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A SHORT PROOF OF THE TWELVE POINTS THEOREM

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Abstract. We present a short elementary proof of the following Twelve Points Theorem: Let M be a convex polygon with vertices at the lattice points, containing a single lattice point in its interior. Denote by m (resp. m^*) the number of lattice points in the boundary of M (resp. in the boundary of the dual polygon). Then

$$m + m^* = 12.$$

The Twelve points theorem is an elegant theorem, which is easy to formulate, but no simple proof was available until now. In this paper we present a short and elementary proof of this result. To state our theorem we need the following

Definition of the dual polygon. Let $M = A_1 A_2 \dots A_n$ be a convex polygon all of whose vertices lie in the lattice of points with integer coordinates (Figure 1 on the left). Suppose that O is the only lattice point in the interior of M. Draw the vectors $\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \dots \overrightarrow{A_nA_1}$ from the point O. Choose on each of the obtained segments the nearest to O lattice point distinct from O. Connecting the n chosen points consequtively, we get a polygon M^*, *dual* to the original polygon (Figure 1 on the right). Denote by m the number of lattice points in the boundary of M, and by m^* — in the boundary of the dual polygon.

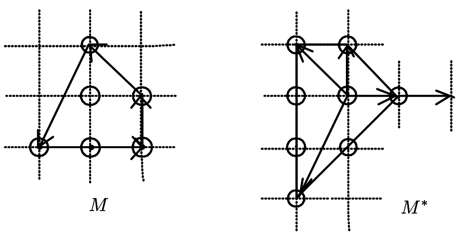

Figure 1.

Key words and phrases. Lattice, lattice polygon, dual polygon, the Pick formula, toric varietes.

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The Twelve Points Theorem. *Suppose that O is the only lattice point in the interior of a convex polygon M; then*

$$m + m^* = 12.$$

This theorem appeared in [Ful93]. There are only some hints for the proof, applying the theory of toric varietes. In an interesting paper [PRV00], completely dedicated to the 12 points theorem, even four different proofs are discussed. Two of them are rather long and they use toric varieties and modular forms respectively. There are also outlined two proofs applying only linear algebra. The first of them is exhausting (there are 16 different types of polygons M in our theorem up to $\mathrm{SL}_2(\mathbb{Z})$). The idea of the second one is very close to the proof of recent paper.

Our elementary proof is analogous to one of the proofs of the Pick formula. We reduce the Twelve points theorem to the specific case when M is a parallelogram and $m = 4$. Let us begin with this latter case.

(1) *If $M = ABCD$ ia a parallelogram without lattice points in its sides then $m + m^* = 12$ (Figure 2).*

Indeed, in this case $O = AC \cap BD$ because the point symmetric to the point O with respect to $AC \cap BD$ is a lattice point and belongs to the interior of $ABCD$, so it coincides with O. It is easy to show that M^* is a parallelogram with sides obtained from the diagonals AC and BD by parallel translations with vectors $\pm\overrightarrow{OB}$ nd $\pm\overrightarrow{OA}$, respectively. Since a unique lattice point O belongs to these diagonals, then any side of the parallelogram M^* contains one lattice point, hence, $b + b^* = 4 + 8 = 12$.

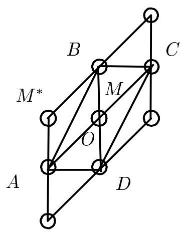

Figure 2.

Now suppose that $M = A_1A_2 \dots A_n$. Let us assume that all the lattice boundary points of M are vertices (possibly, with the angle 180°). This does not affect the definition of M^*. Assume that some triangle $A_{i-1}A_iA_{i+1}$ is *simple*, i. e. it contains no lattice points except its vertices (neither in the interior nor in the boundary). *Deleting a triangle* is cutting off from polygon M the triangle $A_{i-1}A_iA_{i+1}$. The reverse operation is called *adding a triangle*. Our reduction is based on the following assertion:

(2) *The value $m + m^*$ is preserved under delleting or adding a triangle.*

It is sufficient to prove that deleting a simple triangle, say $A_1A_2A_3$, from M gives adding a simple triangle $A_{12}A_{13}A_{23}$ to M^* (Figure 3). Here by A_{kl} we denote the point such that $\overrightarrow{OA_{kl}} = \overrightarrow{A_kA_l}$. In particular, if $l = k + 1$ then A_{kl} is a vertex of the polygon M^*. Delete $A_1A_2A_3$. Then one should delete from M^* the vertices

A_{12} and A_{23}, and add to it a new vertex A_{13}. The last vertex shold be joined by segments with A_{n1} and A_{34}.

