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Preprint series, Vol. 42 (2004), 953

ON A POINT STABILISER IN A PERMUTATION GROUP

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ISSN 1318-4865

December 29, 2004

Ljubljana, December 29, 2004

On a point stabiliser in a permutation group

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[1]This author gratefully acknowledges support by the US Department of State and the Fulbright Scholar Programme, and thanks Northern Arizona University for hospitality during his visit in Spring 2004.

Abstract: If V is a (possibly infinite) set, G a permutation group on V, $v \in V$, and Ω an orbit of the stabiliser G_v, let G_v^Ω denote the permutation group induced by the action of G_v on Ω, and let $N(G)$ be the normaliser of G in $\mathrm{Sym}(V)$. In the article, we discuss a relationship between the structures of G_v and G_v^Ω. In particular, we prove that if G is transitive and if the $N(G)$-orbital $\{(v^g, u^g) \mid u \in \Omega, g \in N(G)\}$ is strongly connected (when viewed as a digraph on V), then every simple homomorphic image of a subgroup of G_v is also a homomorphic image of a subgroup of G_v^Ω. This generalises a result of Wielandt concerning finite primite permutation groups.

Keywords: permutation group, digraph, graph, stabiliser.

Slovenski naslov: O stabilizatorju točke v permutacijski grupi

Povzetek: Za množico V (lahko neskončno), permutacijsko grupo G na V, točko $v \in V$ in orbito Ω delovanja stabilizatorja G_v, naj G_v^Ω predstavlja permutacijsko grupo na Ω, inducirano z delovanjem grupe G_v na Ω, in naj $N(G)$ predstavlja normalizator grupe G v simetrični grupi $\mathrm{Sym}(V)$. V članku obravnavamo povezavo med strukturama grup G_v in G_v^Ω. Med drugim dokažemo, da je v primeru, ko je G tranzitivna in je $N(G)$-orbitala $\{(v^g, u^g) \mid u \in \Omega, g \in N(G)\}$ krepko povezana (razumljena kot usmerjen graf na množici točk V), vsaka enostavna homomorfna slika grupe G_v hkrati tudi homomorfna slika grupe G_v^Ω. S tem posplošimo soroden rezultat Wielandta o končnih primitivnih permutacijskih grupah.

Ključne besede: permutacijska grupa, usmerjen graf, graf, stabilizator.

1 Introduction

It is well known that every transitive permutation group G with a point stabiliser $H \leq G$ is permutation isomorphic to the natural action of G on the cosets G/H by multiplication. It is therefore not surprising that determining the size and the structure of a point stabiliser in a transitive permutation group is one of the central problems in the theory of permutation groups. Particular attention has been given to the relationship between the size of a point stabiliser G_v, and the size of an orbit $\Omega \neq \{v\}$ of G_v (called a *G-suborbit*). In particular, the question whether the size of G_v in a finite primitive permutation group is bounded above in terms of the size of G-suborbits, was raised by Sims, and finally answered affirmatively in [2]. As we show in this paper, the structure of a vertex stabiliser G_v seems to remain strongly related to the structure of the permutation group G_v^{Ω} (induced by the action of G_v on a G_v-orbit Ω) even if the group G is not primitive (or finite).

Let V be a (possibly infinite) set, let $G \leq \mathrm{Sym}(V)$, let $v \in V$, and let Ω be an orbit of the stabiliser G_v. Further, let $K = \mathrm{Ker}(G_v \to G_v^{\Omega})$ denote the group of all permutations in G_v fixing every element of Ω. Then G_v^{Ω} (viewed as an abstract group) is isomorphic to the quotient group G_v/K. Since, in principle, the kernel K might be a rather complicated group, the vertex stabiliser G_v could be considerably more complicated than G_v^{Ω}. However, it can be shown that under certain assumptions on G and Ω, the structure of G_v (at least in the sense of the Jordan-Hölder decomposition), is fully determined by the structure of the group G_v^{Ω}. For example, the following generalisation of Wielandt's theorem [6, Theorem 18.2] has been proved recently by Betten, Delandtsheer, Niemeyer, and Praeger. (In the statement of his result, Wielandt assumed that G itself is finite and primitive.)

