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# TOPOLOGY OF MANIFOLDS MODELED ON COUNTABLE DIRECT LIMITS OF MENGER COMPACTA

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# TOPOLOGY OF MANIFOLDS MODELED ON COUNTABLE DIRECT LIMITS OF MENGER COMPACTA

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Abstract. We construct $n$-dimensional counterparts of manifolds modeled on the space $\ell^2$ equipped by the bounded weak topology ($\mu_n^\infty$-manifolds). For $\mu_n^\infty$-manifolds we prove the characterization, triangulation and classification theorems. In addition, a universal map of $\mu_n^\infty$ onto $Q^\infty$ (the countable direct limit of Hilbert cubes and $Z$-embeddings) is constructed and characterized.

## 1. Introduction

Theory of manifolds modeled on universal $n$-dimensional Menger compacta $\mu_n$ (Menger manifolds; $\mu_n$-manifolds), whose backgrounds were created by Bestvina [4], has been widely developed in the papers of Dranishnikov [8], Chigogidze [6, 7], Sakai [13], Ageev and Repovš [1] and others. As the results demonstrate, the Menger manifolds are closer to the $Q$-manifolds (i.e. the manifolds modeled on the Hilbert cube $Q$; see [5]) than to the finite-dimensional Euclidean manifolds.

In this paper we consider manifolds modeled on the countable direct limits $\mu_n^\infty$ of Menger compacta. These manifolds can be considered as $n$-dimensional counterparts of the manifolds modeled on the countable direct limits $Q^\infty$ of sequences of Hilbert cubes (a series of papers [11, 12, 16] is devoted to the latter). Note that the model space $Q^\infty$ naturally appears in functional analysis as a separable Hilbert space $\ell^2$ endowed with the bounded weak (bw) topology: a set in $(\ell^2, \mathrm{bw})$ is closed if and only if its intersection with every closed ball is closed in the weak topology [10]. Therefore, the space $\mu_n^\infty$ can serve as an $n$-dimensional counterpart of the space $(\ell^2, \mathrm{bw})$.

The theory of $\mu_n^\infty$-manifolds can be pursued slightly further than that of $\mu_n$-manifolds. To the universal Dranishnikov map, which plays an important role in formulations (as well as proofs) of the stability theorem and triangulation theorem, there corresponds, in the case of $\mu_n^\infty$-manifolds, a map $\varphi_n\colon \mu_n^\infty \to Q^\infty$, which can be uniquely, up to a homeomorphism, characterized by means of its fundamental properties. Note that there is no characterization theorem for the universal Dranishnikov map $f_n$.

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The paper is organized as follows. Section 3 is devoted to the characterization theorem. In Section 4 we construct the universal map from $\mu_n^\infty$ onto $Q^\infty$ and in Section 5 we use the universal map to formulate and prove the triangulation and stability theorem.

## 2. Preliminaries

2.1. **$n$-invertible and $n$-soft maps.** The notions of $n$-invertible and $n$-soft map were introduced by Shchepin [15]. A map $f\colon X \to Y$ is said to be *$n$-invertible* provided that for every map $g\colon Z \to Y$ with $\dim Z \le n$ there exists a map $h\colon Z \to X$ such that $fh = g$.

A map $f\colon X \to Y$ is said to be *$n$-soft* provided that for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \big\downarrow & & \big\downarrow f \\ Z & \xrightarrow[\psi]{} & Y \end{array}$$

such that $\dim Z \le n$ and $A$ is a closed subset of $Z$ there exists a map $\Phi\colon Z \to X$ such that $f\Phi = \psi$ and $\Phi|A = \varphi$.

If in the latter definition we require that $Z$ is a polyhedron, then $f$ is said to be a *polyhedrally $n$-soft* map.

We say that two maps, $f_1, f_2\colon X \to Y$ are *$n$-homotopic* (written $f_1 \simeq_n f_2$) if for any space $Z$ with $\dim Z \le n$ and any map $g\colon Z \to X$ the maps $f_1 g$ and $f_2 g$ are homotopic (see e.g. [8]).

**Lemma 2.1.** *Let $f, g\colon A \to X$ be $n$-homotopic maps of a metrizable compactum $A$ into a space $X \in \mathcal{MC}^\infty$. Then there exists a compactum $C \subset X$ such that $C \supset f(A) \cup g(A)$ and the maps $f, g\colon A \to C$ are $n$-homotopic.*

*Proof.* There exists an $n$-dimensional compactum and $n$-invertible map $h\colon B \to A$ (see [8]). Then the maps $fh$ and $gh$ are homotopic; denote by $H\colon B \times I \to X$ the homotopy which connects them and let $C = H(B \times I)$.

If $\dim B' \le n$ and a map $h'\colon B' \to A$ is given, then there exists a map $\alpha\colon B' \to B$ such that $h\alpha = h'$. Then $H(\alpha \times \mathrm{id}_I)$ is a homotopy of the maps $fh'$ and $gh'$. Thus, the maps $f, g\colon A \to C$ are $n$-homotopic. □

**Lemma 2.2.** *Suppose that a map $f\colon X \to Y$ of metric compacta induces an isomorphism of the homotopy groups of dimension $\le n$, $Y \in \mathrm{LC}^n$, $(P, L)$ is a polyhedral pair, $\dim P \le n$ and $\alpha\colon P \to Y$, $\beta\colon L \to X$ are maps such that $f\beta = \alpha|L$. Then there exists a map $\hat\beta\colon P \to X$ such that $\hat\beta|L = \beta$ and $f\hat\beta \simeq_{n-1} \alpha$.*

*Proof.* This is essentially Lemma 2.8.7 from [4]. Here we only use the notion of $(n-1)$-homotopy instead of that of $\mu$-homotopy in [4]. □