Let us show that the points A_{12} and A_{23} beling to these segments. Indeed, since O is the only lattice point inside M, it follows that the triangles A_1OA_3, A_2OA_3, A_4OA_3 are simple. By the Pick formula their areas are equal to $1/2$. Since they have a common base OA_3, it follows that the projections of the vectors $\overrightarrow{A_1A_3}$, $\overrightarrow{A_1A_3}$ and $\overrightarrow{A_1A_3}$ on the direction normal to OA_3 are equal.

This implies that the points A_{13}, A_{23} and A_{34} belong to the same line, and A_{23} lies between the two others, because M is convex. It can be proved analogously that A_{12} belongs to the segment $A_{n1}A_{13}$. Therefore the transformation of M^* is just adding the triangle $A_{12}A_{13}A_{23}$.

Now note that the triangle $OA_{12}A_{13}$ is obtained from a simple triangle $A_1A_2A_3$ by a parallel translation, and $OA_{23}A_{13}$ is obtained from it by a cental symmetry. So the triangle $A_{12}A_{13}A_{23}$ is simple, and assertion (2) is proved.

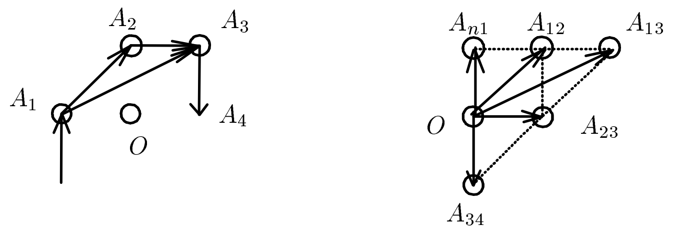

Figure 3.

For the proof of our theorem it remains to notice the following:

(3) *From any polygon M one can obtain a parallelogram without lattice points in the sides by a sequence of deleting and adding triangles.*

Indeed, first assume that M has a diagonal not passing through O. Cut M along this diagonal and consider the obtained part not containing O.

This part necessarily contains a simple triangle of the form $A_{i-1}A_iA_{i+1}$. Deleting it we decrease the number m. Repeat this operation until it is possible. Repetition is impossible only in the following 3 cases (when such a diagonal does not exist):
A) $m = 4$, $M = ABCD$, $O = AC \cap BD$. Since the segments OA, OB, OC and OD do not contain lattice points, then $OA = OC$ and $OB = OD$, that is $ABCD$ is the required parallelogram.
B) $m = 4$, $M = ABD$, and C belongs to the segment BD. In this case let us denote by D' the point symmetric to D with respect to O, and denote by E the middle point of $D'B$. The required sequence of deleting/adding of triangles has the form:

$$ABCD \to AEBCD \to AD'EBCD \to AD'ECD \to AD'CD \quad \text{(Figure 4 on the left).}$$

C) $m = 3$, $M = ABC$. In this case denote by A' and C' the points symmetric to A and C, respectively, with respect to O. The required sequence of deleting/adding of triangles has the form:

$$ABC \to AC'BC \to AC'BA'C \to AC'A'C \quad \text{(Figure 4 on the right).}$$

So in each case we obtain the required parallelogram, that completes the proof of our theorem.

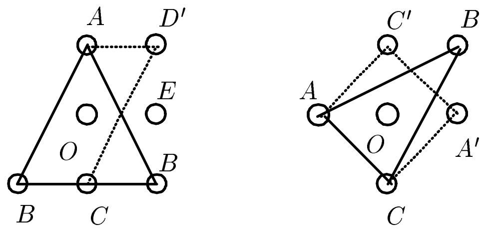

Figure 4.

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