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LINEAR CONNECTIVITY FORCES LARGE COMPLETE BIPARTITE MINORS

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Linear Connectivity Forces Large Complete Bipartite Minors

Dedicated to Professor Neil Robertson on the occasion of his 65th birthday

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Abstract

Let a be an integer. It is proved that for any s and k, there exists a constant $N = N(s,k,a)$ such that every $\frac{31}{2}(a+1)$-connected graph with at least N vertices either contains a subdivision of $K_{a,sk}$ or a minor isomorphic to s disjoint copies of $K_{a,k}$. In fact, we prove that connectivity $3a+2$ and minimum degree at least $\frac{31}{2}(a+1)-3$ are enough. The condition "a subdivision of $K_{a,sk}$" is necessary since G could be a complete bipartite graph $K_{\frac{31}{2}(a+1),m}$, where m could be arbitrarily large. The requirement on $N(s,k,a)$ vertices is necessary since there exist graphs without K_a-minor whose connectivity is $\Theta(a\sqrt{\log a})$.

When $s=1$ and $k=a$, this implies that every $\frac{31}{2}(a+1)$-connected graph with at least $N(a)$ vertices has a K_a-minor. This is the first result where a linear lower bound on the connectivity in terms of a forces a K_a-minor. This was also conjectured in [68, 47, 69, 39]. Our result generalizes a recent result of Böhme and Kostochka [4] and resolves a conjecture of Fon-Der-Flaass [16].

Our result together with a recent result in [25] also implies that there exists an absolute constant c such that there are only finitely many ck-contraction-critical graphs without K_k as a minor and there are only finitely many ck-connected ck-color-critical graphs without K_k-minors. These results are related to the well-known conjecture of Hadwiger [17].

Our result was also motivated by the well-known result of Erdős and Pósa [15]. Suppose that G is $\frac{31}{2}(a+1)$-connected and without a subdivision of $K_{a,t}$. Then there exists an integer $F(s,k,a,t)$ such that either there are s disjoint copies of $K_{a,k}$-minor in G, or G has a vertex set F of order at most $F(s,k,a,t)$ such that $G-F$ has no minor isomorphic to $K_{a,k}$.

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1 Introduction

In this paper, all graphs are finite and may have loops and multiple edges. A graph H is a *minor* of a graph K if H can be obtained from a subgraph of K by contracting edges. A graph H is a *topological minor* of a graph K if K contains a subgraph which is isomorphic to a graph that can be obtained from H by subdividing some edges. In such a case, we also say that K contains a *subdivision* of H.

The study of graphs containing a given graph as a minor, or as a topological minor, has long history. Starting with Wagner's classification of graphs without a K_5-minor [73], there are many results concerning the structure of graphs that do not contain certain graph as a minor. These excluded graphs include $K_{3,3}$ [73], V_8 [52], the 3-cube [40], the octahedron [41], graphs with single crossing [56], and K_6^- [24]. See also [8], [65], [20], and [43].

There are several well-known structures which guarantee that certain minor exists in a graph G if G is large enough. For instance, any 5-connected graph on at least 11 vertices contains the 3-cube as a minor [40]. Any 5-connected non-planar graph on at least 8 vertices contains a V_8-minor [52]. In addition, there are Ramsey-type results similar to the fact that any sufficiently large connected graph contains either a k-path or a k-star. Oporowski, Oxley and Thomas [48] proved that any large 4-connected graph must have a large minor from a set of four families of 4-connected graphs. Moreover, they found a similar result for large 3-connected graphs. Recently, Kawarabayashi [26] proved a similar result for large 5-connected graphs. Ding [11] has characterized large graphs that do not contain a $K_{2,k}$ minor. A corollary of his result is that any large 5-connected graph contains a $K_{2,k}$ minor.

There is another direction for the study of graph minors: Wagner and Mader studied the maximum size of graphs not having K_k as a (topological) minor. Wagner [74] showed that a sufficiently large chromatic number (which depends only on k) guarantees K_k as a minor, and Mader [37] showed that a sufficiently large average degree will do the same.

Later, Kostochka [33, 34] and Thomason [67] independently proved that $\Theta(k\sqrt{\log k})$ is the correct order of the average degree forcing K_k as a minor. Recently, Thomason [68] found the asymptotically best possible value of this "extremal" function.

These results show that if the minimum degree of given graph G is a linear function of k, then G does not necessarily contain a K_k-minor. This does not improve even if we add a connectivity condition. Only the connectivity of order $\Theta(k\sqrt{\log k})$ forces the presence of K_k-minors.

However, as Thomason [68] pointed out, extremal graphs are more or less exactly vertex disjoint unions of suitable dense random graphs. Such graphs cannot have too many vertices. This fact also motivated Mader [39] (see [68, 69]) to ask the following.

Question (Mader). Suppose that G is a large ck-connected graph without K_k-minor, where c is some constant. What does G look like?

Motivated by this question and the results stated above, we prove the following theorem, which answers the question of Mader.

Theorem 1.1 *For any integers a, s and k, there exists a constant $N(s,k,a)$ such that every $(3a+2)$-connected graph of minimum degree at least $\frac{31}{2}(a+1)-3$ and with at least $N(s,k,a)$ vertices either contains $K_{a,sk}$ as a topological minor or a minor isomorphic to s disjoint copies of $K_{a,k}$.*

The proof of this result occupies whole Sections 3 and 5.

It is necessary to include the possibility of having $K_{a,sk}$ as a subdivision since G could be a complete bipartite graph $K_{\frac{31}{2}(a+1)-3,m}$, where m could be arbitrarily large. Recently, several extremal results concerning existence of complete bipartite graph minors have appeared [35, 36, 46, 47], but none of them implies that a linear connectivity in terms of a suffices to force $K_{a,k}$-minors for large values of k.

For $s=1$ and $k=a$, Theorem 1.1 immediately gives the following corollary.

Corollary 1.2 *For any a, there exists a constant $N(a)$ such that every $\frac{31}{2}(a+1)$-connected graph with at least $N(a)$ vertices has a K_a-minor.*

Again, this is the first result showing that a linear function of connectivity guarantees the existence of K_a-minors. (Actually, we prove a somewhat stronger result as stated in Theorem 1.1.) This settles a conjecture of Thomason [68, 47]. Notice that the extremal number of edges for K_a-minors are known only for $a \leq 9$. For up to K_7-minors, these are due to Mader [37]. For the K_8-minor, this is due to Jørgensen [21]. Recently, the K_9-minor case was settled by Song and Thomas [64]. Corollary 1.2 also implies the following result which is closely related to a recent result due to Böhme and Kostochka [4].

Corollary 1.3 *For every positive integers a and s, there is a number $N(a,s)$ such that every $\frac{31}{2}(a+1)$-connected graph with at least $N(a,s)$ vertices either contains a subdivision of $K_{a,s}$ or a minor isomorphic to s disjoint copies of K_a.*

Since $K_{a,sk}$ contains vertices of degree sk, Theorem 1.1 also implies the following result, which answers a question by Fon-Der-Flaass [16].

Corollary 1.4 *For every positive integers a, k and s, there exists a constant $N(k,s,a)$ such that every $\frac{31}{2}(a+1)$-connected graph with maximum degree at most $ks-1$ and with at least $N(k,s,a)$ vertices has a minor isomorphic to s disjoint copies of $K_{a,k}$.*

Our research is also motivated by Hadwiger's Conjecture from 1943 which suggests a far reaching generalization of the Four Color Theorem [1, 2, 63] and is one of the most interesting open problems in graph theory.

Conjecture 1.5 (Hadwiger [17]) *For every $k \geq 1$, every graph with chromatic number at least k contains the complete graph K_k as a minor.*

For $k = 1, 2, 3$, this is easy to prove, and for $k = 4$, Hadwiger himself [17] and Dirac [12] proved it. For $k = 5$, however, it becomes extremely difficult. In 1937, Wagner [73] proved that the case $k = 5$ is equivalent to the Four Color Theorem. So, assuming the Four Color Theorem [1, 2, 63], the case $k = 5$ in Hadwiger's Conjecture holds. Robertson, Seymour and Thomas [61] proved that a minimal counterexample to the case $k = 6$ is a graph G that has a vertex v such that $G - v$ is planar. Hence, assuming the Four Color Theorem, the case $k = 6$ of Hadwiger's Conjecture holds. This result is the deepest in this research area. So far, the conjecture is open for every $k \geq 7$. For the case $k = 7$, Kawarabayashi and Toft [32] proved that any

7-chromatic graph has K_7 or $K_{4,4}$ as a minor, and recently, Kawarabayashi [27] proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor.

It is not even known if there exists an absolute constant c such that any ck-chromatic graph has K_k as a minor. So far, it is known that there exists a constant c such that any $ck\sqrt{\log k}$-chromatic graph has K_k as a minor. Again, this follows from the results in [67, 68, 33, 34]. So it would be of great interest to prove that a linear function of the chromatic number is sufficient to force a K_k-minor. Let us observe that Reed and Seymour [51] proved the fractional version of this conjecture.