2.2. $\mu_n$**-manifolds.** Recall the construction of the standard universal $n$-dimensional Menger compactum $\mu_n$ (see e.g. [9]). Let $\mathcal{K}_i$, $i = 0, 1, 2, \dots$, be the family of $3^{mi}$ congruent cubes obtained by means of partition of the unit $m$-dimensional cube $I^m$, $m \geq n$, by $(m-1)$-dimensional affine subspaces in $\mathbb{R}^m$ given by the equations $x_j = k/3^i$, $j = 1, 2, \dots, m$ and $0 \leq k \leq 3^i$. Denote by $\mathcal{S}_n(K)$ the family of all faces of dimension $\leq n$ of the cube $K$ and for every subfamily $\mathcal{K} \subset \mathcal{K}_i$ let $\mathcal{K}' = \{K \in \mathcal{K}_{i+1} \mid K \subset \cup\mathcal{K}\}$. Taking $\mathcal{F}_0 = \{I^m\}$ $F_0 = \cup\{\mathcal{F}_0\}$ and assuming that $\mathcal{F}_i$ $F_i$ are already defined for all $i < k$, set

$$\mathcal{F}_k = \{K \in \mathcal{F}_{k-1} \mid K \cap (\cup\mathcal{S}_n(F_{k-1})) \neq \emptyset\},\ F_k = \cup\mathcal{F}_k.$$

Finally, let $\mu_n = \cap_{i=0}^{\infty} F_i \subset I^m$.

For $m \geq 2n+1$ and $n$ fixed, all spaces $\mu_n^m$ are homeomorphic [4]. Let $\mu_n = \mu_n^{2n+1}$.

A paracompact space $X$ is said to be a $\mu_n$*-manifold* if there exists a base of the topology of $X$ consisting of sets homeomorphic to open subsets in $\mu_n$. We assume that the $\mu_n$-manifolds under consideration are separable.

Recall that a map $f\colon X \to Y$ is said to be a $Z$*-embedding* if the image $f(X)$ is a $Z$-set in $Y$; the latter means that the identity map $1_Y$ can be approximated by the maps whose image misses $f(X)$ (see e.g. [3]).

**Theorem 2.3** ($Z$-embedding extension theorem [4])**.** *Let* $(A, B)$ *be a compact metrizable pair,* $\dim A \leq n$*. For every* $Z$*-embedding* $f\colon B \to \mu_n$ *there exists an extension to an embedding* $\bar{f}\colon A \to \mu_n$*.*

2.3. By $\mathcal{MC}$ (respectively $\mathcal{MC}(n)$) we will denote the class of metrizable compacta (respectively the class of metrizable compacta of dimension $\leq n$). Given a class $\mathcal{C}$ of topological spaces, we denote by $\mathcal{C}^\infty$ the class of spaces which can be represented as countable direct limits of sequences of spaces $X_1 \hookrightarrow X_2 \hookrightarrow \dots$, where $X_i \in \mathcal{C}$.

By $Q$ we will denote the Hilbert cube, $Q = \prod_{i=1}^{\infty}[-1, 1]_i$. Let $Q^\infty$ denote the direct limit of the sequence

$$Q \to Q \times \{0\} \hookrightarrow Q \times Q \to Q \times Q \times \{0\} \hookrightarrow Q \times Q \times Q \to \dots.$$

By $\mathbb{R}^\infty$ we denote the direct limit of the sequence

$$\mathbb{R} \to \mathbb{R} \times \{0\} \hookrightarrow \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \{0\} \hookrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \dots.$$

## 3. A characterization theorem

3.1. $\mu_n^\infty$**-manifolds.** Denote by $\mu_n^\infty$ the direct limit of the sequence

$$\mu_n^{(1)} \hookrightarrow \mu_n^{(2)} \hookrightarrow \mu_n^{(3)} \hookrightarrow \dots, \tag{3.1}$$

in which all the spaces $\mu_n^{(i)}$ are topological copies of $\mu_n$ and all the embeddings are $Z$-embeddings.

A paracompact space $X$ is said to be a $\mu_n^\infty$*-manifold* if there exists an open cover of the space $X$ with all elements homeomorphic to $\mu_n^\infty$. We assume that all $\mu_n^\infty$-manifolds under consideration are separable.

Recall that by $\mathcal{MC}(n)^\infty$ we denote the class of spaces for which there exists a representation as the direct limit of a sequence of the form

$$Y_1 \hookrightarrow Y_2 \hookrightarrow Y_3 \hookrightarrow \dots,$$

where $Y_i$ are metrizable compacta with $\dim Y_i \le n$ for every $i$.

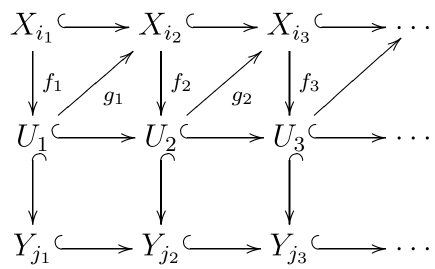
A space $Y$ is said to be *strongly (neighborhood) $n$-universal* if for every compact metrizable pair $(A, B)$, where $\dim A \le n$, and every embedding $f\colon B \to Y$ there exists an embedding $\bar{f}\colon A \to Y$ (respectively an embedding $\bar{f}\colon U \to Y$ of some neighborhood $U$ of the set $B$ in $A$) which extends $f$.

**Theorem 3.1.** *A space $X \in \mathcal{MC}(n)^\infty$ is homeomorphic to $\mu_n^\infty$ (respectively is a $\mu_n^\infty$-manifold) if and only if $X$ is strongly $n$-universal (respectively strongly neighborhood $n$-universal).*

*Proof.* Let $X = \varinjlim X_i$, where $X_i$ are compact metrizable spaces with $\dim X_i \le n$. Write $\mu_n^\infty = \varinjlim Y_j$, where $Y_j$ are homeomorphic to $\mu_n$ and every embedding $Y_j \hookrightarrow Y_{j+1}$ is a $Z$-embedding.