Theorem 1.1 ([1, Theorem[2.1]) *Suppose that G is a transitive subgroup of $\mathrm{Sym}(V)$ whose normaliser in $\mathrm{Sym}(V)$ is finite and primitive. Let $v \in V$ and let $\Omega \subseteq V \setminus \{v\}$ be an orbit of G_v. Then every composition factor (in the sense of the Jordan-Hölder Theorem) of the group G_v is isomorphic to a composition factor of some subgroup of G_v^{Ω}.*

It this paper we generalise the above result and show that under certain restrictions the statement of Theorem 1.1 remains valid for (possibly infinite) permutation groups with an imprimitive normaliser.

Even though the topic of the paper is essentially group theoretical, graph theoretical language and techniques will prove useful, and enable us to understand Wielandt's result in a more general setting.

By a *digraph* on a *vertex set* V, we mean any non-empty subset Γ of the set $V^{(2)} = (V \times V) \setminus \{(v,v) \mid v \in V\}$. For example, if $G \leq \mathrm{Sym}(V)$ is a permutation group, and Δ is a *G-orbital* (that is, an orbit of G on $V^{(2)}$), then Δ can be viewed as a digraph on V. The *automorphism group of* Γ, denoted by $\mathrm{Aut}(\Gamma)$, is the group of all permutations on V that (when viewed as permutations on $V^{(2)}$) preserve Γ. Clearly, a permutation group $G \leq \mathrm{Sym}(V)$ is a subgroup of $\mathrm{Aut}(\Gamma)$ if and only if Γ is a union of G-orbitals. For a set V and $v \in V$, consider the function Φ_v mapping a digraph $\Gamma \subseteq V^{(2)}$ to the *neighbourhood* $\Gamma(v) = \{u \mid (v,u) \in \Gamma\}$ of v in Γ. If G is a transitive permutation group on V, then Φ_v induces a bijective correspondence between digraphs Γ on V for which $G \leq \mathrm{Aut}(\Gamma)$, and unions of G_v-orbits on $V \setminus \{v\}$. In this correspondence, G-orbitals (that is, digraphs on which G acts transitively) bijectively correspond to G_v-orbits. This shows that every statement concerning the action of G_v on a G_v-orbit can be expressed in terms of the action of G as a group of digraph automorphisms.

A digraph Γ is *locally finite* if $\Gamma(v)$ is finite for every $v \in V$, and is *finite* if its vertex set is finite. For a digraph $\Gamma \subseteq V^{(2)}$, we let $\Gamma^* = \{(u,v) \mid (v,u) \in \Gamma\}$ and call Γ a *graph* if $\Gamma = \Gamma^*$. A *directed path of length* n in Γ between vertices $u, v \in V$ is a finite sequence $u = v_0, v_1, \ldots, v_n = v$ such that $(v_{i-1}, v_i) \in \Gamma$ for every $i \in \{1, \ldots, n\}$. A *path* in Γ is a directed path in $\Gamma \cup \Gamma^*$. A digraph Γ is (*strongly*) *connected* if for every $(u,v) \in V^{(2)}$ there exists a (directed) path between u and v.

Recall that an abstract group is a *section* of a group G if it is a homomorphic image of a subgroup of G. A section is called *simple* if it is a simple group. Note that a group is a simple section of a finite group G if and only if it is a composition factor of a subgroup of G. We can now state the following generalisation of Wielandt's theorem:

Theorem 1.2 *Suppose that G is a (possibly infinite) transitive permutation group on a set V, $N(G)$ the normaliser of G in* $\mathrm{Sym}(V)$, *$\Delta \subseteq V^{(2)}$ a G-orbital, Γ the $N(G)$-orbital containing Δ, and $v \in V$. If Γ is strongly connected (when viewed as a digraph on V), then every simple section of G_v is also a section of $G_v^{\Delta(v)}$.*

It is well known that a finite transitive permuation group $G \leq \mathrm{Sym}(V)$ is primitive if and only if every G-orbital is strongly connected (see, for example, [3, Theorem 3.2A and Lemma 3.2A]). It is now clear that Theorem 1.2 implies the statement of Theorem 1.1. Namely, if G, V, v and Ω are as in Theorem 1.1, and $N(G)$ is the normaliser of G in $\mathrm{Sym}(V)$, then the $N(G)$-orbital $\Gamma = \{(v^g, u^g) \mid u \in \Omega, g \in N(G)\}$ is strongly connected (since $N(G)$ is primitive), Γ contains the G-orbital $\Delta = \{(v^g, u^g) \mid u \in \Omega, g \in G\}$, and $\Delta(v) = \Omega$. Hence, by Theorem 1.2, every simple section of G_v is also a simple section of G_v^{Ω}. Since simple sections of a finite group are nothing but composition factors of its subgroups, the assertion of Theorem 1.1 follows.