We hope that our result may be the first step to prove that conjecture since by Mader's result [38], any minimal counterexample to Hadwiger's conjecture has a "highly" connected subgraph. (Actually, Kawarabayashi [25] proved that any minimal counterexample to Hadwiger's conjecture is $\frac{k}{23}$-connected.) So if this graph were larger than $N(k)$ in Corollary 1.2, this would imply that there exists an absolute constant c such that any ck-chromatic graph has K_k as a minor. However, it is not clear whether this graph is large or not. Our result only implies that a minimum counterexample to the conjecture has "small" order. Our result also implies that there exist absolute constants c_1 and c_2 with $c_1 \geq c_2$ such that there are only finitely many c_1k-connected c_2k-color-critical graphs without K_k as a minor. This fact is related to Thomassen's result [70] which says that there are only finitely many 6-color-critical graphs on a fixed surface. Notice that the set of graphs embeddable on a fixed surface is closed under taking minors. More generally, Mohar [44] conjectured the following.

Conjecture 1.6 *There are only finitely many* 3*-connected* k*-color-critical graphs without* K_k *as a minor.*

Note that the above conjecture without the condition on 3-connectivity would be equivalent to Hadwiger's Conjecture since, as observed by Toft [72], if we have one such graph, then we would have infinitely many by applying the Hajós' construction. Hadwiger's conjecture suggests that there are no k-color-critical graphs without K_k as a minor. Since every 4-color-critical planar graph joined with the complete graph K_{k-5} gives rise to a $(k-1)$-color-critical graph without K_k-minor, the number k of colors is necessary. So, this conjecture weakens Hadwiger's conjecture in a sense, and our result implies that the linear chromatic number and connectivity are enough in Conjecture 1.6.

Let G be a graph satisfying the following conditions:

(i) G is k-chromatic.

(ii) G is minimal with respect to the minor-relation in the class of all k-chromatic graphs.

Any graph satisfying (i) and (ii) is said to be *k-contraction-critical*. Such graphs were first defined and studied by Dirac [13, 14]. Corollary 1.2 together with the main result of [25] implies that there exists a constant c such that there are only finitely many ck-contraction-critical graphs without K_k-minor.

Actually, our result implies the following.

Corollary 1.7 *There is a constant $c > 0$ and a polynomial time algorithm for deciding either that*

(1) *a given graph G is k-colorable, or*

(2) *G contains K_{ck}-minor, or*

(3) *G contains a minor H without K_{ck}-minor and with no k-coloring.*

Observe that if c would be 1, then H in (3) would be a counterexample to Hadwiger's conjecture.

For the history and other problems concerning Hadwiger's Conjecture, we refer the reader to [19] or [71].

A graph H is said to have the *Erdős-Pósa property*, if for every integer k there is an integer $f(k, H)$ such that every graph G contains k vertex-disjoint subgraphs, each containing an H-minor, or a set C of at most $f(k, H)$ vertices such that $G-C$ has no H-minor. The term Erdős-Pósa property arose because in [15], Erdős and Pósa proved that the cycle C_3 has this property.

Robertson and Seymour [54] proved that the Erdős-Pósa property holds for a graph H if and only if H is planar. Hence in general, the Erdős-Pósa property does not always hold. But if we restrict our attention to graphs that are "highly" connected or have large minimum degree, then the situation changes. For instance, the result in [31] says that if the minimum degree is at least 7, then either G contains a minor isomorphic to k disjoint copies of K_5 or there is a vertex set F of cardinality at most $f(k)$ such that $G - F$ is 5-degenerate, i.e., every induced subgraph of $G - F$ has a vertex of degree at most 5.

Theorem 1.1 implies the following general result.

Corollary 1.8 *Suppose G is $\frac{31}{2}(a+1)$-connected without a subdivision of $K_{a,sk}$. Then either there are s disjoint copies of $K_{a,k}$-minor or else there exists a constant $f(s,k,a)$ such that G has a vertex set F of order at most $f(s,k,a)$ such that $G - F$ has no minor isomorphic to $K_{a,k}$.*

How can one prove Theorem 1.1? We cannot use "extremal" results like those used in [67, 68] since these do not give a linear function of a. Instead, we will make use of "tree-width" and apply some deep results of Robertson and Seymour from [58, 59]. Tree-width was introduced by Halin in [18], but it went unnoticed until it was rediscovered by Robertson and Seymour [53] and, independently, by Arnborg and Proskurowski [3]. Tree-width was used not only for Graph Minor Theory [54, 57, 58, 59], but also for some structural graph theory results [54, 48, 62, 50, 10, 5]. In particular, three of us [5] proved the following result.

Theorem 1.9 ([5]) *For any positive integers k and w, there exists a constant $N = N(k,w)$ such that every 7-connected graph of tree-width at most w and of order at least N contains $K_{3,k}$ as a minor.*

In another paper [6], we extended Theorem 1.9 to the following result using the Robertson-Seymour structure theorems [58, 59].

Theorem 1.10 ([6]) *For any positive integer k, there exists a constant $N = N(k)$ such that every 7-connected graph of order at least N contains $K_{3,k}$ as a minor.*

In the forthcoming paper [30], we will develop further, and prove the following result.

Theorem 1.11 ([30]) *For any positive integer k, there exists a constant $N = N(k)$ such that every 9-connected graph of order at least N contains $K_{4,k}$ as a minor.*

In [5] it is also proved that for any $a \geq 3$ the following holds. For any positive integers k, a and w there exists a constant $N = N(k,w)$ such that every $265a$-connected graph of tree-width at most w and of order at least N contains $K_{a,k}$ as a minor. We improve this statement to the following result:

Theorem 1.12 *For any positive integers a, k, s and w, there exists a constant $N = N(a,k,s,w)$ such that every $(3a+1)$-connected graph with minimum degree at least $\frac{27}{2}(a+1)$, of tree-width at most w and of order at least N, either contains s disjoint $K_{a,k}$ minors or contains a subdivision of $K_{a,sk}$.*

The proof of Theorem 1.12 is given in Section 3.

By proving Theorem 1.1, we extend this result by omitting the tree-width condition. The basic approach is similar to that of [5], but it is more involved since we improve connectivity $265a$ used in [5] to $3a+1$ and, in addition, we either find s disjoint copies of $K_{a,k}$-minor or a subdivision of $K_{a,sk}$.

Theorem 1.9 is sharp in the sense that the 7-connectivity condition cannot be relaxed. Moreover, the function of the connectivity in Theorems 1.12 and 1.1 must be at least $2a+1$. These facts follow from a construction of a family of arbitrarily large $2a$-connected graphs (of tree-width $3a-1$) none of which contains a $K_{a,2a+1}$-minor; see [5].

Similarly, the following example shows that connectivity $3a+1$ in Theorem 1.12 is almost best possible.

Proposition 1.13 *For every positive integer a, there exist arbitrarily large $(3a-1)$-connected graphs of minimum degree $4a-2$ and tree-width $4a-2$ that neither contain $K_{a,k}$-subdivision nor they contain a minor isomorphic to a disjoint copies of $K_{a,k}$ for $k \geq 4a-1$.*

Proof. Let $C(a,n)$ be the graph with vertex set $V = \{(i,j) \mid 1 \leq i \leq a, 0 \leq j \leq n-1\}$ in which two distinct vertices (i,j) and (i',j') are adjacent if and only if $j-j'$ is 0 or ± 1 modulo n. The degree of each vertex of $C(a,n)$ is $3a-1$. It can be shown that $C(a,n)$ has tree-width $3a-1$ (when n is large enough) and that $C(a,n)$ does not contain $K_{a,k}$-minors if $k > 2a+1$. The proof of these facts can be found in [5].

Let $\tilde{C}(a,n)$ be the graph obtained from $C(a,n)$ by adding $a-1$ additional vertices, each of which is completely joined to $C(a,n)$. Clearly, $\tilde{C}(a,n)$ is $(3a-1)$-connected, its minimum degree is $4a-2$ and its tree-width is $(3a-1)+(a-1) = 4a-2$.

$\tilde{C}(a,n)$ has as many vertices as we want, just take sufficiently large n. Since it has only $a-1$ vertices of degree more than $4a-2$, it does not contain a $K_{a,4a-1}$-subdivision. If it would contain a minor isomorphic to a disjoint copies of $K_{a,k}$, one of them would be contained in $C(a,n)$ which is not possible as mentioned above. This contradiction completes the proof. □

2 Highly linked subgraphs

A graph L is said to be *k-linked* if it has at least $2k$ vertices and for any ordered k-tuples $(s_1,\ldots,s_k)$ and $(t_1,\ldots,t_k)$ of $2k$ distinct vertices of L, there exist pairwise disjoint paths $P_1, \ldots, P_k$ such that for $i = 1, \ldots, k$, the path P_i connects s_i and t_i. Such collection of paths is called a *linkage* from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$.

An important tool will be the following theorem due to Thomas and Wollan [66].

Theorem 2.1 *Every $2k$-connected graph G with at least $5k|V(G)|$ edges is k-linked.*

Theorem 2.1 implies that every $10k$-connected graph is k-linked. Bollobás and Thomason [7] proved that every $22k$-connected graph is k-linked, and Kawarabayashi, Kostochka and Yu [28] proved that every $12k$-connected graph is k-linked.

