}$$

where the left horizontal maps are compositions from diagram (3.4) and the right square is pullback (see [1], [10], and [11]). In particular, we have isomorphisms

$$\pi_n(\mathbb{L}\mathbb{S}(F)) \cong LS_n(F), \quad \pi_n(\mathbb{L}\mathbb{P}(F)) \cong LP_n(F).$$

A topological normal map [8, §7.2]

$$((f, b), (g, c)) : (M, N) \rightarrow (X, Y)$$

to the manifold pair (X, Y, ξ) is represented by a normal map (f, b) to the manifold X which is transversal to Y with $N = f^{-1}(Y)$, and (M, N) is a topological manifold pair with a normal fibration

$$\nu : N \xrightarrow{f|_N} Y \xrightarrow{\xi} \widetilde{BTOP}(q).$$

Additionally the following conditions are satisfied:

(i) the restriction

$$(f, b)|_N = (g, c) : N \rightarrow Y$$

is a normal map;

(ii) the restriction

$$(f, b)|_P = (h, d) : (P, S(\nu)) \rightarrow (Z, S(\xi))$$

is a normal map to the pair $(Z, S(\xi))$, where

$$P = \overline{M \setminus D(\nu)}, \quad Z = \overline{X \setminus D(\xi)};$$

(iii) the restriction

$$(h, d)|_{S(\nu)} : S(\nu) \rightarrow S(\xi)$$

coincides with the induced map

$$(g, c)^\dagger : S(\nu) \rightarrow S(\xi),$$

and $(f, b) = (g, c)^\dagger \cup (h, d)$.

The normal maps to (X, Y, ξ) are called t -triangulations of the manifold pair (X, Y) and the set of concordance classes of t -triangulations of the pair (X, Y, ξ) coincides with the set of t -triangulations of the manifold X [8, Proposition 7.2.3].

An s -triangulation of a manifold pair (X, Y, ξ) in topological category [8, p. 571] is a t -triangulation of this pair for which the maps

$$(3.6) \quad f : M \rightarrow X, \quad g : N \rightarrow Y, \quad \text{and} \quad (P, S(\nu)) \rightarrow (Z, S(\xi))$$

are simple homotopy equivalences (s -triangulations). The set of concordance classes of s -triangulations is denoted by $\mathcal{S}^{TOP}(X, Y, \xi)$ (see [8, page 571]). Natural forgetful maps

$$(3.7) \quad \mathcal{S}^{TOP}(X, Y, \xi) \rightarrow \mathcal{S}^{TOP}(X) \quad \text{and} \quad \mathcal{S}^{TOP}(X, Y, \xi) \rightarrow \mathcal{T}^{TOP}(X)$$

are defined (see [8, §7.2]). We have also the maps of taking obstruction (see [8, page 572])

$$(3.8) \quad \mathcal{S}^{TOP}(X, Y, \xi) \rightarrow LS_{n-q}(F) \quad \text{and} \quad \mathcal{T}^{TOP}(X) \rightarrow LP_{n-q}(F).$$

The maps in (3.7) and (3.8) are realized on spectra level (see [1], [8 §7.2], and [11]). We shall denote by $\mathbb{S}(X, Y, \xi)$ a spectrum with homotopy groups (see [8, page 583])

$$(3.9) \quad \pi_i(\mathbb{S}(X, Y, \xi)) = \mathcal{S}_i(X, Y, \xi), \quad \text{and} \quad \mathcal{S}_{n+1}(X, Y, \xi) \cong \mathcal{S}^{TOP}(X, Y, \xi).$$