In Section 3, we prove Theorem 1.2, by first proving a stronger, albeit rather technical result (see Lemma 3.1). Several group and graph theoretical consequences of the latter are discussed in Section 4. But first, we review some basic notions and facts needed later.

2 Preliminaries

Since we assume no finitness conditions on the groups, sets and digraphs, some use of transfinite set theory is unavoidable. We refer the reader to [4] for a detailed account on ordinal numbers and other set theoretical notions not defined here. The usual well-ordering of the set of ordinals will be denoted by $<$, and $\alpha \leq \beta$ will mean that either $\alpha < \beta$ or $\alpha = \beta$. We shall use the symbol α^+ to denote the successor of an ordinal α. The minimal ordinal number shall be denoted by 0 and its successor by 1. For two ordinals α and β, we let $[\alpha, \beta)$ denote the set of ordinals η such that $\alpha \leq \eta < \beta$, and by $[\alpha, \beta]$ we mean $[\alpha, \beta^+)$ (note that by this definition, $\beta = [0, \beta)$ and $\beta^+ = [0, \beta]$ for every ordinal β). Recall that a non-zero ordinal is a *limit ordinal* if it is not the successor of any other ordinal.

The reader should be aware that the assumption of the Axiom of Choice is needed to prove our results in the most general form. In particular, we shall assume that every set can be well ordered, and consequently, that for every set V there exists a well-ordering of V and an order-preserving bijection from V to an ordinal α. In this case, we shall say that V (together with the well-ordering of V) *gives rise to the ordinal* α.

Let α be an ordinal number, let H be a subgroup of a group G, let $\mathcal{G}$ be a set of groups containing H and contained in G, and let $f\colon [0, \alpha] \to \mathcal{G}$, $f(\beta) = G_\beta$, be a surjective function. Following [5], we say that

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_\alpha = H$$

is a *complete descending series* between G and H of *type* α whenever:

1. $G_\alpha = H$, $G = G_0$, and $G_\beta^+ \lhd G_\beta$ for every $\beta \in [0, \alpha)$; and
2. $\bigcap_{\beta<\lambda} G_\beta = G_\lambda$ for every limit ordinal number $\lambda \in [0, \alpha]$.

The quotients G_β/G_{β^+}, $\beta \in [0, \alpha)$, shall be referred to as the *factors* of the complete descending series. A complete descending series between a group G and the trivial group 1, will be called a *complete descending series* of G. Clearly, if α is a finite ordinal number, then a complete descending series of G is a normal series of G in the usual sense. The following lemma will be needed in the proof of Theorem 1.2.

Lemma 2.1 *Let S be a simple section of a group G and let*

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_\alpha = 1$$

be a complete descending series. Then S is a section of some quotient group G_β/G_{β^+}, $\beta \in [0, \alpha)$.