As in [11], we apply the "back and forth" argument. Set $i_1 = j_1 = 1$. There exists an embedding $f_1\colon X_{i_1} \to Y_{j_1}$. By the strong $n$-universality property of $X$, there exists an embedding $g_1\colon U_1 \to X$ of a closed neigborhood $U_1$ of $f_1(X_{i_1})$ in $Y_{j_1}$ such that $g_1|f_1(X_{i_1}) = f_1^{-1}$.

Since $U_1$ is compact, there exists $i_2 > i_1$ such that $U_1 \subset X_{i_2}$. Proceeding similarly one obtains the commutative diagram




in which the maps $f_k\colon X_{i_k} \to U_k$, $g_k\colon U_k \to X_{i_{k+1}}$ are embeddings and $U_{k+1}$ is a closed neighborhood of $f_k(X_{i_k})$ in $Y_{i_k}$. Then

$$X = \varinjlim X_i = \varinjlim X_{i_k} = \varinjlim\{\, X_{i_1} \xrightarrow{f_1} U_1 \xrightarrow{g_1} X_{i_2} \xrightarrow{f_2} U_2 \xrightarrow{g_2} \cdots \,\} = \varinjlim U_i$$

and the latter set, $U = \varinjlim U_i$, is an open subset of $\mu_n^\infty$. Therefore, $X$ is a $\mu_n^\infty$-manifold. $\square$

In fact, we have proven the following stronger result.

**Theorem 3.2** (Open embedding theorem)**.** *Every $\mu_n^\infty$-manifold admits an open embedding into $\mu_n^\infty$.*

The following is a consequence of the characterization theorem.

**Theorem 3.3.** *Every $\mu_n^\infty$-manifold is homeomorphic to the countable direct limit of $\mu_n$-manifolds and $Z$-embeddings.*

*Proof.* Let $X$ be a $\mu_n^\infty$-manifold, $X = \varinjlim X_i$, where $X_i$ are compacta. There exists an embedding $i_1\colon X_1 \to \mu_n$. By the strong neighborhood $n$-universality property, there exists a closed neighborhood $U_1$ of the set $i_1(X_1)$ in $\mu_n$ such that the embedding $i_1^{-1}\colon i_1(X_1) \to X_1 \subset \mu_n^\infty$ can be extended to an embedding $j_1\colon U_1 \to \mu_n^\infty$. Without loss of generality, one can assume that $U_1$ is a $\mu_n$-manifold. Put $V_1 = j_1(U_1)$.

Suppose that compact $\mu_n$-manifolds $V_1 \subset V_2 \subset \cdots \subset V_k \subset X$ are chosen so that $V_i$ is a $Z$-set in $V_{i+1}$, for every $i = 1, 2, \dots, k-1$. There exists $l \geq k$ such that $V_k \subset X_l$. There exists a $Z$-embedding $i_l\colon X_l \to \mu_n$. Similarly as above, it follows from the strong neighborhood $n$-universality property that there exists a closed neighborhood $U_{k+1}$ of the set $i_l(X_l)$ in $\mu_n$ such that $U_{k+1}$ is a $\mu_n$-manifold and the embedding $i_l^{-1}\colon i_l(X_l) \to X_l \subset \mu_n^\infty$ can be extended to an embedding $j_l\colon U_{k+1} \to \mu_n^\infty$. Put $V_{k+1} = j_l(U_{k+1})$. It follows from the properties of $Z$-sets in $\mu_n$ that $V_k$ is a $Z$-set in $V_{k+1}$. Obviously, $X = \varinjlim V_i$.

□

**Theorem 3.4.** *Every $\mu_n^\infty$-manifold admits a closed embedding into $\mu_n^\infty$.*

*Proof.* Let $X$ be a $\mu_n^\infty$-manifold, $X = \varinjlim X_i$, where $X_i$ are compacta. Let $i_1\colon X_1 \to \mu_n^{(1)}$ be an embedding (recall that, as in (3.1), $\mu_n^\infty = \varinjlim \mu_n^{(i)}$). Suppose that, for every $j < k$, embeddings $i_j\colon \mu_n^{(j)} \to \mu_n^\infty$ are defined so that the following conditions hold:

(i) $i_{j+1}|X_j = i_j$ for every $j < k-1$; and

(ii) $i_{j+1}(X_j) \cap \mu_n^{(j)} = i_j(X_j)$ for every $j < k-1$.

In order to construct an embedding $i_k$, note that, since $\mu_n^{(k)}$ is an $AE(n)$-space, there is an extension, $\tilde{i}_k$, of the map $X_{k-1} \xrightarrow{i_{k-1}} \mu_n^{(k-1)} \hookrightarrow \mu_n^{(k)}$ onto $X_k$. Applying the $Z$-set approximation theorem for $\mu_n$-manifolds, one can approximate $\tilde{i}_k$ by maps $i_k$ so that $i_k(X_k) \cap \mu_n^{(k-1)} = i_k(X_{k-1})$.

It is easy te see that the map $\varinjlim i_k$ is a closed embedding of $X$ into $\mu_n^\infty$.

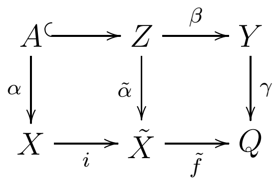
□

## 4. Universal maps

Dranishnikov constructed in [8] $n$-invertible maps $f_n\colon \mu_n \to Q$ and $g_n\colon \mu_n \to \mu_n$ which, in addition to other properties, posses also the following universality property: every map of metric compacta $h\colon X \to Y$, where $\dim X \le n$ (respectively $\dim X \le n$, $\dim Y \le n$) can be embedded into the map $f_n$ (respectively into $g_n$).