Let G be a graph and let A, B be subgraphs of G. We say that the pair (A, B) is a *separation* of G if $A \cup B = G$, $V(A) - V(B) \neq \emptyset$, and $V(B) - V(A) \neq \emptyset$. The *order* of a separation (A, B) is $|V(A) \cap V(B)|$.

The following result is a variation of an old theorem of Mader [38].

Theorem 2.2 *Let G be a graph and k an integer such that*

(a) $|V(G)| \geq \frac{5}{2}k$ *and*

(b) $|E(G)| \geq \frac{25}{4}k|V(G)| - \frac{25}{2}k^2$.

Then $|V(G)| \geq 10k + 2$ and G contains a $2k$-connected subgraph H with at least $5k|V(H)|$ edges.

Proof. Clearly, if G is a graph on n vertices with at least $\frac{25}{4}kn - \frac{25}{2}k^2$ edges, then $\frac{25}{4}kn - \frac{25}{2}k^2 \leq \binom{n}{2}$. Hence, either $n \leq \frac{25}{4}k + \frac{1}{2} - \frac{1}{4}\sqrt{(25k+2)^2 - 400k^2} < \frac{5}{2}k$ or $n \geq \frac{25}{4}k + \frac{1}{2} + \frac{1}{4}\sqrt{(25k+2)^2 - 400k^2} > 10k + 1$. Since $|V(G)| \geq \frac{5}{2}k$, we get the following:

Claim 1. $|V(G)| \geq 10k + 2$.

Suppose now that the theorem is false. Let G be a graph with n vertices and m edges, and let k be an integer such that (a) and (b) are satisfied. Suppose, moreover, that

(c) G contains no $2k$-connected subgraph H with at least $5k|V(H)|$ edges, and

(d) n is minimal subject to (a), (b) and (c).

Claim 2. *The minimum degree of G is more than $\frac{25}{4}k$.*

Suppose that G has a vertex v with degree at most $\frac{25}{4}k$, and let G' be the graph obtained from G by deleting v. By (c), G' does not contain a $2k$-connected subgraph H with at least $5k|V(H)|$ edges. Claim 1 implies that $|V(G')| = n - 1 \geq \frac{5}{2}k$. Finally, $|E(G')| \geq m - \frac{25}{4}k \geq \frac{25}{4}k|V(G')| - \frac{25}{2}k^2$. Since $|V(G')| < n$, this contradicts (d) and the claim follows.

Claim 3. *$m \geq 5kn$.*

The claim follows easily from (b) by using Claim 1.

By Claim 3 and (c), G is not $2k$-connected. Since $n > 2k$, this implies that G has a separation (A_1, A_2) such that $A_1 \setminus A_2 \neq \emptyset \neq A_2 \setminus A_1$ and $|A_1 \cap A_2| \leq 2k - 1$. By Claim 2, $|A_i| \geq \frac{25}{4}k + 1$. For $i \in \{1, 2\}$, let G_i be a subgraph of G with vertex set A_i such that $G = G_1 \cup G_2$ and $E(G_1 \cap G_2) = \emptyset$. Suppose that $|E(G_i)| < \frac{25}{4}k|V(G_i)| - \frac{25}{2}k^2$ for $i = 1, 2$. Then

$$\begin{aligned} \frac{25}{4}kn - \frac{25}{2}k^2 &\leq m = |E(G_1)| + |E(G_2)| \\ &< \frac{25}{4}k(n + |A_1 \cap A_2|) - 25k^2 \\ &\leq \frac{25}{4}kn - \frac{25}{2}k^2, \end{aligned}$$

a contradiction. Hence, we may assume that $|E(G_1)| \geq \frac{25}{4}k|V(G_1)| - \frac{25}{2}k^2$. Since $n > |V(G_1)| \geq \frac{25}{4}k + 1$ and G_1 contains no $2k$-connected subgraph H with at least $5k|V(H)|$ edges, this contradicts (d), and the proposition is proved. □

By Theorem 2.1, every $2k$-connected graph G with at least $5k|V(G)|$ edges is k-linked. Hence, Theorem 2.2 implies the following:

Corollary 2.3 *Let G be a graph and k an integer such that*

(a) $|V(G)| \geq \frac{5}{2}k$ *and*

(b) $|E(G)| \geq \frac{25}{4}k|V(G)| - \frac{25}{2}k^2$.

Then G contains a k-linked subgraph.

3 Bounded tree-width structure

In this section, we consider the bounded tree-width case and prove Theorem 1.12.

A *tree decomposition* of a graph G is a pair (T, Y), where T is a tree and Y is a family $\{Y_t \mid t \in V(T)\}$ of vertex sets $Y_t \subseteq V(G)$, such that the following two properties hold:

(W1) $\bigcup_{t \in V(T)} Y_t = V(G)$, and every edge of G has both ends in some Y_t.

(W2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The *width* of a tree decomposition (T, Y) is $\max_{t \in V(T)}(|Y_t| - 1)$. It was shown in [48] that if a graph G has a tree decomposition of width at most w then G has a tree decomposition of width at most w that further satisfies:

(W3) For every two vertices t, t' of T and every positive integer k, either there are k disjoint paths in G between Y_t and $Y_{t'}$, or there is a vertex t'' of T on the path between t and t' such that $|Y_{t''}| < k$.

(W4) If t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$.

(W5) If $t_0 \in V(T)$ and B is a component of $T - t_0$, then $V_1 = \bigcup_{t \in V(B)} Y_t \setminus Y_{t_0} \neq \emptyset$.

In the rest of this section, we give a proof of Theorem 1.12. Let a, k, s and w be given positive integers. Let G be a connected graph with a tree decomposition (T, Y) of width at most w that satisfies (W1)–(W5).

We will develop a structure that is similar to that used in [48] and in [5]. First, we define the constants that will be used in the proofs:

$$\begin{aligned}
n_1 &= g^{n_2}, \quad \text{where } g = (sk-1)\binom{w+1}{a} \\
n_2 &= n_3^{w+1} \\
n_3 &= (2n_4)^p, \quad \text{where } p = 2^{w+1} \\
n_4 &= n_5^q, \quad \text{where } q = 2^{w(w+1)/2} \\
n_5 &= 2s\, n_6 \\
n_6 &= (29a+6)k\binom{w+1}{a}.
\end{aligned}$$

We assume that $|V(G)| = N \geq (w+1)n_1$ and that G has neither s disjoint $K_{a,k}$-minors nor $K_{a,sk}$-subdivision.

Claim 3.1 *If G is a-connected, then $|V(T)| \geq n_1$ and every vertex of T has degree at most $g = (sk-1)\binom{w+1}{a}$. Consequently, T contains a path R of length $|E(R)| \geq n_2$.*

Proof. The first inequality follows from (W1). Suppose that $t_0 \in V(T)$ has degree at least $g+1$. Let $\mathcal{C}$ be the set of components of $G - Y_{t_0}$. By (W2) and (W5), it is clear that $|\mathcal{C}| \geq g+1$. For $C \in \mathcal{C}$, let v be a vertex in C. Since G is a-connected, there exist a internally disjoint paths connecting v with a distinct vertices in Y_{t_0}. Let $S(C)$ be the union of these paths, and let $X(C)$ be the set of their endvertices in Y_{t_0}. By the Pigeonhole Principle, there is a set $\mathcal{C}' \subseteq \mathcal{C}$ of sk components for which $X(C)$ contains the same set of a vertices of Y_{t_0}. Now it is clear that the union of $S(C)$ for $C \in \mathcal{C}'$ is a subdivision of $K_{a,sk}$ in G. □

From this point on we will no longer need the assumption that the parts Y_t of the tree decomposition have at most $w+1$ vertices. What we will need is the long path R and the assumption that the *adhesion* along R is bounded, where the adhesion is defined as

$$\max\{|Y_t \cap Y_{t'}|\,;\, t, t' \in V(R)\}. \tag{1}$$

This weaker assumption will allow us to use the subsequent conclusions of this section in the analysis of the long vortex structure in Section 5.

For $t \in V(R)$, let t' be its successor on R. Let $\bar{S}_t = Y_t \cap Y_{t'}$. By (W5), every $\bar{S}_t$ separates G. In particular, $|\bar{S}_t| \geq c$ if G is c-connected. The next claim, whose proof can be found in [48] or [5], enables us to assume that there are arbitrarily many such separators $\bar{S}_t$ of the same size and that there is a linkage through all of them.

Claim 3.2 *There is a subsequence $r_1, r_2, \ldots, r_{n_3}$ of length n_3 of the vertices of R such that for some $q \geq 1$, $|\bar{S}_{r_i}| = q$ for $i = 1, 2, \ldots, n_3$, and for every vertex t of R between r_1 and r_{n_3}, $|\bar{S}_t| \geq q$.*

From now on we replace R by the subpath from r_1 to r_{n_3}. Note that $q \leq w+1$.