The maps from (3.8) and (3.9) fit in a lot of diagrams of exact sequences which are given in [8, Proposition 7.2.6]. The diagram

$$(3.10) \quad \begin{array}{ccccccc} \rightarrow & L_{n+1}(\pi_1(X)) & \rightarrow & LS_{n-q}(F) & \rightarrow & \mathcal{S}_n(X, Y, \xi) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & \mathcal{S}_{n+1}(X) & & LP_{n-q}(F) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & \mathcal{S}_{n+1}(X, Y, \xi) & \rightarrow & H_n(X; \mathbf{L}_\bullet) & \rightarrow & L_n(\pi_1(X)) & \rightarrow \end{array},$$

from [8, Proposition 7.2.6] is realized on the spectra level with the left part ($i \geq n$) that is isomorphic to geometrically defined diagram (see [8, page 582]) containing structure sets $\mathcal{S}^{TOP}(X)$, $\mathcal{S}^{TOP}(X, Y, \xi)$, $\mathcal{T}^{TOP}(X)$ in accordance with isomorphisms (2.11) and (3.10). Note here that the geometric version of diagram (3.10) contains also the maps from (3.7) and (3.8).

Let

$$(3.11) \quad (Y, \partial Y) \subset (X, \partial X)$$

be a codimension q manifold pair with boundaries. A manifold pair (3.11) with boundaries defines a pair of closed manifolds $\partial Y \subset \partial X$ with a pushout square

$$(3.12) \quad F_\partial = \begin{pmatrix} \pi_1(\mathcal{S}(\partial\xi)) & \longrightarrow & \pi_1(\partial X \setminus \partial Y) \\ \downarrow & & \downarrow \\ \pi_1(\partial Y) & \longrightarrow & \pi_1(\partial X) \end{pmatrix}$$

of fundamental groups for the splitting problem. A natural inclusion $\delta : \partial X \rightarrow X$ induces a map of $\Delta : F_\partial \rightarrow F$ of squares of fundamental groups.

In the *rel* ∂ -case we consider *t*-triangulations

$$(3.13) \quad (f, \partial f) : (M, \partial M) \rightarrow (X, \partial X)$$

which are split on the boundary along ∂Y . The classes of concordance relative to the boundary of such maps give the set $\mathcal{T}_\partial^{TOP}(X, \partial X)$ (see [8, §7.2]) and the map

$$(3.14) \quad \mathcal{T}_\partial^{TOP}(X, \partial X) \rightarrow LP_{n-q}(F)$$

defines a *rel* ∂ codimension q splitting obstruction along $Y \subset X$ (see [8, §7.2]).

In a similar way (see [8, §7.2]) we can consider an *s*-triangulation of pairs (3.12) which is split along the boundary. The set of concordance *rel* ∂ classes is $\mathcal{S}_\partial^{TOP}(X, \partial X)$ and a *rel* ∂ codimension q splitting obstruction gives a map

$$(3.15) \quad \mathcal{S}_\partial^{TOP}(X, \partial X) \rightarrow LS_{n-q}(F).$$

As in the case of closed manifolds denote by $\mathcal{S}_\partial^{TOP}(X, Y, \xi)$ the set of classes of concordance *rel* ∂ maps which are split along $Y \subset X$.

The algebraic version of surgery exact sequence (2.2) and algebraic versions of the maps (3.14) and (3.15) fit in the commutative braid of exact sequences

$$(3.16) \quad \begin{array}{ccccccc} \rightarrow & L_{n+1}(\pi_1(X)) & \rightarrow & LS_{n-q}(F) & \rightarrow & \mathcal{S}_n^\partial(X, Y, \xi) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & \mathcal{S}_{n+1}^\partial(X, \partial X) & & LP_{n-q}(F) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & \mathcal{S}_{n+1}^\partial(X, Y, \xi) & \rightarrow & H_n(X; \mathbf{L}_\bullet) & \rightarrow & L_n(\pi_1(X)) & \rightarrow \end{array}$$

Diagram (3.16) is realized on the level of spectra and for $\partial X = \emptyset$ coincides with the diagram (3.10). The left part ($i \geq n$) of diagram (3.16) is isomorphic to geometrically defined diagram similarly to diagram (3.10). In particular,

$$(3.17) \quad \mathcal{S}_i^\partial(X, Y, \xi) = \pi_i(\mathbb{S}^\partial(X, Y, \xi)), \text{ and } \mathcal{S}_{n+1}^\partial(X, Y, \xi) \cong \mathcal{S}_\partial^{TOP}(X, Y, \xi).$$