Proof. By definition, there exists a subgroup $H \leq G$ and a group epimorphism $\pi\colon H \to S$. Let K denote the kernel of π. For every $\beta \in [0,\alpha]$ let $H_\beta = G_\beta \cap H$ and $S_\beta = \pi(H_\beta)$. Then it is easy to see that $H = H_0 \rhd H_1 \rhd \cdots \rhd H_\alpha = 1$ and $S = S_0 \rhd S_1 \rhd \cdots \rhd S_\alpha = 1$ are complete descending series of H and S, respectively. We shall now show that there exists $\beta \in [0,\alpha)$ such that $S_\beta = S$ and $S_{\beta^+} = 1$. Consider the set $\{\eta \in [0,\alpha] \mid S_\eta \neq S\}$. Since $S_\alpha = 1$, this set is not empty, hence it has a minimal element μ. If μ were a limit ordinal, then the obvious equality $S_\mu = \bigcap_{\beta\in[0,\mu)} S_\beta = \bigcap_{\beta\in[0,\mu)} S = S$ would lead to contradiction. Therefore, there exists $\beta \in [0,\alpha)$ such that $\mu = \beta^+$. Since S is simple and $S \neq S_\mu \lhd S_\beta = S$, it follows that $S_\mu = S_{\beta^+} = 1$ and $S_\beta = S$, as claimed. This clearly implies that $H_{\beta^+} \leq K$ and $H_\beta/(H_\beta \cap K) \cong S$. But then the Third Isomorphism Theorem implies that $S \cong (H_\beta/H_{\beta^+})/((H_\beta \cap K)/H_{\beta^+})$, and therefore S is a homomorphic image of the group H_β/H_{β^+}. Now, by the Second Isomorphism Theorem, we have that $H_\beta/H_{\beta^+} = (G_\beta \cap H)/(G_{\beta^+} \cap H) \cong (G_\beta \cap H)G_{\beta^+}/G_{\beta^+}$, which is a subgroup of G_β/G_{β^+}. Hence, S is a section of G_β/G_{β^+}. ■

A *base* of a permutation group $G \leq \mathrm{Sym}(V)$ is a subset $B \subseteq V$ such that the pointwise stabiliser $G_{(B)}$ of B in G is trivial (see for example [3, Section 3.3]). Bases of permutation groups prove to be a useful tool in the analysis of finite primitive permutation groups. In Definition 2.2 below, we introduce a variation of this notion. For a digraph $\Gamma \subseteq V^{(2)}$ and a set of vertices U in V, we let $\Gamma(U)$ denote the union of U and all the neighbourhoods $\Gamma(u)$, $u \in U$. (Note that by this definition $\Gamma(u) \neq \Gamma(\{u\})$; in fact, $\Gamma(\{u\}) = \{u\} \cup \Gamma(u)$.) By $\Gamma[U]$ we denote the subdigraph of Γ spanned by U, that is, the digraph $\Gamma \cap U^{(2)}$ on the vertex set U.

Definition 2.2 Let Γ be a digraph on a vertex set V, let γ be an ordinal number, and let $\delta\colon [0,\gamma) \to V$, $\eta \mapsto v_\eta$, be an arbitrary function. If for every $\eta \in [1,\gamma)$ there exists $\eta' < \eta$ such that $v_\eta \in \Gamma(v_{\eta'})$, then we say that δ is a *directed sequence* in Γ of *type* γ. The vertex $\delta(0)$ will than be called the *beginning of* δ. If $\Gamma(\mathrm{Im}(\delta))$ is a base of a group $G \leq \mathrm{Aut}(\Gamma)$, then we say that δ is a *G-leash* in Γ.

With $\mathbb{N} = \{0, 1, \ldots\}$ we mean the set of natural numbers, well-ordered in the usual sense. (The ordinal arising from this well-ordering of $\mathbb{N}$ is denoted by ω.) For a digraph Γ and for $i \in \mathbb{N}$, let $\Gamma^{(i)}(v)$ denote the set of all vertices u for which the shortest directed path from v to u has length i. In particular, $\Gamma^{(0)}(v) = \{v\}$ and $\Gamma^{(1)}(v) = \Gamma(v)$.

Lemma 2.3 *If Γ is a strongly connected digraph on a vertex set V, $G \leq \mathrm{Aut}(\Gamma)$, and $v \in V$, then there exists a G-leash of type $\sum_{i\in\mathbb{N}} \gamma_i$ with the beginning in v, where γ_i, $i \in \mathbb{N}$, is the ordinal number arising from a well-ordering of $\Gamma^{(i)}(v)$.*

Proof. For every $i \in \mathbb{N}$, there exists a well-ordering $<_i$ on $\Gamma^{(i)}(v)$ giving rise to an ordinal number γ_i. Since Γ is strongly connected, we have $V = \bigcup_{i\in\mathbb{N}} \Gamma^{(i)}(v)$.