**Lemma 4.1.** *For any map $f\colon X \to Q$, where $X$ is a metrizable compactum, $\dim X \le n$, there exists a map $\tilde{f}\colon \tilde{X} \to X$, where $\tilde{X}$ is a metrizable compactum, $\dim \tilde{X} \le n$, and an embedding $i\colon X \to \tilde{X}$ such that $\tilde{f} \circ i = f$ and the following condition holds:*

*$(*)$ for every compact metrizable pair $(Z, A)$, where $\dim Z \le n$, every metrizable compactum $Y$, an embedding $\alpha\colon A \to X$ and maps $\beta\colon Z \to Y$, $\gamma\colon Y \to Q$ such that $f \circ \alpha = \gamma \circ \beta|A$, there exists an embedding $\bar{\alpha}\colon Z \to \tilde{X}$ for which the diagram*




*is commutative.*

*Proof.* Denote by $\mathfrak{A}$ the set of all possible sixtuples $S = (Z, A, Y, \alpha, \beta, \gamma)$, in which $Z, Y$ are metrizable compacta, $\dim Z \le n$, $A$ is a closed subset in $Z$, $\alpha\colon A \to X$ is an embedding, and $\beta\colon Z \to Y$, $\gamma\colon Y \to Q$ are maps such that $f \circ \alpha = \gamma \circ \beta|A$.

For every $S \in \mathfrak{A}$, choose an $n$-invertible map $h_S\colon K_S \to Q$, where $K_S$ is an $n$-dimensional metrizable compactum [8]. Fix a map $g_S\colon Z \to K_S$ such that $h_S \circ g_S = \gamma \circ \beta$.

In the space $T = X \sqcup (\sqcup\{K_S \mid S \in \mathfrak{A}\})$ consider the equivalence relation $\sim$ defined by the condition $\alpha(a) \sim g_S(a)$ for every $S = (Z, A, Y, \alpha, \beta, \gamma) \in \mathfrak{A}$ and every $a \in A$. Denote by $H$ the quotient space of the space $T$, and by $q\colon T \to H$ the quotient map.

It is easy to see that the map $q$ is closed and thus $H$ is a normal space. It follows from the Dowker theorem [9] that $\dim H = n$, therefore $\dim \beta H = n$ (see [9]; as usual, by $\beta H$ we denote the Stone-Čech compact extension of a space $H$).

Denote by $j\colon X \to \beta H$ and $j_S\colon K_S \to \beta H$, $S \in \mathfrak{A}$, the natural embeddings. There exists a map $h\colon H \to Q$ such that $h \circ j = f$ and $h \circ j_S = h_S$ for every $S \in \mathfrak{A}$. Denote by $\hat{h}\colon \beta H \to Q$ the unique extension of the map $h$.

It can be easily deduced from Shchepin's Spectral Theorem [15] that there exists an $n$-dimensional metrizable compactum $X_1$ and a map $h_1\colon \beta H \to X_1$, $f_1\colon X_1 \to Q$ such that $\hat{h} = f_1 \circ h_1$. Let $s\colon X_1 \times Q \to Q$ be an embedding and

$$f'_n = f_n|f_n^{-1}(s(X_1 \times Q))\colon f_n^{-1}(s(X_1 \times Q)) \to s(X_1 \times Q)$$

(here $f_n\colon \mu_n \to Q$ is the universal Dranishnikov map [8]). Denote by $\mathcal{R}$ the partition of the space $f_n^{-1}(s(X_1 \times Q))$, whose only nontrivial elements are the sets of the form $f_n^{-1}(s(x,0))$, $x \in X$. Let $\tilde{X} = f_n^{-1}(s(X_1 \times Q))/\mathcal{R}$ and denote by $q_1\colon f_n^{-1}(s(X_1 \times Q)) \to \tilde{X}$ the quotient map. Let $g\colon \tilde{X} \to X_1 \times Q$ be a map such that $s \circ g \circ q_1 = f_n'$.

Let $\tilde{f} = f_1 \circ \mathrm{pr}_1 \circ g$ and define an embedding $i_1\colon X_1 \to \tilde{X}$ by the formula $i_1(x) = q_1(f_n^{-1}(s(x,0)))$, $x \in X_1$. Let $i = i_1 \circ h_1 \circ j$.

Let $S = (Z, A, Y, \alpha, \beta, \gamma) \in \mathfrak{A}$. Define a map $\alpha_1\colon Z \to X_1$ as $\alpha_1 = h_1 \circ j_S \circ g_S$. Let $p\colon Z \to Z/A$ be the quotient map and let $\eta\colon Z/A \to Q$ be an embedding such that $\eta(\{A\}) = 0$.

Define an embedding $\theta\colon Z \to X_1 \times Q$ by the formula $\theta(z) = (\alpha_1(z), \eta \circ p(z))$, $z \in Z$. From $n$-invertibility of the map $f_n'$ it follows that there exists a map $\bar{\theta}\colon Z \to f_n^{-1}(s(X_1 \times Q))$ such that $f_n' \circ \bar{\theta} = s \circ \theta$. Set $\bar{\alpha} = q_1 \circ \bar{\theta}$ and show that condition (*) holds.

