University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1111 Ljubljana, Slovenia

Preprint series, Vol. 42 (2004), 940

BAR-MAGNET POLYHEDRA AND NS-ORIENTATIONS OF MAPS

Bojan Mohar

ISSN 1318-4865

October 20, 2004

Ljubljana, October 20, 2004

Bar-magnet polyhedra and NS-orientations of maps

Bojan Mohar*
Department of Mathematics,
University of Ljubljana,
1000 Ljubljana, Slovenia
bojan.mohar@uni-lj.si

Abstract

At the CCCG 2001 open-problem session [2], J. O'Rourke asked which polyhedra can be represented by bars and magnets. This problem can be phrased as follows: which 3-connected planar graphs may have their edges directed so that the directions "alternate" around each vertex (with one exception of non-alternation if the degree is odd). In this note we solve O'Rourke's problem and generalize it to arbitrary maps on general surfaces. Obstructions to existence of such orientations can be expressed algebraically by a new homology invariant of perfect matchings in the related graph of cofacial odd vertices.

1 Bar-magnet polyhedra

A toy called "Roger's Connection" provides a collection of magnetic bars and steel balls that can be used to construct polyhedra. The structures are most stable when around each vertex (a steel ball), the North and South poles of magnetic bars meeting at that vertex are alternating. This toy motivated the definition of *bar-magnet polyhedra* as those 3-connected plane graphs whose edges can be directed so that the directions "alternate" around each vertex, where one non-alternation is allowed if the degree of the vertex is odd.

Let G be a map, i.e., a graph that is 2-cell embedded in some surface. We refer to [4] for basic definitions concerning graphs on surfaces. An *NS-orientation* of G is an orientation of the edges so that in the clockwise order

*Supported in part by the Ministry of Education, Science and Sport of Slovenia, Research Program P1–0507–0101.

around every vertex v, the incoming and outgoing edges alternate, except when $\deg(v)$ is odd when one violation of this condition is allowed. We say that the pair of consecutive edges in the clockwise order around v where non-alternation occurs is the *exceptional transition* at v. We say that G is *NS-orientable* if it admits an NS-orientation.

At the open-problem session at the 13th Canadian Conference on Computational Geometry in 2001 [2], O'Rourke asked which 3-connected planar maps are NS-orientable. At the corresponding web page of open problems, maintained by Demaine, Mitchell, and O'Rourke [1], it is noted that after the presentation of the problem, Therese Biedl proved that the polyhedron formed by gluing together two tetrahedra with congruent bases is not NS-orientable. Erik Demaine proved that a polyhedron all of whose vertices have even degree is NS-orientable: the graph has a face 2-coloring, and the edges of the faces of color 1 can be oriented counterclockwise, which then orients each face of color 2 clockwise. Demaine also observed that every cubic graph with a perfect matching admits an NS-orientation: orient every cycle in the complement of the perfect matching cyclically and orient the edges of the perfect matching arbitrarily.

In this note we obtain a complete solution to O'Rourke's problem by providing a simple characterization of NS-orientability of planar maps (see Corollary 2.3). We also obtain a generalization of this result to arbitrary maps on general surfaces. Obstructions to existence of NS-orientations depend on certain perfect matchings in an associated graph of cofacial odd vertices and give rise to a new homology invariant of perfect matchings.

By a *map* we mean a graph together with a 2-cell embedding in some closed surface. Underlying graphs of maps are allowed to have loops and multiple edges. They are necessarily connected. If Σ is a surface, the number $g = 2 - \chi(\Sigma)$ is called the *Euler genus* of Σ, where $\chi(\Sigma)$ denotes the Euler characteristic of the surface.

2 NS-orientations and matchings

A graph (or a map) is *Eulerian* if all its vertices have even degree.