Hence, there exists a well-ordering $<$ on V giving rise to the ordinal number $\gamma = \sum_{i\in\mathbb{N}} \gamma_i$. (That is, for each $u \in \Gamma^{(i)}(V)$ and $w \in \Gamma^{(j)}(V)$, we let $u < w$ whenever either $i < j$, or $i = j$ and $u <_i w$.) Moreover, there exists an order-preserving bijection $\delta\colon [0,\gamma) \to V$. It is now clear that δ is a directed sequence in Γ. Since the image $\mathrm{Im}(\delta)$ of δ is V, this directed sequence is also a G-leash for every $G \leq \mathrm{Aut}(\Gamma)$. ■

3 A lemma and a proof of Theorem 1.2

We will now state and prove a general lemma about vertex stablisers in groups of digraph automorphisms. After that, Theorem 1.2 will follow easily.

Lemma 3.1 *Let α be an ordinal number, let $\{\Gamma_\beta \mid \beta \in [0,\alpha)\}$ be a set of pairwise disjoint digraphs on a vertex set V, let $G \leq \bigcap\{\mathrm{Aut}(\Gamma_\beta) \mid \beta \in [0,\alpha)\}$, and let $\Gamma = \bigcup\{\Gamma_\beta \mid \beta \in [0,\alpha)\}$. Suppose that there exists a G-leash δ in Γ of type γ. For $\eta \in [0,\gamma)$, let $v_\eta = \delta(\eta)$, and let $V_\eta = \{v_\iota \mid \iota \in [0,\eta)\}$. Further, let $G_0 = G_{v_0}$, and for every $\eta \in [1,\gamma]$, let $G_\eta = G_{(\Gamma(V_\eta))}$. Then the following holds:*

(i) *$G_{v_0} = G_0 \rhd G_1 \rhd \cdots \rhd G_\gamma = 1$ is a complete descending series for G_{v_0} of type γ;*

(ii) *$G_\eta/G_{\eta^+} \cong G_\eta^{\Gamma(v_\eta)} \leq G_{v_\eta}^{\Gamma(v_\eta)}$ for every $\eta \in [0,\gamma)$.*

Further, for $\eta \in [0,\gamma)$ and $\beta \in [1,\alpha]$, let $G_{0,\eta} = G_\eta$ and $G_{\beta,\eta} = \bigcap\{G_\eta \cap G_{(\Gamma_\iota(v_\eta))} \mid \iota \in [0,\beta)\}$. Then the following holds:

(iii) *$G_\eta = G_{0,\eta} \rhd G_{1,\eta} \rhd \cdots \rhd G_{\alpha,\eta} = G_{\eta^+}$ is a complete descending series between G_η and G_{η^+} of type α, with all the terms $G_{\beta,\eta}$, $\beta \in [0,\alpha]$, normal in G_η;*

(iv) *$G_{\beta,\eta}/G_{\beta^+,\eta} \cong G_{\beta,\eta}^{\Gamma_\beta(v_\eta)} \leq G_{v_\eta}^{\Gamma_\beta(v_\eta)}$ for every $\beta \in [0,\alpha)$.*

Consequently, there exists a complete descending series for G_{v_0} of type $\alpha\gamma$, with each of its factors isomorphic to a subgroup of some local group $G_u^{\Gamma_\beta(u)}$, $\beta \in [0,\alpha)$, $u \in V$.

Proof. We shall first prove that for every $\eta \in [0,\gamma)$ we have $G_\eta \leq G_{v_\eta}$. This is clearly true for $\eta = 0$. Now, if $0 < \eta$, then, by definition of a directed sequence, there exists $\eta' < \eta$ such that $v_\eta \in \Gamma(v_{\eta'})$. On the other hand, G_η is contained in every point-wise stabiliser $G_{(\Gamma(v_\iota))}$ for $\iota < \eta$, and thus also in $G_{(\Gamma(v_{\eta'}))}$. Therefore, G_η fixes v_η, as claimed.

In particular, there exists a natural action of G_η on $\Gamma(v_\eta)$ (as well as on $\Gamma_\beta(v_\eta)$ for every $\beta \in [0,\alpha)$). Since $G_{\eta^+} = G_\eta \cap G_{\Gamma(v_\eta)} = \mathrm{Ker}(G_\eta \to G_\eta^{\Gamma(v_\eta)})$, it follows that $G_{\eta^+} \lhd G_\eta$ and $G_\eta/G_{\eta^+} \cong G_\eta^{\Gamma(v_\eta)}$, completing the proof of Part (ii).