First of all, it is obvious that the map $\bar{\alpha}$ is an embedding. If $a \in A$, then

$$\begin{aligned}\bar{\alpha}(a) =& q_1 \circ \bar{\theta}(a) = q_1(f_n^{-1}(s(\alpha_1(a), 0))) = i_1 \circ \alpha_1(a) \\ =& i_1 \circ h_1 \circ j_S \circ g_S(a) = i_1 \circ h_1 \circ j \circ \alpha(a) = i \circ \alpha(a).\end{aligned}$$

Moreover,

$$\begin{aligned}\tilde{f} \circ i(x) =& f_1 \circ \mathrm{pr}_1 \circ g \circ i_1 \circ h_1 \circ j(x) \\ =& f_1 \circ \mathrm{pr}_1 \circ s^{-1} \circ s \circ g \circ q_1 \circ f_n^{-1}(s(h_1 \circ j(x), 0)) \\ =& f_1 \circ \mathrm{pr}_1 \circ s^{-1} \circ f_n' \circ f_n^{-1} \circ s(h_1 \circ j(x), 0) = f_1 \circ \mathrm{pr}_1(h_1 \circ j(x), 0) \\ =& f_1 \circ h_1 \circ j(x) = f(x)\end{aligned}$$

and

$$\begin{aligned}\tilde{f} \circ \bar{\alpha} =& \tilde{f} \circ q_1 \circ \bar{\theta} = f_1 \circ \mathrm{pr}_1 \circ g \circ q_1 \circ \bar{\theta} = f_1 \circ \mathrm{pr}_1 \circ s^{-1} \circ f_n' \circ \bar{\theta} = f_1 \circ \mathrm{pr}_1 \circ \theta \\ =& f_1 \circ \alpha_1 = f_1 \circ h_1 \circ j_S \circ g_S = \hat{h} \circ j_S \circ g_S = h_S \circ g_S = \gamma \circ \beta.\end{aligned}$$

□

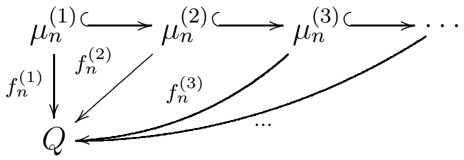
**Definition 4.2.** A map $f\colon X \to Y$ is said to be *strongly* $(n,\infty)$*-universal* (respectively, *strongly* $(n,n)$*-universal*, *strongly* $(n,\omega)$*-universal*), if for every compact metrizable pair $(Z, A)$, where $\dim Z \le n$, and a metrizable compactum $C$ (respectively metrizable compactum $C$ of dimension $\le n$, finite-dimensional metrizable compactum $C$), every embedding $\alpha\colon A \to X$ and maps $\beta\colon Z \to C$, $\gamma\colon C \to Y$ such that $f \circ \alpha = \gamma \circ \beta|A$, there exists an embedding $\bar{\alpha}\colon Z \to X$ such that $\bar{\alpha}|A = \alpha$ and $f \circ \bar{\alpha} = \gamma \circ \beta$.

Recall that a *homeomorphism* of a map $f\colon X \to Y$ into a map $f'\colon X' \to Y'$ consists of a pair of homeomorphisms $g\colon X \to X'$, $h\colon Y \to Y'$ such that $f'g = hf$.

**Theorem 4.3.** *There exists a unique (up to homeomorphism) strongly* $(n,\infty)$*-universal map* $\varphi_n\colon \mu_n^\infty \to Q^\infty$.

*Proof.* Let $f_n\colon \mu_n \to Q$ be the universal Dranishnikov map (see [8]). Using Lemma 4.1, define a sequence of maps $f_n^{(i)}\colon \mu_n^{(i)} \to Q$ and embeddings $\mu_n^{(i)} \hookrightarrow \mu_n^{(i+1)}$ for which the following conditions hold:

(1) $f_n^{(1)} = f_n$;
(2) the diagram




is commutative;
(3) for every compact metrizable pair $(X,A)$, where $\dim X \le n$, metric compactum $Y$, and maps $\alpha\colon X\to Y$, $\psi\colon Y\to Q$ and $\varphi\colon X\to \mu_n^{(i)}$ such that $\psi\circ\alpha = f_n^{(i)}\circ\varphi$ and $\varphi|A$ is an embedding, there exists an embedding $\bar\varphi\colon X\to\mu_n^{(i+1)}$ such that $\bar\varphi|A = \varphi|A$ and $f_n^{(i+1)}\circ\bar\varphi = \psi\circ\alpha$.

Let

$$Q = \prod_{j=1}^{\infty}[-1,1]_j,\ \ Q^{(i)} = \prod_{j=1}^{\infty}\left[-1+\frac{1}{i+1}, 1-\frac{1}{i+1}\right]_j,\ \ i\ge 1.$$

The set $Y = \mathrm{rint} Q = \bigcup\{Q^{(i)} \mid i\ge 1\}$ is called the *radial pseudointerior* of the Hilbert cube $Q$.

Let $X_i = (f_n^{(i)})^{-1}(Q^{(i)})$, $X = \bigcup\{X_i \mid i \ge 1\}$, and let $\varphi_n\colon X\to Y$ be a map such that $\varphi_n|X_i = f_n^{(i)}|X_i$, $i\ge 1$. Topologize the sets $X$ and $Y$ as the countable direct limits, $\varinjlim\{X_i\}$, $\varinjlim\{Q^{(i)}\}$; the resulting spaces are denoted by $\hat X$ and $\hat Y$, respectively. It is easy to see that the map $\varphi_n\colon \hat X\to\hat Y$ is continuous. It follows from characterization theorem 3.1 and the Sakai characterization theorem [11] that $\hat X \cong \mu_n^\infty$, $\hat Y\cong Q^\infty$.

The strong $(n,\infty)$-universality of the map $\varphi_n\colon \mu_n^\infty \cong \hat X \to \hat Y \cong Q^\infty$ is a consequence of condition 3.

We are going to show that the map $\varphi_n$ is unique up to a homeomorphism. Let $f\colon \mu_n^\infty \to Q^\infty$ be a strongly $n$-universal map. Write $\mu_n^\infty = \varinjlim A_i$, $Q^\infty = \varinjlim B_i$, where $A_i$, $B_i$ are compacta and $f(A_i)\subset B_i$ (we will denote by $f_i\colon A_i\to B_i$ the restriction of $f$). Assume that $A_1 = \{x_0\}$, $B_1 = \{y_0\}$.