Let G be an Eulerian map on a surface S and let G^* be its dual map. Suppose that the embedding of G^* is represented by means of local rotations and signatures (cf. [4] for details). We say that G is *orientably partitionable* if the faces of G can be partitioned into two classes $F' \cup F''$ such that two adjacent faces are in the same class if and only if the corresponding edge in G^* has negative signature. It is easy to see that G is orientably partitionable if and only if for every cycle C in the dual G^*, the number of edges on C with positive signature is even, i.e., the number of edges with negative signature has the same parity as the length of C. Alternatively, every even cycle of G^* is orientation preserving and every odd cycle is orientation reversing on the surface. This implies that being orientably partitionable is independent of the choice of the rotation-signature representation of G^*. In particular, if the surface S is orientable, then all signatures can be taken positive, and being orientably partitionable is equivalent to being face 2-colorable.

Proposition 2.1 *Let G be an Eulerian map. Then G is NS-orientable if and only if it is orientably partitionable.*

Proof. Under any NS-orientation, facial walks are directed closed walks. Adjacent facial walks are oppositely oriented when compared with respect to a common edge. In an orientable surface, this means that faces are either positively or negatively oriented, and this yields a bipartition of the faces. If the surface is nonorientable, then cycles of G^* of even length behave similarly as on orientable surfaces and they must be orientation preserving, while cycles of odd length in G^* must be orientation reversing. These facts easily yield the stated equivalence. $\square$

Theorem 2.2 *Let G be a map. Define a new graph R whose nodes are the vertices of odd degree in G, with two nodes of R adjacent if they are cofacial in G (i.e., they lie on a common facial walk). Then G has an NS-orientation if and only if R has a perfect matching M such that G can be extended to an orientably partitionable map in the same surface, whose underlying graph of is $G + M$.*

Proof. Suppose that we have an NS-orientation of G. Let F be a facial walk. When we traverse the facial walk F, the edges all head in the same direction until an exceptional transition is met. From that point on, all edges head in the other direction until the next exceptional transition is reached, when the direction changes again. This implies

Claim 1 *Every facial walk has an even number of exceptional transitions.*

Let G' be the map obtained from G by adding a new vertex v_F in every face F with at least one exceptional transition and joining v_F to all vertices in F corresponding to exceptional transitions.

Claim 2 *G' is an Eulerian graph and the NS-orientation of G uniquely extends to an NS-orientation of G'. Under this orientation, all faces of G' are directed closed walks, and hence G' is orientably partitionable.*

Let us consider a facial walk $F = v_1e_1v_2e_2 \dots v_ke_kv_1$ where v_i and e_i are consecutive vertices and edges (respectively) on F. If the edges e_{i-1} and e_i form an exceptional transition at the vertex v_i, then G' has the edge $e'_i = v_Fv_i$ that is inserted between e_{i-1} and e_i in the local clockwise rotation around v_i on the surface. Clearly, there is a unique way of orienting e'_i so that none of the transitions e_{i-1}, e'_i and e'_i, e_i is exceptional. It is also clear if the exceptional transition of G in F that immediately follows the transition at v_i occurs at v_j, then the orientation (at v_F) of the inserted edge $e'_j = v_Fv_j$ is opposite to that of e'_i. This shows that all exceptional transitions of G disappear in G' and that no exceptional transitions arise at new vertices v_F. In particular, all vertices of G' have even degree, i.e. G' is Eulerian. By Proposition 2.1, G' is orientably partitionable. This completes the proof of Claim 2.

Let us now consider a vertex v_F of G'. Suppose that its neighbors on F in the clockwise order are $u_1, \dots, u_{2k}$. Let M_F be the set of edges $u_1u_2, \dots, u_{2k-1}u_{2k}$. Finally, let M be the union of all M_F, where F is any face of G. Since each odd vertex of G has precisely one exceptional transition (and vertices of even degree have none), M is a perfect matching in R. Since G' is orientably partitionable, it is clear that the map $\tilde{G}$ obtained from G by adding all edges of every M_F in F (following the boundary of F) is also orientable partitionable. This completes the first part of the proof.