Further, by the definition of a G-leash, it is obvious that $G_\gamma = \bigcap\{G_{(\Gamma(v_\iota))} \mid \iota \in [0,\gamma)\} = G_{(\Gamma(V_\Gamma))} = G_{(\Gamma(\mathrm{Im}(\delta)))} = 1$. Now, let $\lambda \in [1,\gamma]$, and observe that $\bigcap_{\eta\in[0,\lambda)} G_\eta = \bigcap_{\eta\in[0,\lambda)} G_{(\Gamma(V_\eta))} = G_{(\Gamma(V_\lambda))} = G_\lambda$. Part (i) now follows if we apply this equality for every limit ordinal number $\lambda \leq \gamma$.

Similarly, for any $\eta \in [0,\gamma)$ we have $G_{\alpha,\eta} = G_\eta \cap G_{(\cup\{\Gamma_\iota(v_\eta)|\iota\in[0,\alpha)\})} = G_\eta \cap G_{(\Gamma(v_\eta))} = G_{\eta^+}$. Observe also that for every $\lambda \in [1,\alpha]$, we have $\bigcap_{\beta\in[0,\lambda)} G_{\beta,\eta} = G_{\lambda,\eta}$. This fact (when applied for limit ordinals λ) is needed in the proof of Part (iii).

Now, let $\beta \in [0,\alpha)$. Then $G_{\beta^+,\eta}$ can be written either as $G_\eta \cap G_{(\cup\{\Gamma_\iota(v_\eta)|\iota\in[0,\beta]\})}$, or as $G_{\beta,\eta} \cap G_{(\Gamma_\beta(v_\eta))}$. This shows that $G_{\beta^+,\eta} = \mathrm{Ker}(G_\eta \to G_\eta^{\cup\{\Gamma_\iota(v_\eta)|\iota\in[0,\beta]\}}) = \mathrm{Ker}(G_{\beta,\eta} \to G_{\beta,\eta}^{\Gamma_\beta(v_\eta)})$. In particular, $G_{\beta^+,\eta}$ is normal in $G_{\beta,\eta}$ as well as in G_η. Moreover, $G_{\beta,\eta}/G_{\beta^+,\eta} \cong G_{\beta,\eta}^{\Gamma_\beta(v_\eta)}$, as claimed in Part (iv). To complete the proof of Part (iii), we need to show that $G_{\lambda,\eta}$ is normal in G_η for every limit ordinal $\lambda \leq \alpha$. But, since $G_{\lambda,\eta}$ is the intersection of all $G_{\beta^+,\eta}$, $\beta < \lambda$, and since the latter are proven to be normal in G_η, so is $G_{\lambda,\eta}$. The last assertion of the lemma now follows easily from Parts (i) - (iv). ∎

Proof of Theorem 1.2. Let G, $N(G)$, Δ, Γ, v, and V be as in the statement of Therorem 1.2. Since G is normal in $N(G)$, every element of the set $\mathcal{D} = \{\Delta^g \mid g \in N(G)\}$ is a G-orbital. Moreover, the sets in $\mathcal{D}$ are pairwise disjoint and their union is Γ. Let α be the ordinal number arising from a well-ordering of $\mathcal{D}$. Then we can label the elements of $\mathcal{D}$ by the ordinals $\beta \in [0,\alpha)$, that is, $\mathcal{D} = \{\Gamma_\beta \mid \beta \in [0,\alpha)\}$.

Since G is a transitive subgroup of $N(G)$, the group $N(G)$ can be written as a product of G and the stabiliser $N(G)_v$. Consequently, $\mathcal{D} = \{\Delta^g \mid g \in N(G)_v\}$. For $u \in V$ and $g \in N(G)_v$, consider the local group $G_u^{\Delta^g(u)} \cong G_u/G_{(\Delta^g(\{u\}))}$. Since G is transitive, there exists an element $h \in G$ such that $u^h = v$. Observe that the conjugation by hg^{-1} maps G_u to G_v and $G_{(\Delta^g(\{u\}))}$ to $G_{(\Delta(\{v\}))}$, and thus induces an isomorphism between the local groups $G_u^{\Delta^g(u)}$ and $G_v^{\Delta(v)}$. Hence, all the local groups $G_u^{\Gamma_\beta}(u)$, $u \in U$ and $\beta \in [0,\alpha)$, are isomorphic to $G_v^{\Delta(v)}$.