By (W3) and Claim 3.2, there are q disjoint paths in G from Y_{r_1} to $Y_{r_{n_3}}$. Fix these paths, denote them by $P_1, \ldots, P_q$, and put $Z = P_1 \cup \cdots \cup P_q$. Since G is 3-connected, these paths can be chosen such that every Z-bridge in G is attached to at least two of the paths (cf., e.g., [23]), which we assume henceforth. Let us recall that a *Z-bridge* in G is either an edge $e \in E(G) \setminus E(Z)$ whose endvertices are both in Z, or a subgraph of G

consisting of a connected component C of $G - Z$ together with all edges joining C and Z. The vertices of a Z-bridge B in $Z \cap B$ are called *vertices of attachment* of B, and we say that B is *attached* to Z at these vertices.

Denote the subpath of P_j with one end in $\bar{S}_t$ and the other end in $\bar{S}_{t'}$ by $P_j(t, t')$ for any $t, t' \in \{r_1, \ldots, r_{n_3}\}$. Let $p_1, \ldots, p_n$ be a subsequence of $r_1, \ldots, r_{n_3}$. The path P_j is said to be *trivial* if $P_j(p_1, p_n)$ is a single vertex, and it is said to be *everywhere nontrivial* (*almost nontrivial*) w.r.t. the sequence $p_1, \ldots, p_n$ if $P_j(p_i, p_{i+1})$ contains at least three (respectively, at least two) vertices for every $i = 1, \ldots, n-1$. The paths P_j and P_l are said to be *everywhere bridge connected* (resp. *everywhere bridge disconnected*) with respect to $p_1, \ldots, p_n$ if for every $i = 1, \ldots, n-1$, there exists (resp. does not exist) a Z-bridge which has a vertex of attachment in $P_j(p_i, p_{i+1})$ and a vertex of attachment in $P_l(p_i, p_{i+1})$.

The following claim can be found in [5].

Claim 3.3 *There is a subsequence $p_1, p_2, \ldots, p_{n_5}$ of $r_1, r_2, \ldots, r_{n_3}$ of length n_5 such that for each $j = 1, \ldots, q$, $P_j(p_1, p_{n_5})$ is either trivial or everywhere nontrivial (w.r.t. the subsequence). Moreover, for every pair of distinct indices $j, l \in \{1, \ldots, q\}$, $P_j(p_1, p_{n_5})$ and $P_l(p_1, p_{n_5})$ are either everywhere bridge connected or everywhere bridge disconnected (w.r.t. the new subsequence).*

Proof. Clearly, there is a subsequence of $r_1, \ldots, r_{n_3}$ of length $\sqrt{n_3}$ such that the corresponding segment of P_1 is either trivial or everywhere almost nontrivial with respect to the subsequence. By repeating this argument on the subsequence for $P_2, \ldots, P_q$, respectively, we end up with a sequence of length at least $2n_4$ such that every path is either trivial or everywhere almost nontrivial. By taking every second element of this sequence, a subsequence of length n_4 satisfying the first part of the claim is obtained. Starting with that subsequence, one can obtain a subsequence of length n_5 satisfying also the second part of the claim by using similar arguments as above, except that we have to repeat the subsequence argument $\binom{q}{2} \leq \binom{w+1}{2}$ times. $\square$

Following [5], we introduce the *auxiliary graph* Γ. It has vertex set $V(\Gamma) = \{P_1, \ldots, P_q\}$, and the paths P_j and P_l are adjacent vertices in Γ if they are everywhere bridge connected w.r.t. $p_1, \ldots, p_{n_5}$.

At least one of the paths is everywhere nontrivial, say P_1. Let Γ_1 be the induced subgraph of Γ on the everywhere nontrivial paths. Let Γ_0 be the connected component of Γ_1 containing P_1. Note that Γ_0 contains none of the everywhere trivial paths. Let $\{P_1, \ldots, P_{q_0}\}$ ($q_0 = |V(\Gamma_0)|$) be the paths in $V(\Gamma_0)$.

For $i = 1, 2, \ldots, n_5 - 1$, denote by $Z'(i)$ the union of $P_j(p_i, p_{i+1})$ for $j = 1, \ldots, q_0$ together with all those trivial paths that are everywhere bridge connected to some path $P_j \in V(\Gamma_0)$. Let $\hat{Z}_i$ be the subgraph of G obtained by taking the union of $Z'(i)$ and all those Z-bridges B that have all vertices of attachment in $Z'(i)$. Finally, let Z_i be the subgraph of $\hat{Z}_i$ obtained by deleting all vertices corresponding to the trivial paths. Furthermore, we write $S_i = \bar{S}_{p_i} \cap (P_1 \cup \cdots \cup P_{q_0})$.

Let $r = 25a + 2$. For $i = 1, 2, \ldots, n_5 - r$, let $H_i = \bigcup_{j=0}^{r-1} Z_{i+j}$ and $\hat{H}_i = \bigcup_{j=0}^{r-1} \hat{Z}_{i+j}$.

Claim 3.4 *If G is a-connected, then at most $a-1$ trivial paths are adjacent to Γ_0 in Γ.*

Proof. Let A be the set of vertices of those everywhere trivial paths that are adjacent to Γ_0 in Γ. Suppose that $|A| \geq a$. For the purpose of this proof, let us say that $\hat{H}_i$ is separable if there is a separation (A_i, B_i) of $\hat{H}_i$ of order at most $a - 1$ such that $A \subseteq A_i$ and $B_i - A_i$ contains a vertex in Z_{i+2a}. Suppose that there exists a set I of $(sk-1)\binom{w+1}{a} + 1$ values of i such that $\hat{H}_i$ is not separable for $i \in I$ and such that any two distinct elements $i, j \in I$ differ by at least $r+1 = 25a+3$. Since $H_i \cap H_j = \emptyset$ whenever $|i-j| \geq r+1$, the corresponding graphs H_i $(i \in I)$ are pairwise disjoint.

For each $i \in I$, choose a internally disjoint paths in $\hat{H}_i$ from a vertex in Z_{i+2a} to a distinct vertices in A. Such paths exist by Menger's theorem since $\hat{H}_i$ is not separable. By the Pigeonhole Principle, there is a subset of sk of such subgraphs $\hat{H}_i$ whose a paths end up at the same a-tuple of vertices in A. Clearly, the internally disjoint paths in these sk subgraphs form a subdivision of $K_{a,sk}$. This contradiction shows that $\hat{H}_i$ is separable for all but at most $(r+1)(sk-1)\binom{w+1}{a}$ values of i.

Since $n_5 - r > (r+1)(sk-1)\binom{w+1}{a}$, there is an i such that $\hat{H}_i$ is separable. Let (A_i, B_i) be a corresponding separation chosen so that $B_i - A_i$ is connected. Since $|A_i \cap B_i| \leq a-1$, there exists $p \in A$ such that $p \in A_i - B_i$. Similarly, we see that there exist j, l where $1 \leq j < 2a$ and $2a < l \leq 4a$ such that neither Z_{i+j} nor Z_{i+l} contains a vertex in $A_i \cap B_i$. Since $\hat{Z}_{i+j} - A_i$ is a connected subgraph of $\hat{H}_i$ that contains p, we conclude that $Z_{i+j} \subseteq A_i - B_i$. Similarly, we see that $Z_{i+l} \subseteq A_i - B_i$. The assumption that $B_i - A_i$ is connected and contains a vertex in Z_{i+2a} implies that $B_i - A_i$ does not intersect $S_i \cup S_{i+r}$. This implies that $A_i \cap B_i$ separates the graph G. This contradicts the assumption that G is a-connected and shows that $|A| \leq a-1$. □

An immediate corollary of Claim 3.4 is

Claim 3.5 *If G is $3a$-connected, then $|V(\Gamma_0)| \geq a+1$.*

Proof. Let $q_0 = |V(\Gamma_0)|$. Since the $2q_0$ vertices in $S_i \cup S_{i+r}$ together with at most $a-1$ vertices of trivial paths adjacent to Γ_0 in Γ separate the graph G, we have $2q_0 + (a-1) \geq 3a$. This implies that $q_0 \geq a+1$. $\square$

Claim 3.6 *Let T_0 be a spanning tree of Γ_0. If $q_0 \geq a+1$, there are vertices $t_0, t_1, \ldots, t_a$ of T_0 such that for $l = 0, \ldots, a$, the vertex t_l has degree 1 or 0 in the subtree $T_0 \setminus \{t_0, \ldots, t_{l-1}\}$.*

For $X \subseteq \{1, \ldots, q_0\}$, we define $X(i) = \{P_x \cap S_i \mid x \in X\}$ as the set of vertices in S_i that lie on the paths whose indices are in X.

Claim 3.7 *Let $X, Y \subseteq \{1, \ldots, q_0\}$, where $|X| = |Y| = a+1$. If $j \geq i+4a+4$, then $Z_i \cup Z_{i+1} \cup \cdots \cup Z_{j-1}$ contains $a+1$ disjoint paths connecting $X(i)$ with $Y(j)$.*

Proof. Let T_0 be a spanning tree of Γ_0, let $t_0, \ldots, t_a$ be as stated in Claim 3.6, and let $U = \{t_0, \ldots, t_a\}$. We will identify the elements of X and Y with the corresponding vertices of T_0.