The proof of the converse implication is essentially the reverse of the above proof, so we omit it. □

Corollary 2.3 *Let G be a graph that is 2-cell embedded in the plane or the 2-sphere. Define a new graph R whose nodes are the vertices of odd degree in G, with two nodes of R adjacent if they are cofacial in G. Then G has an NS-orientation if and only if R has a perfect matching.*

Proof. Suppose that R has a perfect matching M. If $e = uv \in M$, there is a face in which u and v bot appear. For every $e \in M$ choose one such face. For a given face F of G, let M_F be all edges in M that have selected F as their face of cofaciality. Let $u_1, \dots, u_{2k}$ be the endvertices of the edges in M_F in the order as they appear on the facial walk of F. Let $M'_F = \{u_1u_2, \dots, u_{2k-1}u_{2k}\}$ and let M' be the union of all M'_F. Clearly, M'

is a matching of R and the embedding of G can be extended to an embedding of $\tilde{G} = G + M'$, yielding an Eulerian map. A well known consequence of simple connectivity is that every Eulerian map in a simply connected surface is face 2-colorable. So it is $\tilde{G}$, and we are done by Theorem 2.2. $\square$

Since the existence of a perfect matching is polynomially checkable, Corollary 2.3 provides a good characterization in the sense of Edmonds. This is the best one can hope for.

For instance, if we have a planar map with precisely two vertices of odd degree, then an NS-orientation exists if and only if the two odd vertices are cofacial.

3 Maps on general surfaces

An example, the Petersen graph on the projective plane, is shown in Figure 1. The dotted edges form a perfect matching of R and it is easy to check that the extended map is orientably partitionable.

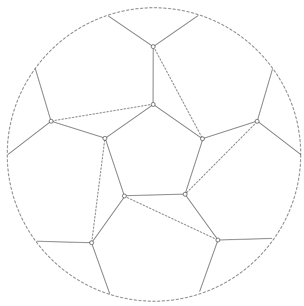


Figure 1: An orientably partitionable extension of the Petersen graph in the projective plane

As mentioned above, existence of a perfect matching is polynomially solvable, but perfect matchings satisfying additional conditions may be harder to detect. Nevertheless, since some kind of parity is involved, we believe that Theorem 2.2 yields a good characterization also for more general maps. This belief is partially supported by our subsequent results (cf. Corollary 3.3), at least if we consider maps on a fixed surface.

It is worthwhile to note that a vertex may appear more than once on a facial walk. In such a case we can get loops and multiple edges in R. Since loops never occur in a matching, they may as well be eliminated from

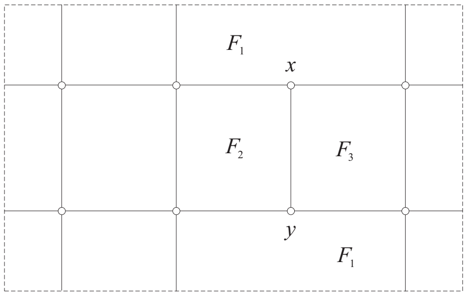


Figure 2: A toroidal map with two extensions

R. However, multiple edges may be kept in R since their inclusion in $\tilde{G}$ corresponding to distinct possibilities of the cofaciality may give rise to different partiteness behavior of $\tilde{G}$. See Figure 2 for an example of a map on the torus, where the choice of the edge xy drawn inside the face F_1 gives rise to an orientably partitionable map, while its inclusion in one of the faces F_2 or F_3 does not give that. Of course, in the case of planar maps, such a distinction is not necessary since all possibilities are equally good as shown by Corollary 2.3.