Denote by $LS_*(\Delta) = LS_*(F_\partial \rightarrow F)$ and $LP_*(\Delta) = LP_*(F_\partial \rightarrow F)$ the relative groups for the map of squares $\Delta : F_\partial \rightarrow F$ which is induced by the natural inclusion $\delta : \partial X \rightarrow X$. It follows from functoriality of diagram (3.5) that these relative groups are realized on spectra level. We have cofibrations of Ω -spectra

$$(3.18) \quad \mathbb{L}\mathbb{S}(F_\partial) \rightarrow \mathbb{L}\mathbb{S}(F) \rightarrow \mathbb{L}\mathbb{S}(\Delta)$$

and

$$(3.19) \quad \mathbb{L}\mathbb{P}(F_\partial) \rightarrow \mathbb{L}\mathbb{P}(F) \rightarrow \mathbb{L}\mathbb{P}(\Delta)$$

where

$$\pi_n(\mathbb{L}\mathbb{S}(\Delta)) \cong LS_n(F_\partial \rightarrow F) \text{ and } \pi_n(\mathbb{L}\mathbb{P}(\Delta)) \cong LP_n(F_\partial \rightarrow F).$$

These groups fit in the commutative diagram of exact sequences

$$(3.20) \quad \begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & & \vdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & L_{n+1}(\rho) & \rightarrow & LS_{n-q}(F_\partial) & \rightarrow & LP_{n-q}(F_\partial) & \rightarrow & L_n(\rho) & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & L_{n+1}(\pi) & \rightarrow & LS_{n-q}(F) & \rightarrow & LP_{n-q}(F) & \rightarrow & L_n(\pi) & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & L_{n+1}(\rho \rightarrow \pi) & \rightarrow & LS_{n-q}(\Delta) & \rightarrow & LP_{n-q}(\Delta) & \rightarrow & L_n(\rho \rightarrow \pi) & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

where $\pi = \pi_1(X)$ and $\rho = \pi_1(\partial X)$. Diagram (3.20) is realized on spectra level and two middle columns are homotopy long exact sequences of cofibrations (3.18) and (3.19).

Now consider relative structure groups for a codimension q manifold pair with boundaries (3.11). We have a normal fibration $(\xi, \partial\xi)$ over the pair $(Y, \partial Y)$ and a decomposition

$$(3.21) \quad (X, \partial X) = (D(\xi) \cup_{S(\xi)} Z, D(\partial\xi) \cup_{S(\partial\xi)} \partial_+ Z)$$

where $(Z; \partial_+ Z, S(\xi); S(\partial\xi))$ is a manifold triad. Note here that $\partial_+ Z = \overline{\partial X \setminus D(\partial\xi)}$.

A topological normal map (3.13) of manifold pairs with boundaries provides a normal fibration $(\nu, \partial\nu)$ over the pair $(N, \partial N)$ where (see [8, p. 570])

$$(N, \partial N) = (f^{-1}(Y), (\partial f)^{-1}(\partial Y)).$$

We have the following decomposition

$$(3.22) \quad (M, \partial M) = (D(\nu) \cup_{S(\nu)} P, D(\partial\nu) \cup_{S(\partial\nu)} \partial_+ P)$$

where $(P; \partial_+ P, S(\nu); S(\partial\nu))$ is a manifold triad.

Let

$$(f, \partial f) : (M, \partial M) \rightarrow (X, \partial X)$$

be a normal map of a codimension q pair with boundary $(N, \partial N) \subset (M, \partial M)$ to a codimension q pair $(Y, \partial Y) \subset (X, \partial X)$. It is an s -triangulation if the constituent maps $f : M \rightarrow X$ and $\partial f : \partial M \rightarrow \partial X$ are s -triangulations of corresponding codimension q pairs. We shall denote the set of concordance classes of s -triangulations of the codimension q manifold pair (3.11) by

$$\mathcal{S}^{TOP}(X, Y; \partial) = \mathcal{S}^{TOP}(X, \partial X; Y, \partial Y; \xi, \partial(\xi)).$$