First, we prove that there are paths connecting $X(i)$ with $U(i+2a+2)$ in $Z_i \cup \cdots \cup Z_{i+2a+1}$. Choose an enumeration $x_0, \ldots, x_a$ of elements of X such that for $l = 0, \ldots, a$, the distance from x_l to t_l in T_0 is minimum among all elements of $X \setminus \{x_0, \ldots, x_{l-1}\}$.

In Z_i we start at $X(i)$ and follow the paths P_l ($l \in X \setminus \{x_0\}$) until S_{i+2}. The path P_{x_0} is re-routed to P_{t_0} as follows. In Z_i, we use Z-bridges corresponding to the edges on the path in T_0 from x_0 to t_0 to get a path from P_{x_0} to P_{t_0}, and then we follow P_{t_0} through all the remaining parts $Z_{i+1}, \ldots, Z_{i+2a+1}$ to reach $U(i+2a+2)$. Since x_0 was selected as a vertex that is closest to t_0 in T_0, the resulting path does not intersect other paths within $Z_i \cup Z_{i+1}$. In the following two parts, $Z_{i+2} \cup Z_{i+3}$, we repeat the process with the remaining paths. All of them, except P_{x_1}, just follow the paths P_l, while P_{x_1} is re-routed to P_{t_1} within Z_{i+2} (using bridges corresponding to the edges on the (x_1, t_1)-path in T_0), and afterwards it just follows P_{t_1} to reach $U(i+2a+2)$. By the choice of x_1, the re-routed path does not intersect other paths. Since x_0 was selected as a leaf, the re-routed path cannot intersect P_{t_0}. This process is repeated for the remaining paths, P_{x_j} being re-routed in parts Z_{i+2j} and Z_{i+2j+1}. Re-routing never intersects the subsequent paths since x_j was selected to be closest to t_j in T_0, and does not intersect with any of the previous ones (namely $P_{t_0}, \ldots, P_{t_{j-1}}$) since

$t_0, \dots, t_a$ have been selected according to Claim 3.6. Therefore the process yields desired paths to $U(i+2a+2)$.

In the same way we can connect $Y(j)$ with $U(j-2a-2)$ in $Z_{j-1} \cup \cdots \cup Z_{j-2a-2}$ (going in the "backwards" direction). Since $i+2a+1 < j-2a-2$, we can link $U(i+2a+2)$ with $U(j-2a-2)$ so that the resulting collection of $a+1$ paths from $X(i)$ to $Y(j)$ are pairwise disjoint. $\square$

We shall prove that every subsequence of length n_6 of our sequence $p_1, \dots, p_{n_5}$ gives rise to a $K_{a,k}$-minor in the union of the corresponding subgraphs H_i. This will show that there are s disjoint $K_{a,k}$-minors in G. Therefore, it suffices to consider the initial subsequence for $i = 1, \dots, n_6$ and prove that there is a $K_{a,k}$-minor.

Claim 3.8 *Suppose that G is a-connected. If the minimum degree of G is at least $\frac{27}{2}(a+1)$, then the average degree of H_i is at least $\frac{25}{2}(a+1)$.*

Proof. By Claim 3.4, every vertex in $H_i - (S_i \cup S_{i+r})$ has at least $\frac{27}{2}(a+1) - (a-1) = \frac{25}{2}(a+1) + 2$ neighbors in H_i. Therefore, if h is the number of vertices of H_i, the average degree of H_i is at least

$$\frac{(\frac{25}{2}(a+1)+2)(h-2q_0)}{h}. \tag{2}$$

Since $h \geq rq_0$, we have

$$\frac{h-2q_0}{h} \geq 1 - \frac{2}{r} = \frac{25a}{25a+2}. \tag{3}$$

Now, (2) and (3) easily imply the conclusion of the claim. $\square$

From now on we assume that the minimum degree of G is at least $\frac{27}{2}(a+1)$ and that G is $(3a+1)$-connected. By Corollary 2.3 and Claim 3.8 we conclude:

Claim 3.9 *For every i, H_i contains an $(a+1)$-linked subgraph M_i.*

Claim 3.10 *There are $2a+2$ pairwise disjoint paths $Q_0, \dots, Q_a$ and $Q'_0, \dots, Q'_a$ in H_i such that the following properties hold:*

(a) *For $l = 0, \dots, a$, the path Q_l starts in M_i and ends in S_{i+r}.*

(b) *For $l = 0, \dots, a$, the path Q'_l starts in S_i and ends in M_i.*

Proof. Let A be the set of vertices of everywhere trivial paths that are adjacent to Γ_0 in Γ. We take a set of $2a+2$ disjoint paths $\mathcal{W} = \{W_1, \ldots, W_{2a+2}\}$ joining M_i with $S_i \cup S_{i+r}$ in H_i such that:

(i) The number of edges in $\bigcup_{l=1}^{2a+2} E(W_l) \setminus \bigcup_{j=0}^{r-1} E(Z'(i+j))$ is minimum.

(ii) Let n_L be the number of paths W_l ending in S_i, and let n_R be the number of paths W_l ending in S_{i+r}. Subject to (i), we assume that $|n_L - n_R|$ is minimum.

By Claim 3.4, $|A| \leq a-1$. Since G is $(3a+1)$-connected, $G - A$ is $(2a+2)$-connected. By applying Menger's theorem to $G - A$, we see that such a collection of paths $\mathcal{W}$ exists. Let us observe that some of the paths may be trivial since M_i may contain vertices in $S_i \cup S_{i+r}$.

If at least two paths in $\mathcal{W}$ intersect a path P_j, let W and W' be the paths that intersect P_j as close as possible (on P_j) to S_i and S_{i+r}, respectively. If $W = W'$, suppose that the intersection u of W with P_j nearest S_i (say) comes before the intersection nearest S_{i+r}. By (i), W ends at S_i, i.e., its segment from u to its end coincides with the segment $P_j(S_i, u)$ of P_j. This shows that $W \neq W'$. Then the path W (resp. W') must end at S_i (resp. S_{i+r}) by (i).

Suppose that precisely one path, say $W \in \mathcal{W}$, intersects a path P_j. In this case, we can elect to have W ending at $P_j \cap S_i$ or at $P_j \cap S_{i+r}$ by following the path P_j.

This implies that the value $|n_L - n_R|$ in (ii) can be made to be zero or one. However, since $n_L + n_R = 2a+2$ is even, we conclude that $n_L - n_R = 0$.

Now let the $a+1$ paths in $\mathcal{W}$ that end in S_i be called $Q'_0, \ldots, Q'_a$ and the $a+1$ paths in $\mathcal{W}$ that end in S_{i+r} be called $Q_0, \ldots, Q_a$. This completes the proof. □

Define $\alpha = r + 4a + 4$ and for $t = 1, \ldots, ak$ set $i_t = 1 + (t-1)\alpha$. Observe that $i_{ak} \leq n_6 - r$.

We shall now construct disjoint paths $\mathcal{P}_l^\circ$ ($l = 0, \ldots, a$) from S_1 to S_{n_6} satisfying the following additional condition. For $t = 1, \ldots, ak$, the subgraph Z_{i_t+r+1} contains a path D_t which connects $\mathcal{P}_0^\circ$ with $\mathcal{P}_j^\circ$, where $j \in \{1, \ldots, a\}$ is congruent to t modulo a and D_t is internally disjoint from the paths $\mathcal{P}_l^\circ$. Having such a collection of paths, a $K_{a,k}$-minor is easily constructed. First, by contracting the paths $\mathcal{P}_l^\circ$ for $l = 1, \ldots, a$, we get a vertices that will play the role of the vertices of degree k in the $K_{a,k}$-minor. To get the vertices of the other class, we divide $\mathcal{P}_0^\circ$ into k segments, each containing parts of the path in subgraphs Z_{i_t+r+1} for a consecutive values of t. By contracting each of these k segments of $\mathcal{P}_0^\circ$, the paths D_t can be used to get the desired $K_{a,k}$-minor.

It remains to see how to obtain the paths $\mathcal{P}_l^\circ$ and D_t. In each H_{i_t} we take $a+1$ paths joining S_{i_t} with S_{i_t+r} and passing through the $(a+1)$-linked subgraph M_{i_t}. They can be obtained by Claim 3.10: by using paths $Q'_0, \ldots, Q'_a$ we join S_{i_t} with M_{i_t}, and by using $Q_0, \ldots, Q_a$ we join M_{i_t} with S_{i_t+r}. Since M_{i_t} is $(a+1)$-linked, the endvertices of $Q'_0, \ldots, Q'_a$ in M_{i_t} can be linked to the endvertices of $Q_0, \ldots, Q_a$ in M_{i_t}. At this moment we do not yet specify which vertex is actually linked to which one under this linkage, since we will need this freedom in order to prove that appropriate paths D_t exist.