By Lemma 2.3, there exists a G-leash $\delta\colon [0,\gamma) \to V$, such that $\delta(0) = v$. Moreover, by Lemma 3.1, there exists a complete descending series for G_v with each of its factors isomorphic to a subgroup of some local group $G_u^{\Gamma_\beta(u)}$, which is isomorphic to $G_v^{\Delta(v)}$, as shown above. Finally, by Lemma 2.1, every simple section of G_v is then also a section of $G_v^{\Delta(v)}$. ∎

4 Some implications of Lemma 3.1

The rather weak assumptions of Lemma 3.1 allow us to specialise the result in numerous different directions. A few of them are considered below in a series of remarks. Throughout this section the notation and all the assumptions of Lemma 3.1

are maintained without further notice. In addition, we let $v = v_0$. We start with few remarks regarding different finitness conditions.

Remark 1. Assume that all the digraphs Γ_β, $\beta \in [0, \alpha)$, are locally finite. Then every local group $G_u^{\Gamma_\beta(u)}$ is finite, and the last assertion of Lemma 3.1 implies that there exists a complete descending series between of G_v of type $\alpha\gamma$ with finite factors. In the standard group theoretical terminology (see for example [5]) this means that G_v is hypo-finite. If, in addition, the ordinal α is finite, then Γ is also locally finite and the type of this descending series is either γ (if γ is infinite), or finite (if γ is finite).

Remark 2. Assume now, that the ordinal γ is finite (that is, assume that the G-leash in Γ is finite). Then the complete descending series in Part (i) of Lemma 3.1 is finite, and hence a normal series for G_v. If, in addition, Γ is locally finite, then the factors of this normal series are finite. Moreover:

Corollary 4.1 *Let Γ be a disjoint union of finitely many locally finite digraphs $\Gamma_1, \ldots, \Gamma_m$ on a vertex set V, let $v \in V$, and let $G \le \mathrm{Aut}(\Gamma_i)$ for every $i \in \{1, \ldots, m\}$. If there exists a finite G-leash in Γ with the beginning in v, then the following holds:*

(i) *G_v is a finite group;*

(ii) *if p is a prime divisor of $|G_v|$, then p divides $|G_u^{\Gamma_i(u)}|$ for some $u \in V$ and some $i \in \{1, \ldots, m\}$;*

(iii) *if $G_u^{\Gamma_i(u)}$ is solvable for every $u \in V$ and every $i \in \{1, \ldots, m\}$, then G_v is solvable;*

(iv) *if $|\Gamma_i(u)| \le 4$ for every $u \in V$ and every $i \in \{1, \ldots, m\}$, then G_v is solvable.*

Remark 3. If the digraph Γ is strongly connected, then Lemma 2.3 guarantees the existence of a G-leash with the beginning in any prescribed vertex. Moreover, if α is finite, then this G-leash can be chosen so that its type is at most ω. It follows from Remark 1, that in this case, for every $v \in V$, there exists a complete descending series of G_v of type at most ω with finite factors.

Remark 4. Assume now that Γ is a graph. Let $\eta \in [1, \gamma)$ and $\eta' < \eta$ such that $v_\eta \in \Gamma(v_{\eta'})$. Since Γ is a graph, then also $v_{\eta'} \in \Gamma(v_\eta)$, and thus $G_\eta^{\Gamma(v_\eta)} \le G_{v_\eta v_{\eta'}}^{\Gamma(v_\eta)} \cong (G_{v_\eta}^{\Gamma(v_\eta)})_{v_{\eta'}}$. That is, every factor of the complete descending series for G_v in Part (i) of Lemma 3.1, with the sole exception of the first one G_1/G_0 (which is isomorphic to $G_v^{\Gamma(v)}$), is a subgroup of a vertex stabiliser in a local group $G_u^{\Gamma(u)}$. This has further interesting consequences.