The relative surgery theory (see [8, §7.2], [10, §11], and [11]) provides that this structure set fits in the following exact sequences

$$(3.23) \quad \cdots \rightarrow \mathcal{S}^{TOP}(X, Y; \partial) \rightarrow \mathcal{T}^{TOP}(X, \partial X) \rightarrow LP_{n-q}(\Delta)$$

and

$$(3.24) \quad \cdots \rightarrow \mathcal{S}^{TOP}(X, Y; \partial) \rightarrow \mathcal{S}^{TOP}(X, \partial X) \rightarrow LS_{n-q}(\Delta).$$

Proposition 1. *There exists an Ω -spectrum $\mathbb{S}(X, Y; \partial)$ with homotopy groups*

$$(3.25) \quad \mathcal{S}_i(X, Y; \partial) \cong \pi_i(\mathbb{S}(X, Y; \partial)) \text{ and } \mathcal{S}_{n+1}(X, Y; \partial) \cong \mathcal{S}^{TOP}(X, Y; \partial).$$

There defined algebraic versions of exact sequences (3.23) and (3.24)

$$(3.26) \quad \cdots \rightarrow \mathcal{S}_{n+1}(X, Y; \partial) \rightarrow H_n(X, \partial X; \mathbf{L}_\bullet) \rightarrow LP_{n-q}(\Delta) \rightarrow \cdots$$

and

$$(3.27) \quad \cdots \rightarrow \mathcal{S}_{n+1}(X, Y; \partial) \rightarrow \mathcal{S}_{n+1}(X, \partial X) \rightarrow LS_{n-q}(\Delta) \rightarrow \cdots$$

which are realized on the spectra level by cofibrations

$$(3.29) \quad (X/\partial X)_+ \wedge \mathbf{L}_\bullet \rightarrow \mathbb{L}\mathbb{P}(\Delta) \rightarrow \mathbb{S}(X, Y; \partial)$$

and

$$(3.30) \quad \mathbb{S}(X, \partial X) \rightarrow \mathbb{L}\mathbb{S}(\Delta) \rightarrow \mathbb{S}(X, Y; \partial),$$

respectively.

Proof. Commutative diagram (2.12) is generated by a homotopy commutative diagram of spectra

$$(3.31) \quad \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & (\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\rho) & \longrightarrow & \mathbb{S}(\partial X) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\pi) & \longrightarrow & \mathbb{S}^\partial(X, \partial X) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & (X/\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\rho \rightarrow \pi) & \longrightarrow & \mathbb{S}(X, \partial X) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

in which each row and column is a cofibration sequence. Commutative diagram (3.20) is generated by a homotopy commutative diagram of spectra

$$(3.32) \quad \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathbb{L}\mathbb{S}(F_\delta) & \longrightarrow & \mathbb{L}\mathbb{P}(F_\delta) & \longrightarrow & \Sigma^{-q}\mathbb{L}(\rho) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{L}\mathbb{S}(F) & \longrightarrow & \mathbb{L}\mathbb{P}(F) & \longrightarrow & \Sigma^{-q}\mathbb{L}(\pi) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{L}\mathbb{S}(\Delta) & \longrightarrow & \mathbb{L}\mathbb{P}(\Delta) & \longrightarrow & \Sigma^{-q}\mathbb{L}(\rho \rightarrow \pi) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

in which each row and column is a cofibration sequence. Consider a homotopy commutative square of spectra

$$(3.33) \quad \begin{array}{ccc} (\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{L}\mathbb{P}(F_\partial) \\ \downarrow & & \downarrow \\ X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{L}\mathbb{P}(F) \end{array}$$

in which vertical maps are induced by inclusion δ and horizontal maps follows from diagram (3.10) for the manifold pair $\partial Y \subset \partial X$ and from diagram (3.16), respectively. Denote by $\mathbb{S}(X, Y; \partial)$ a spectrum fitting in the diagram extending the square (3.33) by cofibration sequences

$$(3.34) \quad \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & (\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{L}\mathbb{P}(F_\partial) & \longrightarrow & \mathbb{S}(\partial X, \partial Y, \partial \xi) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{L}\mathbb{P}(F) & \longrightarrow & \mathbb{S}^\partial(X, Y, \xi) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & (X/\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{L}\mathbb{P}(\Delta) & \longrightarrow & \mathbb{S}(X, Y; \partial) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Let $\mathcal{S}_i(X, Y; \partial) = \pi_i(\mathbb{S}(X, Y; \partial))$. Now applying functor π_0 to the bottom cofibration in (3.34) we obtain exact sequence (3.26). Since homotopical fiber (cofiber)

of the map of spectra is correctly and functorially defined (see [10, §8.31]) we obtain an isomorphism between (3.23) and (3.26). The case of exact sequence (3.27) follows from a homotopy commutative diagram of cofibrations