Claim 3.7 can be used to link the ends of the paths $Q_0, \ldots, Q_a$ in S_{i_t+r} with the initial vertices in $S_{i_{t+1}}$ of the paths constructed in $H_{i_{t+1}}$, $t = 1, \ldots, ak-1$. In the subgraph Z_{i_t+r+1}, there exists a path D_t joining two of the constructed paths. Now, the linkage in M_{i_t} can be chosen in such a way that D_t will connect $\mathcal{P}_0^\circ$ with $\mathcal{P}_j^\circ$, where $j \in \{1, \ldots, a\}$ is congruent to t modulo a. This gives rise to appropriate paths.

This completes the proof of Theorem 1.12.

4 The Excluded Minor Theorem

Hereby, we shall consider the case when the tree-width is arbitrarily large. We shall make use of Robertson-Seymour's Excluded Minor Theorem [58] which describes the structure of graphs that do no contain a given graph as a minor. A strengthened version of that theorem was proved in [59]. This version enables us to apply the method used in the case of bounded tree-width in the part of the proof when we consider the vortex structure.

Let (T,Y) be a tree decomposition of a graph G. For an edge $tt' \in E(T)$, let $Z_{tt'} = Y_t \cap Y_{t'}$. Let us recall that the *adhesion* of a tree decomposition (T,Y) is $\max |Z_{tt'}|$ taken over all edges $tt' \in E(T)$. If T is a path, then (T,Y) is also said to be a *path decomposition* of G.

It is easy to see that for every tree decomposition (T^0, Y^0) of G there exists a tree decomposition (T,Y) of G having the same width and not larger adhesion than (T^0, Y^0) satisfying (W4) and (W5) of the tree-decomposition.

Let $t_1 t_2 \in E(T)$ and let V_1 be the vertex set defined in (W5). Define similarly the set V_2. Then also $V_2 \neq \emptyset$ and hence $Z_{t_1 t_2}$ is a separating set of G which separates V_1 and V_2 in G.

Let G be a graph and let $W = \{w_0, \ldots, w_n\}$, $n = |W| - 1$, be a linearly ordered subset of its vertices such that w_i precedes w_j in the linear order if and only if $i < j$. The pair (G, W) is called a *vortex* of *length* n, W is the *society* of the vortex and all vertices in W are called *society vertices*. Suppose that for $i = 0, \ldots, n$, there exist vertex sets $X_i \subseteq V(G)$ with the following properties:

(V1) $w_i \in X_i$ for $i = 0, \ldots, n$,

(V2) $\cup_{0 \le i \le n} X_i = V(G)$,

(V3) every edge of G has both endvertices in some X_i,

(V4) if $i \le j \le k$, then $X_i \cap X_k \subseteq X_j$, and

(V5) if $j \notin \{i, i+1\}$, then $w_j \notin X_i$.

Then the family $(X_i \mid i = 0, \ldots, n)$ is a *vortex decomposition* of the vortex (G, W). For $i = 1, \ldots, n$, denote by $Z_i = (X_{i-1} \cap X_i) \setminus W$. The *adhesion* of the vortex decomposition is the maximum of $|Z_i|$, for $i = 1, \ldots, n$. The vortex decomposition is *linked* if for $i = 1, \ldots, n-1$, the subgraph of G induced on the vertex set $X_i \setminus W$ contains a collection of disjoint paths linking Z_i with Z_{i+1}. Clearly, in that case $|Z_i| = |Z_{i+1}|$, and the paths corresponding to $Z_i \cap Z_{i+1}$ are trivial. Note that every vortex admits a linked decomposition since we can take $X_i = (V(G) \setminus W) \cup \{w_i, w_{i+1}\}$ (where $w_{n+1} := w_n$). The *adhesion of the vortex* is the minimum adhesion taken over all linked decompositions of the vortex. Let us observe that in a linked decomposition of adhesion q, there are q disjoint paths linking Z_1 with Z_n in $G - W$.

Let H be a subgraph of a graph G_0. If G_0 can be written as $G_1 \cup G_2$, where $G_1 \cap G_2 = \{v_1, \ldots, v_t\} \subset V(G_0)$, $1 \le t \le 3$, $V(G_2) \setminus V(G_1) \ne \emptyset$, and every vertex of H in $G_2 - \{v_1, \ldots, v_t\}$ has degree 2 in H, then we replace G_0 by the graph G' obtained from G_1 by adding all edges $v_i v_j$ $(1 \le i < j \le t)$ that are not already in G_1. If $H \cap G_2$ has a path in G_2 connecting v_i and v_j, then we replace that path in H by the edge $v_i v_j$. The resulting graph H' is a subgraph of G', and we say that the pair (G', H') was obtained from (G_0, H) by an *elementary reduction*. Every pair (G'', H'') that can be obtained from (G_0, H) by a sequence of elementary reductions is a *reduction* of (G_0, H).

A *surface* is a compact connected 2-manifold (with boundary). The surface is *closed* if the boundary is empty. The components of the boundary are called the *cuffs*. If H is a subgraph of a graph G_0, we say that the pair (G_0, H) can be *embedded* in a surface Σ *up to* 3*-separations* if there is a reduction (G'', H'') of (G_0, H) such that G'' has an embedding in Σ.

Let G be a graph, H a subgraph of G, Σ a surface, and $\alpha \geq 0$ an integer. We say that the pair (G, H) can be *α-nearly embedded* in Σ if there is a set of at most α cuffs in Σ, $C_1, \ldots, C_b$ ($b \leq \alpha$), and there is a set A of at most α vertices of G such that $G - A$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_b$ where:

(N1) H is a subgraph of G_0, and (G_0, H) can be embedded in Σ up to 3-separations.

(N2) If $1 \leq i < j \leq b$, then $V(G_i) \cap V(G_j) = \emptyset$.

(N3) $W_i = V(G_0) \cap V(G_i) = V(G_0) \cap C_i$ for every $i = 1, \ldots, b$.

(N4) For every $i = 1, \ldots, b$, the pair (G_i, W_i) is a vortex of adhesion less than α, where the ordering of W_i is determined by the order of these vertices on C_i.

The vertices in A are called the *apex vertices* of the *α-near embedding*. It may happen that $A = V(G)$, and $G - A$ is empty. In that case we say that the α-near embedding of G in Σ is *trivial*. Otherwise, G_0 is nonempty. The subgraph G_0 of G is said to be the *embedded subgraph* with respect to the α-near embedding and the decomposition $G_0, G_1, \ldots, G_b$. The pairs (G_i, W_i), $i = 1, \ldots, b$, are the *vortices* of the α-near embedding. The vortex (G_i, W_i) is said to *be attached to the cuff* C_i of Σ containing W_i.

Let us recall that an *r-wall* is a graph which is isomorphic to a subdivision of the graph W_r with vertex set $V(W_r) = \{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq r\}$ in which two vertices (i, j) and (i', j') are adjacent if and only if one of the following possibilities holds:

(1) $i' = i$ and $j' \in \{j - 1, j + 1\}$.

(2) $j' = j$ and $i' = i + (-1)^{i+j}$.

Now we can state the theorem. We shall use a formulation which is a simplified version of one of the cornerstones of Robertson and Seymour's theory of graph minors, the Excluded Minor Theorem, as stated in [59].

Theorem 4.1 *For every positive integer w, there exists a positive integer r such that the following holds. Let R be a graph and let G be a graph that does not contain R as a minor. If G has tree-width at least w, then G contains an r-wall H as a subgraph and there is a constant α (depending only on R) such that the pair (G, H) has an α-near embedding in some surface Σ in which R cannot be embedded. Moreover, $r = r(w)$ is nondecreasing as a function of w and $\lim_{w \to \infty} r(w) = \infty$.*

Some additional remarks should be made at this point:

(1) The r-wall H is *planarly embedded* in Σ, i.e., every cycle in H is contractible in Σ and there is a disk $D \subset \Sigma$ such that H and all 6-faces of the embedding of H in the plane are contained in D. To see this, observe that the cycle space of H is generated by the facial 6-cycles of its planar embedding. If all these cycles are contractible in Σ, then an $(r/2)$-subwall of H is planarly embedded in Σ. If more than $171g$ of the facial 6-cycles of H are noncontractible in Σ, where g is the Euler genus of Σ, then there are $9g$ such cycles, $F_1, \dots, F_{9g}$, such that any two of them are at distance at least 3 in H. This implies, in particular, that $H - (F_1 \cup \dots \cup F_{9g})$ is connected and hence no four cycles among $F_1, \dots, F_{9g}$ are homotopic. Consequently, $F_1, \dots, F_{9g}$ contains a subfamily of $3g$ cycles, no two of which are homotopic. This is not possible (cf., [45, Proposition 4.2.6]). Hence, at most $171g$ of the 6-cycles of H are noncontractible and H contains a large subwall that is planarly embedded, and we can take this subwall instead of H. The size r' of this smaller wall still satisfies the condition that $r' = r'(w) \to \infty$ as w increases.