Remark 5. Suppose that Γ is a graph and that $G_u^{\Gamma(u)}$ is semiregular for every $u \in V$ (recall that a permutation group is *semiregular* if the stabiliser of every point is trivial). Then all the factors G_β/G_{β^+}, $\beta \in [1, \gamma)$, in the series $G_v = G_0 \rhd G_1 \rhd \cdots \rhd G_\gamma = 1$, are trivial. Suppose that G_1 is not trivial, and let $\mu = \min\{\beta \in [2,\gamma) \mid G_\beta \neq G_1\}$. If $\mu = \beta^+$ for some β, then $1 = G_\beta/G_\mu = G_1/G_\mu$, and thus $G_\mu = G_1$, a contradiction. On the other hand, if μ is a limit ordinal, then $G_\mu = \bigcap_{\beta<\mu} G_\beta = \bigcap_{\beta<\mu} G_1 = G_1$, again contradicting the assumption on μ. Therefore, G_1 is trivial, and so $G_v^{\Gamma(v)} \cong G_0/G_1 \cong G_0 = G_v$. By taking Remark 3 into account, we may conclude the following:

Corollary 4.2 *Let Γ be a connected graph on a vertex set V, and let $G \leq \mathrm{Aut}(\Gamma)$ be such that $G_u^{\Gamma(u)}$ is semiregular for every $u \in V$. Then $G_v \cong G_v^{\Gamma(v)}$ for every $v \in V$.*

Remark 6. Under the assumption that Γ is a graph, a variation of Theorem 1.2 can be proved.

Corollary 4.3 *Let Γ be a connected graph on a vertex set V, let $v \in V$, let $G \leq \mathrm{Aut}(\Gamma)$, and let S be a simple section of G_v. Then S is a section of the local group $G_v^{\Gamma(v)}$, or it is a section of the point stabiliser $(G_u^{\Gamma(u)})_w$ in the local group $G_u^{\Gamma(u)}$ for some $u \in V$ and $w \in \Gamma(u)$.*

PROOF. As in the proof of Theorem 1.2, it follows that S is a section of one of the factors G_β/G_{β^+}. If this happens for $\beta = 0$, then S is a section of $G_0/G_1 \cong G_v^{\Gamma(v)}$, as asserted. On the other hand, if $\beta \geq 1$, then G_β/G_{β^+} is a subgroup of $(G_u \cap G_w)^{\Gamma(u)}$, for some $(u, w) \in \Gamma$, and the result follows. ■

Remark 7. We shall conclude this section with a remark on the very special situation where Γ is a finite vertex-transitive digraph. Observe that in this case strong connectivity of Γ is equivalent to connectivity of Γ. Indeed, let Γ be a connected digraph. For a vertex v of Γ, let $S(v)$ denote the set containing v and all the vertices u for which there exists a directed path from v to u. Clearly, $S(u) \subseteq S(v)$ for every $u \in S(v)$, and $S(v)^g = S(v^g)$ for every $g \in \mathrm{Aut}(\Gamma)$. Since $\mathrm{Aut}(\Gamma)$ is transitive, this implies that $|S(u)| = |S(v)|$ for every two vertices u, v. In particular, $S(u) = S(v)$ for every $u \in S(v)$, and so the induced digraph $\Gamma[S(v)]$ is strongly connected. Hence, if Γ is not strongly connected, then there exists a pair $(w, v) \in \Gamma$ such that $w \notin S(v)$. On the other hand, $v \in S(w)$ implying $S(v) = S(w)$, a contradiction. This shows that the word "strongly connected" in the statements of the results above can be substituted by "connected" whenever Γ is a finite vertex transitive digraph.

References

[1] A. Betten, A. Delandtsheer, A. C. Niemeyer, and C. E. Praeger. On a theorem of Wielandt for finite primitive permutation groups, *J. Group Theory* **6** (2003), 415–420.

[2] P. J. Cameron, C. E. Praeger, J. Saxl, and G. M. Seitz. On the Sims conjecture and distance transitive graphs, *Bull. London Math. Soc.* **15** (1983), 499–506.

[3] J. D. Dixon and B. Mortimer. *Permutation Groups*, Graduate Texts in Mathematics, vol. 163, Springer, New York (1996).

[4] P. R. Halmos. *Naive Set Theory*, The University Series in Undergraduate Mathematics, D. Van Nostrand, New York (1960).

[5] D. J. S. Robinson. *Finitness Conditions and Generalized Slouble Groups*, Part 1, Springer, New York (1972).

[6] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York (1964).