(3.35)

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \longrightarrow & \mathbb{S}(\partial X, \partial Y, \partial \xi) & \longrightarrow & \mathbb{S}(\partial X) & \longrightarrow & \Sigma^{q+1}\mathbb{L}\mathbb{S}(F_\partial) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & \mathbb{S}^\partial(X, Y, \xi) & \longrightarrow & \mathbb{S}^\partial(X, \partial X) & \longrightarrow & \Sigma^{q+1}\mathbb{L}\mathbb{S}(F) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & \mathbb{S}(X, Y; \partial) & \longrightarrow & \mathbb{S}(X, \partial X) & \longrightarrow & \Sigma^{q+1}\mathbb{L}\mathbb{S}(\Delta) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

which is similar to (3.34). Diagram (3.35) follows from consideration of the cofibration sequences of the right upper square in (3.35). \square

Consider the composition

$$(3.36) \quad LS_*(F_\partial) \rightarrow LP_*(F_\partial) \rightarrow LP_*(F)$$

of geometrically defined maps from diagram (3.20). Composition (3.36) is realized by a map of spectra

$$(3.37) \quad \mathbb{L}\mathbb{S}(F_\partial) \rightarrow \mathbb{L}\mathbb{P}_*(F)$$

which is a composition of maps from diagram (3.32). We denote homotopical cofiber of the map (3.37) by $\mathbb{L}\mathbb{P}\mathbb{S}(\Delta)$ and its homotopy groups by

$$(3.38) \quad LPS_i(\Delta) = \pi_i(\mathbb{L}\mathbb{P}\mathbb{S}(\Delta)).$$

In particular, we have a cofibration

$$(3.39) \quad \mathbb{L}\mathbb{S}(F_\partial) \rightarrow \mathbb{L}\mathbb{P}(F) \rightarrow \mathbb{L}\mathbb{P}\mathbb{S}(\Delta).$$

Theorem 2. *There exists the following cofibration of spectra*

$$(3.40) \quad \mathbb{S}(X, Y, \partial) \rightarrow \mathbb{T}\mathbb{S}(X, \partial X) \rightarrow \Sigma^{q+1}\mathbb{L}\mathbb{P}\mathbb{S}(\Delta).$$

Proof. Consider the following homotopy commutative diagram of spectra

$$\begin{array}{ccccc}
 \mathbb{S}(\partial X, \partial Y, \partial \xi) & \longrightarrow & \mathbb{S}^\partial(X, Y, \xi) & \longrightarrow & \mathbb{S}(X, Y; \partial) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{S}(\partial X) & \longrightarrow & \Sigma(X_+ \wedge \mathbf{L}_\bullet) & \longrightarrow & \mathbb{T}\mathbb{S}(X, \partial X) \\
 \downarrow & & \downarrow & & \downarrow \\
 \Sigma^{q+1}\mathbb{L}\mathbb{S}(F_\partial) & \longrightarrow & \Sigma^{q+1}\mathbb{L}\mathbb{P}(F) & \longrightarrow & \Sigma^{q+1}\mathbb{L}\mathbb{P}\mathbb{S}(\Delta)
 \end{array}
 \tag{3.41}$$

in which all rows and columns are cofibration sequences. The left bottom square of (3.41) follows from the commutative diagram