The above example shows that it may be helpful to introduce a graph $\tilde{R} = \tilde{R}(G)$ that will capture not only the combinatorial but also geometric information about cofaciality of odd vertices. The vertices of $\tilde{R}$ are the vertices of odd degree in G. Suppose that u, v are distinct vertices of $\tilde{R}$. If u appears in a facial walk F and v appears in the same facial walk, then we have an edge joining u and v in $\tilde{R}$ for every such pair of common appearances. Observe that a vertex can appear more than once in a facial walk. We say that this edge *corresponds* to the face F and to appropriate appearances of u, v in F. Let v_F be a point on the surface in the interior of the face F, and let $S_1, \ldots, S_r$ be internally disjoint simple arcs joining occurrences of vertices in F with v_F. If S_i and S_j correspond to the appearances of u and v which determine an edge e of $\tilde{R}$, then we say that $S_i \cup S_j$ is a *drawing* of e in F. If $M \subseteq E(\tilde{R})$ is a matching, the drawings of edges in M determine an Eulerian map whose graph consists of G together with edges (and vertices v_F) corresponding to those segments S_i that correspond to edges in M. We denote this map by $G \dotplus M$.

Let B be a map in a surface Σ such that all faces are of even length. Then we say that B is *locally bipartite*. Let $H_1 = H_1(B; GF(2))$ be the first homology group of the map B with coefficients in the group $GF(2)$.

The elements of H_1 can be identified with subsets of $E(B)$ such that every vertex is incident with an even number of edges in the subset. Two such set C_1, C_2 are *equivalent* (and they correspond to the same element of H_1) if there is a collection of facial walks such that C_2 is the symmetric difference of C_1 with the boundaries of these facial walks, cf. [3]. For $C \subseteq E(B)$, let $\sigma(C)$ be equal to the parity of the number of edges in C whose signature is positive. Since $\sigma(C) = 0$ for the edge set C of any facial walk, σ induces a homomorphism

$$\sigma_0 : H_1 \to GF(2).$$

To relate this homomorphism to our previous thoughts, let us observe that an Eulerian map G is orientably partitionable if and only if the homomorphism σ_0 corresponding to the dual map G^* is trivial, i.e. $\sigma_0(H_1) = \{0\}$.

If G is a map and M is a perfect matching in $\tilde{R}(G)$, the map $G_M = G \dot{+} M$ is Eulerian. Let $B = G_M^*$ be its dual map, and let $\sigma_M : H_1(B; GF(2)) \to GF(2)$ be the corresponding homomorphism. We call σ_M the *characteristic mapping* of M. These definitions yield a reformulation of Theorem 2.2:

Theorem 3.1 *A map G is NS-orientable if and only if its cofaciality graph $\tilde{R}(G)$ contains a perfect matching whose characteristic map is trivial.*

If M_1, M_2 are perfect matchings of $\tilde{R}$, we may consider their symmetric difference $L = M_1 + M_2$. In $\tilde{R}$, L is a collection of disjoint cycles. Drawings of these cycles (as defined above) give rise to a collection of closed curves in Σ. We say that M_1 and M_2 are *homologous* if the collection of these curves is 0-homologous on the surface (with respect to $GF(2)$-homology). If $L = M_1 + M_2$ and $L' = M_2 + M_3$ are both 0-homologous on the surface, so is $L + L' = M_1 + M_3$. This implies that the homology of perfect matchings is an equivalence relation.

Lemma 3.2 *Perfect matchings M_1, M_2 of $\tilde{R}(G)$ are homologous if and only if their characteristic mappings are the same.*

Proof. If M_1 and M_2 are homologous, then $L = M_1 + M_2$ is 0-homologous on the surface. Therefore, every simple closed curve crosses edges of L an even number of times. In particular, this holds for cycles in dual maps $G_{M_1}^*$, $G_{M_2}^*$ and, consequently, $\sigma_{M_1} = \sigma_{M_2}$.

On the other hand, if L is not 0-homologous, then there is a cycle in the dual map that intersects L an odd number of times. If γ is the homology class of that cycle, then $\sigma_{M_1}(\gamma) \neq \sigma_{M_2}(\gamma)$. This completes the proof. $\square$

At this point it is clear why the cofaciality graph $\tilde{R}(G)$ did not need geometric information when considering maps in simply connected surfaces. This is because all perfect matchings are homologous and since every homomorphism $\{0\} \to GF(2)$ is trivial.