**Claim.** Let $g\colon \mu_n^\infty \to Q^\infty$ be a strongly $n$-universal map, $h\colon A \to B$ be a map of metrizable compacta, where $\dim A \le n$. For any commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{i'} & \mu_n^\infty \\ {\scriptstyle h|A'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B' & \xrightarrow[j']{} & Q^\infty \end{array}$$

where $A'$, $B'$ are closed subsets in $A$, $B$ respectively, $i'$, $j'$ are embeddings, there exist embeddings $i\colon A \to \mu_n^\infty$, $j\colon B \to Q^\infty$ such that $i|A' = i'$, $j|B' = j'$ and $gi = jh$.

Indeed, there exists an embedding $j\colon B \to Q^\infty$ that extends $j'$. By the strong $(n, \infty)$-universality property, there exists an embedding $i\colon A \to \mu_n^\infty$ such that $i|A' = i'$ and $gi = jh$.

Suppose now that $g\colon C \to D$ is a strongly $(n, \infty)$-universal map, where $C \in \mathcal{MC}(n)^\infty$, $D \in \mathcal{MC}^\infty$. Write $C = \varinjlim C_i$, $D = \varinjlim D_i$, where $C_i$, $D_i$ are compacta and $g(C_i) \subset D_i$ (we denote by $g_i\colon C_i \to D_i$ the restriction of $g$).

Applying Claim one can easily construct a commutative diagram in the category of maps,

$$\begin{array}{ccccccc} f_{k_1} & \hookrightarrow & f_{k_2} & \hookrightarrow & f_{k_3} & \hookrightarrow & \cdots \\ \big\downarrow{\scriptstyle i_1} & \nearrow{\scriptstyle j_1} & \big\downarrow{\scriptstyle i_2} & \nearrow{\scriptstyle j_2} & \big\downarrow{\scriptstyle i_3} & \nearrow & \\ g_{l_1} & \hookrightarrow & g_{l_2} & \hookrightarrow & g_{l_3} & \hookrightarrow & \cdots \end{array}$$

in which $k_1 < k_2 < \dots$, $l_1 < l_2 < \dots$, and the morphisms $i_p$, $j_q$ are embeddings (in the category of maps). Then

$$f = \varinjlim f_{k_p} \simeq \varinjlim \{\, f_{k_1} \xrightarrow{i_1} g_{l_1} \xrightarrow{j_1} f_{k_2} \xrightarrow{i_2} g_{l_2} \xrightarrow{j_2} \cdots \,\} \simeq \varinjlim g_q = g.$$

□

The following result is a counterpart of the Product Theorem of the theory of $Q$-manifolds (see [5]) in the category $\mathcal{MC}(n)^\infty$.

**Theorem 4.4.** *Let $X \subset Q^\infty$, $X \in \mathcal{MC}^\infty$ and $X$ be an absolute neighborhood extensor for the class $\mathcal{MC}(n)$. Then $\varphi_n^{-1}(X)$ is a $\mu_n^\infty$-manifold.*

*Proof.* We verify the conditions of the characterization theorem 3.1 for $\mu_n^\infty$-manifolds. Obviously, $\varphi_n^{-1}(X) \in \mathcal{MC}^\infty$. Given a compact metrizable pair $(A, B)$ with $\dim A \le n$ and an embedding $f\colon B \to \varphi_n^{-1}(X)$, one can extend the map $\varphi f\colon B \to X$ to a map $g\colon A \to X$. It follows from the strong $n$-universality of $\varphi$ that there exists an embedding $\bar f\colon A \to \mu_n^\infty$ such that $\varphi \bar f = g$ and $\bar f|B = f$. Then $\bar f(A) \subset \varphi^{-1}(X)$ and we are done. □

**Theorem 4.5.** *There exists a strongly $(n,n)$-universal map $\psi_n\colon \mu_n^\infty \to \mu_n^\infty$, which is unique up to a homeomorphism.*

*Proof.* We suppose that $\mu_n^\infty \subset Q^\infty$. Let $X = \varphi_n^{-1}(\mu_n^\infty)$. We are going to show that $X$ is homeomorphic to $\mu_n^\infty$. Obviously, $X \in \mathcal{MC}(n)^\infty$. Let $(A,B)$ be a compact metrizable pair with $\dim A \le n$ and $f\colon B \to X$ an embedding. Since $\mu_n^\infty$ is an absolute extensor for metrizable compacta of dimension $\le n$, there exists an extension $g\colon A \to \mu_n^\infty$ of the map $\varphi_n f$. It follows from the strong $(n,\omega)$-universality property that there exists an embedding $\bar f\colon A \to \mu_n^\infty$ such that $\bar f|B = f$ and $\varphi_n \bar f = g$. The latter condition means that $\bar f(A) \subset X$ and, by the characterization theorem, $X \cong \mu_n^\infty$.

The strong $(n,n)$-universality of the map $\psi_n$ is an easy consequence of the strong $(n,\omega)$-universality property of the map $\varphi_n$.

In turn, the uniqueness of the map $\psi_n$ can be derived from its strong $(n,n)$-universality similarly as in the proof of Theorem 4.3.