$$(3.42) \quad \begin{array}{ccccc} \mathbb{S}(\partial X) & \longrightarrow & \Sigma((\partial X)_+ \wedge \mathbf{L}_\bullet) & \longrightarrow & \Sigma(X_+ \wedge \mathbf{L}_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^{q+1}\mathbb{L}\mathbb{S}(F_\partial) & \longrightarrow & \Sigma^{q+1}\mathbb{L}\mathbb{P}(F_\partial) & \longrightarrow & \Sigma^{q+1}\mathbb{L}\mathbb{P}(F) \end{array}$$

in which the left square fits in the diagram (3.10) on the spectra level for the manifold pair $\partial Y \subset \partial X$ and right square fits in (3.34). The left column of (3.41) fits in (3.35). The middle column in (3.41) fits in (3.34). The right vertical maps in (3.41) are defined as map of cofibers of horizontal maps in accordance with (2.17), (3.35), and (3.39). The left upper horizontal map in (3.41) is the composition

$$(3.43) \quad \mathbb{S}(\partial X, \partial Y, \partial \xi) \xrightarrow{\cong} \mathbb{S}(\partial X, \partial Y, \partial \xi) \rightarrow \mathbb{S}^\partial(X, Y, \xi)$$

where the middle spectrum in (3.43) is the homotopical fiber of the map

$$\Sigma((\partial X)_+ \wedge \mathbf{L}_\bullet) \rightarrow \Sigma^{q+1}\mathbb{L}\mathbb{P}(F_\partial)$$

fitting in (3.42). The right column of cofibration (3.41) is cofibration (3.40). \square

Corollary 3. *There exists the following long exact sequence*

$$(3.44) \quad \cdots \rightarrow \mathcal{S}_{n+1}(X, Y; \partial) \rightarrow \mathcal{TS}(X, \partial X) \rightarrow LPS_{n-q-1}(\Delta) \rightarrow \cdots$$

Proof. Exact sequence (3.44) is the homotopy long exact sequence of the cofibration (3.40). \square

Corollary 4. *There is a map*

$$\Theta : \mathcal{TS}^{TOP}(X, \partial X) \rightarrow LPS_{n-q}(\Delta)$$

of obstructions to splitting along the submanifold with boundary $(Y, \partial Y)$. For a representative $z = ((f, b), (\partial f, \partial b))$ we have $\Theta(z) = 0$ if and only if the class of z in $\mathcal{TS}^{TOP}(X, \partial X)$ contains a representative which is split along $(Y, \partial Y) \subset (X, \partial X)$.

Proof. We have the commutative diagram

$$(3.45) \quad \begin{array}{ccc} \mathcal{S}_{n+1}(X, Y, \partial) & \longrightarrow & \mathcal{S}^{TOP}(X, Y; \partial) \\ \downarrow & & \downarrow \\ \mathcal{S}_{n+1}(X, \partial X) & \longrightarrow & \mathcal{TS}^{TOP}(X, \partial X) \end{array}$$

in which the horizontal maps are isomorphisms, the right vertical map is a natural forgetful map, and the left vertical map follows from (3.41). Now the diagram (3.41) and the exact sequence (3.44) provide the result of the Corollary. \square

From now we describe algebraic properties of LPS_* -groups and their relations to splitting and surgery obstruction groups.

Theorem 5. *There exist following braids of exact sequences*

$$(3.46) \quad \begin{array}{ccccccc} \rightarrow & LP_{n+1}(\Delta) & \rightarrow & L_{n+q}(\rho) & \rightarrow & LS_{n-1}(F_\partial) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & LP_n(F_\partial) & & LPS_n(\Delta) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & LS_n(F_\partial) & \longrightarrow & LP_n(F) & \longrightarrow & LP_n(\Delta) & \rightarrow \end{array},$$

$$(3.47) \quad \begin{array}{ccccccc} \rightarrow & L_{n+q+1}(\pi) & \rightarrow & LS_n(\Delta) & \rightarrow & LP_n(\Delta) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & L_{n+q+1}(\rho \rightarrow \pi) & & LPS_n(\Delta) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & LP_{n+1}(\Delta) & \longrightarrow & L_{n+q}(\rho) & \longrightarrow & L_{n+q}(\pi) & \rightarrow \end{array},$$

and

$$(3.48) \quad \begin{array}{ccccccc} \rightarrow & LS_n(F_\partial) & \rightarrow & LP_n(F) & \rightarrow & L_{n+q}(\pi) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & LS_n(F) & & LPS_n(\Delta) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & L_{n+q+1}(\pi) & \longrightarrow & LS_n(\Delta) & \longrightarrow & LS_{n-1}(F_\partial) & \rightarrow \end{array},$$

where $\rho = \pi_1(\partial X)$ and $\pi = \pi_1(X)$. Diagram (3.46) – (3.48) are realized on the level of spectra.