If Σ is a surface of Euler genus g, then H_1 is isomorphic to $GF(2)^g$. This gives

Corollary 3.3 *A map on a surface of Euler genus g has at most* 2^g *distinct homology classes of perfect matchings for cofacial odd vertices.*

Sometimes, checking if extensions are orientably partitionable can be overridden. For instance, if the graph G of the map is cubic (all vertices have degree 3), then NS-orientability is independent of the embedding of G since NS-orientations correspond to orientations of edges such that no vertex is a sink (all edges incoming) or a source (all edges outgoing). Therefore, it makes sense to speak of NS-orientability of the graph. The same holds for *subcubic graphs* in which all vertices have degree at most three. Let us recall that we allow loops and multiple edges.

Proposition 3.4 *Every subcubic graph is NS-orientable.*

Proof. The proof is by induction on the number of edges. If G has a vertex v of degree 1, let u be its neighbor and let $G_1 = G - v$. By the induction hypothesis, G_1 has an NS-orientation. If u has at least one incoming edge under such an orientation, then we orient the edge uv in the direction from u to v; otherwise from v to u. Clearly, this gives rise to an NS-orientation of G. If G has a vertex v of degree 2, and uv, vw are its incident edges, then we apply induction to the graph $G_1 = (G - v) + uw$. An NS-orientation of G_1 clearly gives rise to one in G.

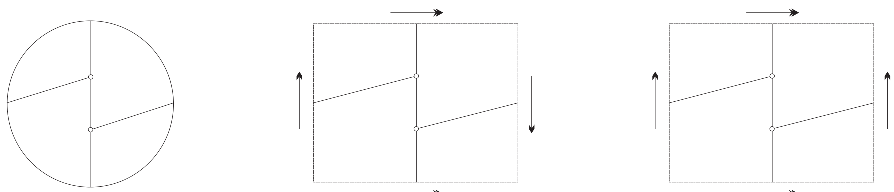
Otherwise, G is a cubic graph. If G has a cutedge uv, we apply induction to $G - uv$ and orient the edge uv arbitrarily. Finally, suppose that G has no cutedges. Then G has a perfect matching M by a well-known theorem of Petersen. The graph $G - M$ consists of a collection of disjoint cycles. By orienting the edges of each such cycle consistently with a chosen direction on the cycle, and orienting edges in M arbitrarily, an NS-orientation is obtained. This completes the proof. $\square$

When considering embedded cubic graphs, we can say much more.

Theorem 3.5 *Let G be a map in a surface Σ of Euler genus g. If every vertex of G has degree at most* 3 *and G is not a cycle that is 2-cell embedded* *in the projective plane, then for every homomorphism* $\sigma : H_1(\Sigma; GF(2)) \to GF(2)$ *there exists a perfect matching in* $\tilde{R}(G)$ *whose characteristic map is equal to* σ. *In other words, perfect matchings of all possible* 2^g *homology classes exist.*

The proof of Theorem 3.5 is deferred until the end of this section.

Let us observe that the map whose graph is the cycle in the projective plane has only one matching of $\tilde{R}$ (the empty matching) and its characteristic map is the trivial homomorphism.

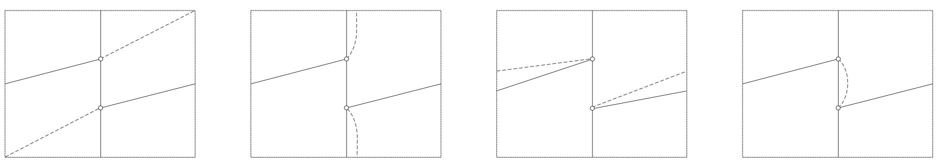


(a) (b) (c)

Figure 3: Three maps with a Θ-graph

A graph is called a Θ-*graph* if it consists of three internally disjoint paths joining two vertices.

Lemma 3.6 *If the graph of* G *is a* Θ*-graph, then* G *satisfies the conclusion of Theorem 3.5.*



00 01 10 11

Figure 4: Four nonhomologous extensions

Proof. Θ-graphs admit precisely four nonequivalent embeddings, one in the plane, projective plane, Klein bottle and the torus; see Figure 3 for the last three of these cases.