(2) We may additionally assume that the face-width (or representativity, see [45] for the definition) of the embedded subgraph G'' in Σ is as large as we want (in terms of R). To see this, suppose that there is a non-contractible closed curve C that intersects G'' only at vertices and $|C \cap V(G'')|$ is small. Then we delete all the vertices in $C \cap V(G'')$ from G'' and add them into the set of apex vertices. Then the genus of Σ goes down, and the number of apex vertices is still bounded. Continuing this procedure, we get the graph on a simpler surface whose face-width is as large as we wanted. See [55, 57, 50] for details. For the survey on the face-width of embeddings, we refer to [45].

5 The large tree-width case

In this section we complete the proof of our main result, Theorem 1.1. We will make use of Theorem 4.1. We let $R = sK_{a,k}$, and apply Theorem 4.1 to G, R and a large value of w that will be specified later. We let $r = r(w)$, Σ, H, and $\alpha = \alpha(a, s, k)$ be the quantities from Theorem 4.1. By taking large enough w, we may assume that r is as large as we want.

We shall use the notation introduced in Section 4. In particular, we let G_0 be the embedded subgraph of G, and (G'', H'') be the corresponding reduction of (G_0, H). Since R cannot be embedded in Σ, Euler genus of Σ is at most $\frac{s(a-2)(k-2)}{2}$. Since the embedded part G'' of G contains the r-wall H'', we may assume that G'' is as large as we want. Suppose G'' has N' vertices. Then $N' \geq r^2$. Let A be the set of apex vertices. Suppose that each vortex has adhesion at most α and that there are $b \leq \alpha$ vortices.

Again, we define the constants that will be used in the proofs:

$$\begin{aligned} n_1 &= 4(ask + 4sk\binom{\alpha}{a} + \alpha n_2) \\ n_2 &= n_3^{2\alpha+1} \\ n_3 &= 3a(2n_4)^p, \quad \text{where } p = 2^{2\alpha+1} \\ n_4 &= n_5^q, \quad \text{where } q = 2^{\alpha(2\alpha+1)} \\ n_5 &= 16(a+1)sk\binom{2\alpha}{a}. \end{aligned}$$

From now on we assume that w is so large that r is large enough to guarantee that $N' \geq r^2 \geq n_1$. In the rest of our proof, we also assume that G is $(3a+2)$-connected and that the minimum degree of G is at least $\frac{31}{2}(a+1) - 3$.

In this section we shall sometimes abuse terminology and speak of paths in a set U (usually a subgraph or just a vertex set), but will always mean paths in the subgraph of G induced by the vertices in U.

First, we will show that only a bounded number of non-society vertices have a or more neighbors in G that are not their neighbors in G''.

Claim 5.1 *There are at most $(sk-1)\binom{\alpha}{a}$ vertices of G'' that can have a or more neighbors in A.*

Proof. Otherwise, by the Pigeonhole Principle, there is a vertex set $C \subseteq V(G'')$ with $|C| \geq sk$ such that each vertex in C has a common neighbors in A. But this gives $K_{a,sk}$ as a subgraph, a contradiction. □

Claim 5.2 *There are at most $3(sk-1)\binom{\alpha}{a}$ vertices of G'' that have been used in the elementary reductions yielding the embedded subgraph G'' from G_0.*

Proof. The argument is similar to the one used in the proof of Claim 5.1. Suppose that we have made t elementary reductions in order to obtain G'' from G_0. Let $G_1^{(i)}$ and $G_2^{(i)}$ be the graphs used in the ith elementary reduction, $i = 1, \ldots, t$. We may assume that vertex sets $V(G_2^{(i)}) \setminus V(G_1^{(i)})$ removed in these reductions are pairwise disjoint. Let $v_i \in V(G_2^{(i)}) \setminus V(G_1^{(i)})$. Since G is $(a+3)$-connected, there exist $a+3$ internally disjoint paths connecting v_i with a vertex v_i' in $V(G_1^{(i)}) \setminus V(G_2^{(i)})$. At most three of these paths reach v_i' through vertices in $V(G_1^{(i)}) \cap V(G_2^{(i)})$, so at least a of them go through A. They give rise to a collection of a paths joining v_i with distinct vertices in A, and these paths are contained in $A \cup V(G_2^{(i)}) \setminus V(G_1^{(i)})$. If more than $3(sk-1)\binom{\alpha}{a}$ vertices of G'' have been involved in the reductions, then $t > (sk-1)\binom{\alpha}{a}$. By the Pigeonhole Principle, there is a set of sk indices $i_1, \ldots, i_{sk}$ such that their a-tuple of paths end in the same a-tuple of vertices in A. Clearly, these paths determine a subdivision of $K_{a,sk}$ in G. $\square$

The following claim is a corollary of the large face-width condition, see remark (2) after Theorem 4.1.

Claim 5.3 *For every cuff C_i $(1 \le i \le b)$, there exists a cycle C_i' in G'' such that C_i' separates a cylinder D_i in Σ whose boundary components are C_i and C_i'. Every vertex in C_i' is cofacial with some vertex in C_i (i.e., they belong to a common facial walk). Moreover, interiors of cylinders D_i are pairwise disjoint for $i = 1, \ldots, b$.*

By Menger's Theorem we have:

Claim 5.4 *For $i = 1, \ldots, b$, let π_i be the maximum number of pairwise disjoint paths connecting $C_i \cap W_i$ with C_i'. Then G'' has a separation (I_i, J_i) of order π_i such that $C_i \subseteq J_i \subseteq D_i$ and $C_i' \subseteq I_i$.*

We shall prove that there is a large vortex with some special properties, to which we will be able to apply similar arguments as used in Section 3. We say that a society vertex $v \in C_i$ is *essential* if $\deg_{G''}(v) \le 4$. We say that the vortex (G_i, W_i) attached to the cuff C_i is *n-wide* if it contains n essential society vertices $w_1, \ldots, w_n \in W_i$ and there are n pairwise disjoint paths in G'' joining $\{w_1, \ldots, w_n\}$ with the cycle C_i'.

Claim 5.5 *There exists an n_2-wide vortex.*

Proof. For each cuff C_i ($1 \le i \le b$), let L_i be all essential vertices in C_i. If for some i, there are at least n_2 disjoint paths from L_i to C_i', then we are done. Otherwise, by Claim 5.4, for each i, there is a separation (I_i, J_i) of order at most $n_2 - 1$ such that J_i contains all the vertices in L_i and $C_i' \subseteq I_i$. Let G_1'' be the graph obtained from G'' by deleting $J_i - I_i$ for all i. Then $G_1'' - \bigcup_{i=1}^{b}(J_i \cap I_i)$ has no essential vertices. Since $C_i' \subseteq G_1''$ for $1 \le i \le b$, and since every vertex in C_i' is cofacial with some vertex in C_i, G_1'' contains an $(r-1)$-subwall of H''. Hence, G_1'' has at least $N'' \ge (r-1)^2$ vertices. By Claims 5.1 and 5.2, at least $N'' - 4(sk-1)\binom{\alpha}{a} - b(n_2 - 1)$ vertices have degree at least $\frac{31}{2}(a+1) - 3 - (a-1)$ in G_1''. On the other hand, the surface Σ has Euler genus at most $s(a-2)(k-2)/2$, and hence, by Euler's formula, G_1'' has at most $3N'' + 3s(a-2)(k-2)/2$ edges. This yields a contradiction. □

Let (G_1, W_1) be an n_2-wide vortex. Let $w_1, \ldots, w_{n_2}$ be the corresponding essential society vertices, and let Q_i be disjoint paths joining w_i with C_1', $i = 1, \ldots, n_2$. The vortex (G_1, W_1) has a linked vortex decomposition. If the adhesion is q, let $P_1, \ldots, P_q$ be the corresponding paths, with the convention that P_1 is a tree, composed of all paths Q_i and the segment of C_1' joining the ends of these paths, starting at Q_1 and passing through $Q_2, Q_3, \ldots$ until reaching Q_{n_2}. After contracting each Q_i to a point, we can think of P_1 as the path with vertices $w_1, \ldots, w_{n_2}$ and think of it as being contained in G_1. From now on, we will only be interested in minors within the vortex, so making contractions of all Q_i is admissible. Only once we shall get a subdivision (and not a minor) of $K_{a,sk}$, but in that case P_1 will not be used. We let $Z = P_1 \cup \cdots \cup P_q$.

It turns out that it is convenient to treat apex vertices as being contained in the vortex. This is achieved by adding A to G_1 and adding all A into every part of the linked vortex decomposition of (G_1, W_1). Each added vertex then determines a (trivial) path in the linked vortex decomposition of the extended vortex. This increases the adhesion at most by α. We assume that this change to the vortex has been made and hence its adhesion is bounded by 2α. In particular, we have $q \le 2\alpha$.

Similarly as in Section 3 (cf. Claims 3.2 and 3.3), we consider a subset of essential society vertices, $\{w_p \mid p \in I\}$ of cardinality n_5 such that the following conditions hold:

(a) $I = \{p_1, \ldots, p_{n_5}\}$, where $1 < p_1 < p_2 < \cdots < p_{n_5} < n_2$.