□

**Theorem 4.6.** *There exists a strongly $(n,\omega)$-universal map $\psi_{n,\infty}\colon \mu_n^\infty \to \mathbb{R}^\infty$, which is unique up to a homeomorphism.*

*Proof.* We suppose that $\mathbb{R}^\infty \subset Q^\infty$. Let $X = \varphi_n^{-1}(\mathbb{R}^\infty)$. The rest of the proof is completely analogous to that of Theorem 4.5. □

## 5. Triangulation and classification theorems for $\mu_n^\infty$-manifolds

**Lemma 5.1.** *For every $\mu_n^\infty$-manifold $X$ there exists a locally finite polyhedron $P$ of dimension $\le n$ and a map $f\colon P \to X$ that induces an isomorphism of the homotopy groups in dimensions $\le n-1$.*

*Proof.* Let $X = \varinjlim\{M_i, s_i\}$, where

$$M_1 \xrightarrow{s_1} M_2 \xrightarrow{s_2} M_3 \xrightarrow{s_3} \cdots$$

is a sequence of compact $\mu_n$-manifolds and embeddings. For every $i$ there exist compact $\mu_n$-manifolds $M_i'$ and $M_i'' \subset M_i'$ such that $M_i, M_i''$ are $Z$-sets in $M_i'$ and there exists a polyhedrally $n$-soft retraction $r_i\colon M_i' \to M_i$ such that $r_i|M_i''\colon M_i'' \to M_i$ is a homeomorphism. This can be easily deduced from the properties of the universal map $g_n\colon \mu_n \to \mu_n$ (see [8]).

Define the space $X'$ as the quotient space of the disjoint union $\sqcup\{M_i' \mid i \in \mathbb{N}\}$ with respect to the equivalence relation that identifies every point $x \in M_i''$ with the point $s_i \circ r_i(x) \in M_{i+1} \subset M_{i+1}'$. By $q\colon \sqcup\{M_i' \mid i \in \mathbb{N}\} \to X'$ we denote the quotient map.

Define a map $h\colon X' \to X$ by the condition: if $x \in M_i$ then $h \circ q(x) = r_i(x) \in M_i \subset X$.

It is not difficult to show that the map $h$ induces an isomorphism of the homotopy groups in dimensions $\le n-1$. Since the space $X$ is locally compact, there exists a locally finite polyhedron $P$ of dimension $\leq n$ and a map $g\colon P \to X'$ that induces an isomorphism of the homotopy groups in dimensions $\leq n-1$. The composition $f = h \circ g$ is the required map.

□

**Lemma 5.2.** *Let $f, g\colon A \to X$ be $(n-1)$-homotopic maps of a metrizable compactum $A$. Then there exists a compactum $C \subset X$ such that $C \supset f(A) \cup g(A)$ and the maps $f, g\colon A \to C$ are $(n-1)$-homotopic.*

*Proof.* There exists an $n$-invertible map $h\colon B \to A$, where $B$ is an $n$-dimensional compactum [8]. Then the maps $fh$ and $gh$ are homotopic; denote by $H\colon B \times I \to X$ a homotopy connecting them. Let $C = H(B \times I)$.

If $\dim B' \leq n$ and a map $h'\colon B' \to A$ is given, then there exists a map $\alpha\colon B' \to B$ such that $h\alpha = h'$. Then $H(\alpha \times \mathrm{id}_I)$ is a homotopy of the maps $fh'$ and $gh'$. Thus, $f, g\colon A \to C$ are $(n-1)$-homotopic. □

The proof of the following lemma is a direct modification of the proof of Lemma 2.8.7 from [4]. Note that in [4] the notion of $\mu$-homotopy was used where we use $(n-1)$-homotopy.

**Lemma 5.3.** *Suppose that a map $f\colon X \to Y$ induces an isomorphism of homotopy groups in dimension $\leq n$, $Y$ is an $\mathrm{AE}(n)$-space, $(P, L)$ is a polyhedral pair with $\dim P \leq n$ and $\alpha\colon P \to Y$, $\beta\colon L \to X$ are maps such that $f\beta = \alpha|L$. Then there exists a map $\hat{\beta}\colon P \to X$ such that $\hat{\beta}|L = \beta$ and $f\hat{\beta} \sim_{n-1} \alpha$.*

**Lemma 5.4.** *Let $f\colon X \to Y$ be a map of $\mu_n^\infty$-manifolds which induces isomorphisms of the homotopy groups in dimension $\leq n-1$. For every compact metrizable pair $(A, B)$, where $\dim A \leq n$, and every pair of maps $\alpha\colon B \to X$, $\beta\colon A \to Y$ such that $\alpha$ is an embedding and $f\alpha \simeq_{n-1} \beta|B$ there exists an embedding $\alpha'\colon A \to X$ such that $\alpha'|B = \alpha$ and $f\alpha' \simeq_{n-1} \beta$.*

*Proof.* There exists an $n$-dimensional finite polyhedral pair $(P, L)$ and maps $g\colon A \to P$, $g'\colon P \to Y$ such that $g'g \simeq_{n-1} \beta$, $g(B) \subset L$ and there exists a map $h\colon L \to X$ such that $hg|B \simeq_{n-1} \alpha$ (see [4]).

By Lemma 5.1, there exists a map $g''\colon P \to X$ such that $g''|L = h$ and $fg'' \simeq_{n-1} g'$. Then, by Lemma 5.2 and Theorem 3.3, there exists a compact $\mu_n$-manifold $M \subset X$ such that $\alpha(B) \cup g''(P) \subset M$ and the maps $hg|B, \alpha\colon B \to M$ are $(n-1)$-homotopic. By [6, Proposition 2.2], there exists a map $\tilde{\alpha}\colon A \to M$ such that $\tilde{\alpha}|B = \alpha$ and $\tilde{\alpha} \simeq_{n-1} g''g$. By Theorem 3.3, there exists a compact $\mu_n$-manifold $M'$ such that $M \subset M' \subset X$ and $M$ is a $Z$-set in $M'$. Then, by the $Z$-set approximation theorem for $\mu_n$-manifolds ([4, Theorem 2.3.8]), there exists an embedding $\alpha'\colon A \to M'$ such that $\alpha' \simeq_{n-1} \tilde{\alpha}$ and $\alpha'|B = \alpha$. Then also

$$f\alpha' \simeq_{n-1} f\tilde{\alpha} \simeq_{n-1} fg''g \simeq_{n-1} g'g \simeq_{n-1} \beta.$$

□

The following result is a classification theorem for $\mu_n^\infty$-manifolds.