Let us consider the Θ-graph map on the torus. As the generators for the fundamental group (and consequently for H_1), we select the horizontal and the vertical simple closed curves α and β (respectively) with respect to the presentation in Figure 3(c). By taking the four possible matching edges of $\tilde{R}(G)$ which are shown in Figure 4, the resulting values of their characteristic maps on α and β are 00, 01, 10, and 11, respectively. This proves the claim in the case of the toroidal map. The proof for the Klein bottle is essentially the same, while the planar and the projective planar cases are obvious. The details are left to the reader. $\square$

Lemma 3.7 *The dumbbell map on the Klein bottle shown in Figure 5(a) satisfies the conclusion of Theorem 3.5.*

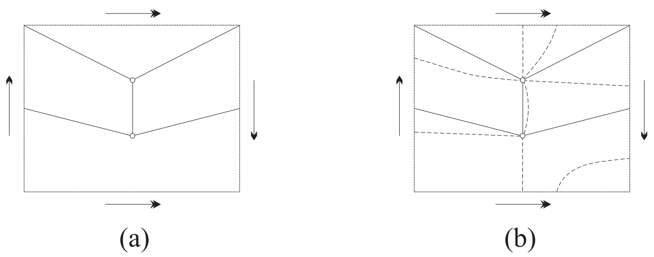


Figure 5: The dumbbell map on the Klein bottle

Proof. Four extensions with distinct characteristic maps are shown by dotted edges in Figure 5(b). $\square$

We are ready for the proof of Theorem 3.5.

Proof (of Theorem 3.5). The proof is by induction on the number of edges of G. If G has a vertex of degree 2, the reduction is exactly the same as in the proof of Proposition 3.4. The same proof can be followed if G has a vertex v of degree 1 and $G - v$ is not the cycle in the projective plane. However, in the latter case it is easy to check that the theorem holds.

Suppose now that all vertices of G have degree 3. If G is not 2-connected, it contains a cutedge $e = uv$. The embedding of G gives rise to two maps G_1, G_2 whose graphs are the two components of $G - e$. Vertices $u \in V(G_1)$ and $v \in V(G_2)$ become vertices of degree 2. If none of G_1, G_2 is just the cycle in the projective plane, then we apply the induction hypothesis for G_1 and G_2. For every perfect matching M_1 in $\tilde{R}(G_1)$ and M_2 of $\tilde{R}(G_2)$, let M be the perfect matching of $\tilde{R}(G)$ consisting of M_1, M_2 and the edge uv

(embedded so that it is homotopic to the edge e in G). By the induction hypothesis, perfect matchings of $\tilde{R}(G_i)$ give rise to all possible 2^{g_i} characteristic mappings (where g_i is the Euler genus of G_i), $i = 1, 2$. Since the Euler genus of G is $g = g_1 + g_2$, it is now easy to see that all combinations of M_1 and M_2 give rise to 2^g distinct characteristic maps corresponding to perfect matchings of $\tilde{R}(G)$.

If G_1 and G_2 are both cycles in the projective plane, then G is the dumbbell map in the Klein bottle, and we apply Lemma 3.7. If G_1 is the cycle in the projective plane but G_2 is not, we proceed in a similar way as above, except that we combine every perfect matching M_2 of $\tilde{R}(G_2)$ first with the edge uv embedded homotopic to e in G, and secondly with the edge uv embedded so that it first follows e from v towards u and then goes across the crosscap, following the cycle G_1. The resulting matchings give rise to $2 \cdot 2^{g_2} = 2^g$ distinct characteristic maps.

From now on we may assume that G is 2-connected. Let $e = uv$ be an edge of G and consider the induced map $G' = G - e$. If G is a Θ-graph, we are done by Lemma 3.6. So, we may assume that G' is not a cycle, and we can apply the induction hypothesis to G'. Observe that vertices u, v are of degree 2 in G'.