(b) For $j = 1, \ldots, q$, either $P_j(p_1 - 1, p_{n_5} + 1)$ is a single vertex (in which case we say that P_j is a *trivial path*), or all segments $P_j(p_i - 1, p_i + 1)$

$(i = 1, \dots, n_5)$ are mutually disjoint (in which case we say that P_j is *nontrivial*). However, we do not request that $P_j(p_i - 1, p_i + 1)$ contains more than one vertex. Let us observe that all paths corresponding to the apex vertices are trivial and that P_1 is nontrivial.

(c) Any two paths P_j, P_l are either *everywhere bridge connected* or *everywhere bridge disconnected*. This means that for all (or for none) of the values $i = 1, \dots, n_5$, there is a Z-bridge in G_1 that is attached to $P_j(p_i - 1, p_i + 1)$ and to $P_l(p_i - 1, p_i + 1)$.

We also introduce the following notation which is similar (but not identical) to the one used in Section 3. We let $Z_i \subseteq G_1$ be the set of segments of paths, $Z(i) = \cup_{j=1}^{q} P_j(p_i - 1, p_i + 1)$, together with all Z-bridges in G_1 that have all their vertices of attachment in $Z(i)$.

By using (c), we define the auxiliary graph Γ and we let Γ_0 be the subgraph consisting of the connected component that contains P_1 and is obtained from Γ after deleting its vertices corresponding to the trivial paths. We assume that $V(\Gamma_0) = \{P_1, \dots, P_{q_0}\}$.

As in Section 3, we introduce the graph $\hat{H}_i \subseteq Z_i$ which consists of all segments $P_j(p_i - 1, p_i + 1)$ for $j = 1, \dots, q_0$ together with all Z-bridges in Z_i that are attached to at least one of the paths $P_1, \dots, P_{q_0}$. Observe that $\hat{H}_i$ may contain vertices of trivial paths, but the only nontrivial paths participating in $\hat{H}_i$ are $P_1, \dots, P_{q_0}$. Finally, we define H_i as the induced subgraph of $\hat{H}_i$ obtained by deleting the trivial paths. For easier notation, we also introduce vertices $z_i = w_{p_i}$. Let

$$
\begin{aligned}
S_i &= V(\hat{H}_i) \cap \big(\cup_{j=1}^{q} P_j(p_i, p_i) \big), \\
S_i^- &= V(\hat{H}_i) \cap \big(\cup_{j=1}^{q} P_j(p_i - 1, p_i - 1) \big), \quad \text{and} \\
S_i^+ &= V(\hat{H}_i) \cap \big(\cup_{j=1}^{q} P_j(p_i + 1, p_i + 1) \big).
\end{aligned}
$$

Let us observe that, unlike in Section 3, S_i, S_i^-, and S_i^+ need not be disjoint. All we can say is that $z_i \in S_i \setminus (S_i^- \cup S_i^+)$.

Unfortunately, we cannot easily prove an analogue of Claim 3.4. Instead, we will be satisfied with the following weaker statement.

Claim 5.6 *For all but at most $2(sk-1)\binom{2\alpha}{a}$ values of i, the following holds:*

(a) *If $v \in V(H_i) - S_i^- - S_i^+$, then v has at most a neighbors in $S_i^- \cap H_i$ and at most a neighbors in $S_i^+ \cap H_i$.*

(b) $\hat{H}_i$ *has a separation* (A_i, B_i) *of order at most* $a-1$ *such that* A_i *contains all vertices of trivial paths in* $\hat{H}_i$ *and such that* $B_i - A_i - S_i^- - S_i^+$ *contains a vertex adjacent to* z_i.

Proof. If $u \in V(H_i)$ is adjacent to $a+1$ vertices in $S_i^- \cap H_i$ or to $a+1$ vertices in $S_i^+ \cap H_i$, then a of these neighbors lie on distinct everywhere nontrivial paths $P_1^i, \ldots, P_a^i$, where $u \notin P_1^i \cup \cdots \cup P_a^i$. If this happens for more than $(sk-1)\binom{\alpha}{a}$ values of i, there are sk values of i for which the a-tuple of paths $P_1^i, \ldots, P_a^i$ is the same. It is easy to see that this gives rise to s disjoint $K_{a,k}$-minors in G.

From now on we exclude all those values of i for which a vertex in $V(H_i) - S_i^- - S_i^+$ has more than a neighbors in $S_i^- \cap H_i$ or more than a neighbors in $S_i^+ \cap H_i$. The society vertex z_i is essential, so it has degree more than $3a+1$ in $\hat{H}_i$. By the same argument as used in the proof of Claim 5.1, z_i has a neighbors in the set of trivial paths for at most $(sk-1)\binom{2\alpha}{a}$ values of i. As assumed above, z_i has at most a neighbors in $S_i^- \cap H_i$ and at most a of them in $S_i^+ \cap H_i$. Therefore, z_i has a neighbor v_i in $H_i - S_i^- - S_i^+$. If there are a internally disjoint paths in $\hat{H}_i$ from v_i to distinct trivial paths, and this happens for more than $(sk-1)\binom{2\alpha}{a}$ values of i, then we get a subdivision of $K_{a,sk}$. Consequently, there is a separation (A_i, B_i) of $\hat{H}_i$ of order at most $a-1$ such that $v_i \in B_i - A_i$ (hence $v_i \in B_i - A_i - S_i^- - S_i^+$), and A_i contains all vertices of trivial paths that are in $\hat{H}_i$. This completes the proof. □

From now on we only consider those values of i for which the properties (a) and (b) of Claim 5.6 hold.

Claim 5.7 $q_0 \geq a+1$.

Proof. Let us consider the vertex $v_i \in B_i - A_i - S_i^- - S_i^+$. The vertices in $S = (A_i \cap B_i) \cup (S_i^- \cap H_i) \cup (S_i^+ \cap H_i) \cup \{z_i\}$ separate v_i from G_0 in G. Therefore, $|S| \geq 3a+2$. Since $|A_i \cap B_i| \leq a-1$, it follows that $|S| \leq a-1+|S_i^- \cap H_i| + |S_i^+ \cap H_i| + 1 = 2q_0 + a$. Combining the two bounds on $|S|$ implies that $q_0 \geq a+1$. □

The last claim can be used to prove an analogue of Claim 3.7.

Claim 5.8 $B_i - A_i - S_i^- - S_i^+ - z_i$ *contains an* $(a+1)$*-linked subgraph* M_i.

Proof. We will apply Corollary 2.3 to the graph $L_i = B_i - A_i - S_i^- - S_i^+ - z_i$. First of all, let us observe that every vertex $v \in V(L_i)$ has degree at least

$\frac{31}{2}(a+1)-3$ and has at most $3a-1$ neighbors in $A_i \cup (S_i^- \cap H_i) \cup (S_i^+ \cap H_i)$ by Claim 5.6. If v has a neighbor $u \in S_i^- \setminus V(H_i)$, then u forms one of the trivial paths, so it belongs to A_i. Consequently, v has at most $3a$ neighbors in $A_i \cup S_i^- \cup S_i^+ \cup \{z_i\}$. Hence the degree of v in L_i is at least $\frac{31}{2}(a+1)-3-3a = \frac{25}{2}(a+1)$. Thus we conclude that $|E(L_i)| \geq \frac{25}{4}(a+1)|V(L_i)|$. Since $v_i \in V(L_i)$, L_i is a nonempty graph and its order is obviously at least the degree of v_i. This shows that Corollary 2.3 can be applied to L_i, and we conclude that M_i exists. □

Finally, we construct s disjoint $K_{a,k}$-minors in the same way as in Section 3. The only difference is that we take $2a+2$ paths $Q_0, \ldots, Q_a, Q_0', \ldots, Q_a'$ from M_i to $S_i^- \cup S_i^+$ in the graph $G - z_i - (A_i \cap B_i)$, and therefore we need connectivity $3a+2$ instead of $3a+1$ because of the additionally removed vertex z_i.

This completes the proof of Theorem 1.1.

6 Conclusion

Let us observe that our proof implies the following.

Theorem 6.1 *For any s, t, a and k, there exists a constant $N(s,k,a,t)$ such that every $(3a+2)$-connected graph of minimum degree at least $\frac{31}{2}(a+1)-3$ and with at least $N(s,k,a,t)$ vertices contains either a subdivision of $K_{a,t}$ or a minor isomorphic to s disjoint copies of $K_{a,k}$.*

This theorem says that in Theorem 1.1, the result holds not only for a topological minor of $K_{a,sk}$ but also for a topological minor of $K_{a,t}$ for any t that does not need to depend on s and k. Hence t could be arbitrarily large compared to sk.

Our final remark is that, as observed in [5], the sequence of graphs $K_{a,k}$, where a is fixed and k tends to infinity, is essentially the only family of graphs for which a result like our Theorem 1.1 holds. More precisely:

Theorem 6.2 ([5]) *Let c and $w \geq c$ be positive integers, and let H_k ($k \geq 1$) be a sequence of graphs such that $\lim_{k\to\infty} |V(H_k)| = \infty$. Suppose that for any positive integer k there exists an integer $N(k)$ such that every c-connected graph of tree-width $\leq w$ and of order at least $N(k)$ contains H_k as a minor. Then H_k is a minor of $K_{c,N(k)}$ for $k \geq 1$.*

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