Proof. It follows from definition of spectra \mathbb{LPS} and diagram (3.32) that we have a homotopy commutative diagram of spectra

$$(3.49) \quad \begin{array}{ccccc} \mathbb{LS}(F_\partial) & \rightarrow & \mathbb{LP}(F_\partial) & \rightarrow & \Sigma^{-q}\mathbb{L}(\rho) \\ \downarrow = & & \downarrow & & \downarrow \\ \mathbb{LS}(F_\partial) & \rightarrow & \mathbb{LP}(F) & \rightarrow & \mathbb{LPS}(\Delta) \end{array}$$

where the rows are cofibrations and the right vertical map is induced by the two left vertical maps (see [9]). The right square in (3.49) is a pullback square since fibers of horizontal maps are naturally homotopy equivalent. Hence the right square in (3.49) is a pushout square and homotopy long exact sequences of this square give the braid of exact sequences (3.46). From diagram (3.32) and [5, Lemma 2] we obtain that the spectrum $\mathbb{LPS}(\Delta)$ fits in the cofibrations of spectra

$$\Sigma^{-q}\mathbb{LP}(\Delta) \rightarrow \Sigma^{-q}\mathbb{L}(\rho) \rightarrow \mathbb{LPS}(\Delta)$$

and

$$\Sigma^{-q-1}\mathbb{L}(\pi) \rightarrow \mathbb{LS}(\Delta) \rightarrow \mathbb{LPS}(\Delta).$$

Now the same line of arguments as for the braid of exact sequences (3.46) provides diagrams (3.47) and (3.48). \square

Corollary 6. *The groups $LPS_*(\Delta)$ fits in the following exact sequences*

$$(3.50) \quad \cdots \rightarrow LS_n(F_\partial) \rightarrow LP_n(F) \rightarrow LPS_n(\Delta) \rightarrow \cdots ,$$

$$(3.51) \quad \cdots \rightarrow LP_{n+1}(\Delta) \rightarrow L_{n+q}(\rho) \rightarrow LPS_n(\Delta) \rightarrow \cdots ,$$

and

$$(3.52) \quad \cdots \rightarrow L_{n+q+1}(\pi) \rightarrow LS_n(\Delta) \rightarrow LPS_n(\Delta) \rightarrow \cdots ,$$

which are realized on the level of spectra.

Proof. These sequences fit in the diagrams of Theorem 6. \square

Corollary 7. *Let $\Delta : F_\partial \rightarrow F$ be an isomorphism of pushout squares. Then we have isomorphisms*

$$LPS_*(\Delta) \cong L_{n+q}(\rho) \cong L_{n+q}(\pi).$$

Proof. The result follows immediately from exact sequences of Corollary 6. \square

The next theorem describes relations between obstruction groups LPS_* and different structure sets which arise naturally for the manifold pair with boundaries.

Theorem 8. *There exist the following braids of exact sequences*

$$(3.53) \quad \begin{array}{ccccccc} \rightarrow & \mathcal{S}_n(X, Y; \partial) & \rightarrow & \mathcal{TS}_n(X, \partial X) & \rightarrow & L_{n-1}(\pi) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & \mathcal{S}_n(X, \partial X) & & LPS_{n-q-1} & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & L_n(\pi) & \longrightarrow & LS_{n-q-1}(\Delta) & \longrightarrow & \mathcal{S}_{n-1}(X, Y; \partial) & \rightarrow \end{array} ,$$

and

$$(3.54) \quad \begin{array}{ccccccc} \rightarrow & \mathcal{S}_{n+1}(X, Y; \partial) & \rightarrow & H_n(X, \partial X; \mathbf{L}_\bullet) & \rightarrow & L_{n-1}(\rho) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & & \mathcal{TS}_{n+1}(X, \partial X) & & LP_{n-q}(\Delta) & & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & L_n(\rho) & \longrightarrow & LPS_{n-q} & \longrightarrow & \mathcal{S}_n(X, Y; \partial) & \rightarrow \end{array} ,$$

where $\rho = \pi_1(\partial X)$ and $\pi = \pi_1(X)$. Diagrams (3.53) and (3.54) are realized on the level of spectra.