We will distinguish three subcases outlined below. Since the choice of e is arbitrary, we may assume (in (2) and (3)) that no edge of G can be chosen so that one of the previous cases would be obtained.

(1) *The edge uv belongs to two distinct facial walks in G.* In this case the Euler genus of G' is the same as that of G. Let F be the facial walk of G' that is the combination of the two facial walks of G containing uv. To every perfect matching M' of $\tilde{R}(G')$, we associate a perfect matching M of $\tilde{R}(G)$ as follows. If $e' \in M'$ corresponds to a face different from F, then we include e' in M. Let $e'_1, \ldots, e'_t$ be the edges in M' that correspond to F. The problem is that these edges may not be edges of $\tilde{R}(G)$. Consider the occurrences of the endpoints of $e'_1, \ldots, e'_t$ on F; denote them by $v_1, \ldots, v_{2t}$ in order in which they appear in F and such that v appears between v_{2t} and v_1 and that u appears between v_j and v_{j+1}, where the considered appearances of u and v correspond to the removed edge uv. If j is even, then we add to M the following edges of $\tilde{R}(G)$: $vv_1, \ldots, v_j u$ and $v_{j+1}v_{j+2}, \ldots, v_{2t-1}v_{2t}$. If j is odd, then we add to M the edges $vv_1, \ldots, v_{j-1}v_j$ and $uv_{j+1}, \ldots, v_{2t-1}v_{2t}$.

(2) *The edge uv belongs to a single facial walk F in G and is traversed on F twice in the same direction.* In this case, the Euler genus of

G' decreases by 1. Excluding the possibility of (1), F is the only facial walk of G and hence G' also has only one facial walk, which we denote by F'. To every perfect matching M' of $\tilde{R}(G')$, we associate two perfect matchings M_1, M_2 of $\tilde{R}(G)$. For M_1, we just combine M' with the edge uv embedded in G homotopic to the removed edge. In M_2 we add uv embedded across the face F (joining the appearance of u at the first traversal of e and the appearance of v at the second traversal of e in F).

(3) *The edge uv belongs to a single facial walk F in G and is traversed twice in the different direction.* In this case, the Euler genus of G' decreases by 2 and F gives rise to two new facial walks F_1, F_2. To every perfect matching M' of $\tilde{R}(G')$, we now associate four perfect matchings M_1, M_2, M_3, M_4 of $\tilde{R}(G)$. For each of them we add to M' the edge uv embedded in G within the face F in a different manner, similarly as shown in the example in Figure 4. More precisely, let v_α and u_α be the appearances of v and u in F at the first traversal of e, and let v_β and u_β be the their appearances at the second traversal of e. Since F is the only face, there exists the third appearance v_γ of v and another appearance u_γ of u. Then we take $M_1 = M' \cup \{u_\alpha, v_\alpha\}$, $M_2 = M' \cup \{u_\alpha, v_\beta\}$, $M_3 = M' \cup \{u_\alpha, v_\gamma\}$, and $M_4 = M' \cup \{u_\gamma, v_\alpha\}$.

By applying the induction hypothesis to G', it is easy to see that in every one of the above possibilities (1)–(3), the obtained perfect matchings of $\tilde{R}(G)$ give rise to 2^g distinct characteristic maps. $\square$

References

[1] Erik D. Demaine, Joseph S. B. Mitchell, and Joseph O'Rourke, The open problems project, `http://cs.smith.edu/~orourke/TOPP/`

[2] Erik D. Demaine and Joseph O'Rourke, Open problems from CCCG 2001, In "Proceedings of the 14th Canadian Conference on Computational Geometry," August 2002.
`http://www.cs.uleth.ca/~wismath/cccg/papers/open.pdf`

[3] P. J. Giblin, Graphs, surfaces and homology. An introduction to algebraic topology, Second edition, Chapman & Hall, London–New York, 1981.

[4] Bojan Mohar and Carsten Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, MD, 2001.