**Theorem 5.5.** *Let $f\colon X \to Y$ be a map of $\mu_n^\infty$-manifolds which induces isomorphisms of homotopy groups in dimension $\leq n-1$. Then the map $f$ is $(n-1)$-homotopic to a homeomorphism.*

*Proof.* Let $X = \varinjlim M_i$, $Y = \varinjlim N_j$ be representations of the spaces $X$ and $Y$ as countable direct limits of compact $\mu_n$-manifolds. By Theorem 3.3, we may require, in addition, that $M_i$ and $N_j$ are $Z$-sets in $M_{i+1}$ and $N_{j+1}$ respectively. Set $M_{i_0} = N_{i_0} = \emptyset$ and define by induction sequences $i_0 < i_1 < i_2 < \dots$ and $j_0 < j_1 < j_2 < \dots$, maps $f_k\colon X \to Y$, $\alpha_k\colon M_{i_k} \to Y$, $\beta_k\colon N_{j_k} \to X$ such that the following holds:

1. $f_{k+1} \simeq_{n-1} f_k$;
2. all $\alpha_k, \beta_k$ are embeddings, $\alpha_k(M_{i_k}) \subset N_{j_k}$, $\beta_k(N_k) \subset M_{i_{k-1}}$, and $\beta_k\alpha_k = \mathrm{id}$, $\alpha_{k+1}\beta_k = \mathrm{id}$, $\alpha_{k+1}|M_{i_k} = \alpha_k$, $\beta_{k+1}|N_{j_k} = \beta_k$; and
3. $f_k|M_{i_k} = \alpha_k$.

Set $f_0 = f$ and suppose that $f_l$, $i_l$, $j_l$, $\alpha_l$, and $\beta_l$ are already constructed for $l < k$. Choose $i_k > i_{k+1}$ so that $\beta_{k-1}(N_{i_{k-1}}) \subset M_{i_k}$. It follows from the $Z$-set approximation theorem that there exists an embedding $\alpha_k\colon M_{i_k} \to Y$ such that $\alpha_k|\beta_{k-1}(N_{j_{k-1}}) = \beta_{k-1}$ and $\alpha_k \simeq_{n-1} f_{k-1}|M_{i_{k-1}}$. By the $n$-homotopy extension property (see [6]), there exists a map $f_k\colon X \to Y$ such that $f_k|M_{i_k} = \alpha_k$ and $f_k \simeq_{n-1} f_{k-1}$. By the construction, $f_k \simeq_{n-1} f_0 = f$ and, therefore, the map $f_k$ induces isomorphisms of homotopy groups in dimension $\leq n-1$ (see [4]).

Choose $j_k > j_{k-1}$ so that $\alpha_k(M_{i_k}) \subset N_{j_k}$. By Lemma 5.4, for the map $f_k$ and embedding $N_{j_k} \hookrightarrow Y$ there exists an embedding $\beta_k\colon N_{j_k} \to X$ such that $\beta_k\alpha_k = \mathrm{id}$. By the construction, $\alpha_k\beta_{k-1} = \mathrm{id}$.

Then the map $\alpha = \varinjlim \alpha_k$ is a homeomorphism from $X = \varinjlim M_{i_k}$ into $Y = \varinjlim N_{j_k}$ with $\beta = \varinjlim \beta_k$ as the inverse. It follows from properies (1) and (2) that $\alpha \simeq_{n-1} f$. □

**Theorem 5.6.** *For every embedding $f$ of a $\mu_n^\infty$-manifold $X$ into $Q^\infty$ we have $X \cong \varphi_n^{-1}(f(X))$.*

*Proof.* It follows from Theorem 4.4 that $\varphi_n^{-1}(f(X))$ is a $\mu_n^\infty$-manifold. Note that the map $\varphi_n|\varphi_n^{-1}(f(X))\colon \varphi_n^{-1}(f(X)) \to f(X)$ induces an isomorphism of homotopy groups in dimensions $\leq n-1$. The result then follows from Theorem 5.5.

□

**Theorem 5.7.** *For every $\mu_n^\infty$-manifold $X$ there exists a locally finite polyhedron $P$ of dimension $\leq n$ such that for every embedding $P \subset Q^\infty$ we have $X \cong \varphi_n^{-1}(P)$.*

*Proof.* By Lemma 5.1, there exists a locally finite polyhedron $P$ of dimension $\leq n$ a map $f\colon P \to X$ that induces an isomorphism of the homotopy groups in dimensions $\leq n-1$. We may assume that $P \subset Q^\infty$, then the map $g = f \circ (\varphi|\varphi_n^{-1}(P))\colon \varphi_n^{-1}(P) \to X$ is a map of $\mu_n^\infty$-manifolds that induces isomorphisms of homotopy groups in dimensions $\leq n-1$. By Theorem 5.5, $g$ is $(n-1)$-homotopic to a homeomorphism. □

## 6. Open questions

There exist counterparts of the spaces $\mu_n^\infty$ which in the class of compact Hausdorff spaces of given weight play the role analogous to that of $\mu_n^\infty$ for the class of metrizable compacta. Namely, Dranishnikov constructed $n$-dimensional spaces $D_n^\tau$ that are universal for the class of compact Hausdorff spaces of weight $\tau$ and of dimension $n$. However, these spaces are not absolute extensors in dimension $n$, because by Dranishnikov's theorem every $n$-dimensional compact absolute extensor in dimension $n$ is metrizable. This does not allow straightforward extension of our results to the case of spaces of weight $\tau$. As a good starting point we propose the open problem of topological characterization of the countable direct limit of a sequence of spaces $D_n^\tau$ and $Z$-embeddings (see related paper [14]).

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