Proof. Consider a homotopy commutative diagram of spectra

$$(3.55) \quad \begin{array}{ccccc} \mathbb{L}(\pi) & \longrightarrow & \mathbb{S}(X, \partial X) & \longrightarrow & \mathbb{TS}(X, \partial X) \\ = \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}(\pi) & \longrightarrow & \Sigma^{q+1}\mathbb{LS}(\Delta) & \longrightarrow & \Sigma^{q+1}\mathbb{LPS}(X, \partial X) \end{array}$$

in which the rows follow from definitions and the columns are obtained from the natural map of diagram (3.31) to diagram (3.32). The right square of (3.55) is pullback and the homotopy long exact sequences of this square give commutative diagram (3.53). The case of diagram (3.54) is similar. \square

4. Examples.

In this section we compute the LPS_* -groups and natural maps for several geometric examples.

Let $(Y^{n-1}, \partial Y) \subset (X^n, \partial X)$, $(n \geq 4)$ be a manifold pair with boundaries, where X is a non-trivial I -bundle over a real projective space $\mathbb{R}P^{n-1}$ and the submanifold Y is a restriction of the I -bundle to projective space $\mathbb{R}P^{n-1} \subset \mathbb{R}P^n$. The pair $\partial Y \subset \partial X$ coincides with $S^{n-2} \subset S^{n-1}$. We have isomorphisms $\pi_1(X^n) = \mathbb{Z}_2^+$ for n odd and $\pi_1(X^n) = \mathbb{Z}_2^-$ for n even. The group \mathbb{Z}_2^+ is a cyclic group of order 2 with trivial homomorphism of orientation and \mathbb{Z}_2^- this group with nontrivial homomorphism of orientation. In the considered case squares (1.2) and (3.12) are the following squares

$$(4.1) \quad F^\pm = \begin{pmatrix} 1 & \rightarrow & 1 \\ \downarrow & & \downarrow \\ \mathbb{Z}_2^\mp & \rightarrow & \mathbb{Z}_2^\pm \end{pmatrix}$$

and

$$(4.2) \quad F_\partial = \begin{pmatrix} 1 \cup 1 & \rightarrow & 1 \cup 1 \\ \downarrow & & \downarrow \\ 1 & \rightarrow & 1 \end{pmatrix}.$$

All horizontal maps in squares (4.1) and (4.2) are isomorphisms.

Theorem 9. *In the considered cases the natural maps*

$$LP_n(F^\pm) \rightarrow LPS_n(F_\partial \rightarrow F^\pm)$$

fitting in diagrams (3.46) and (3.48) are isomorphisms for $n = 0, 1, 2, 3 \pmod 4$. Hence we have

$$(4.3) \quad LPS_n(F_\partial \rightarrow F^+) \cong \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}$$

and

$$(4.4) \quad LPS_n(F_\partial \rightarrow F^-) \cong \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively.

Proof. It follows from [10, page 153] that

$$LS_n(F_\partial) = LN_n(1 \cup 1 \rightarrow 1) = 0$$

for all n . From this result and exact sequence (3.50) the first statement of the theorem follows. We have isomorphisms

$$LP_n(F^\pm) \cong L_{n+1}(i_\mp^!)$$

where

$$i_\mp : 1 \rightarrow \mathbb{Z}_2^\mp$$

is the natural inclusion and $L_{n+1}(i_\mp^!)$ is the relative group of the transfer map (see, for example, [6], [8] and [9]). From this isomorphisms (4.3) and (4.4) follow (see, for example, [9, §3] for the case F^+).

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