KEMPE EQUIVALENCE OF COLORINGS

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Ljubljana, February 4, 2005

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January 5, 2005

Abstract

Several basic theorems about the chromatic number of graphs can be extended to results in which, in addition to the existence of a k-coloring, it is also shown that all k-colorings of the graph in question are Kempe equivalent. Here, it is also proved that for a planar graph with chromatic number less than k, all k-colorings are Kempe equivalent.

1   Introduction

Let G be a graph and $k \geq 1$ an integer. A vertex set $U \subseteq V(G)$ is *independent* if no two vertices of U are adjacent in G. A *k-coloring* of G is a partition of $V(G)$ in k independent sets $U_1, \ldots, U_k$, called *color classes*. If $v \in U_i$ ($i \in \{1, \ldots, k\}$), then v is said to have *color i*. Every k-coloring can be identified with a mapping $c : V(G) \to \{1, \ldots, k\}$ where $c(v)$ is the color of v. The *chromatic number* of G is denoted by $\chi(G)$.

Let $a, b \in \{1, \ldots, k\}$ be distinct colors. Denote by $G(a, b)$ the subgraph of G induced on vertices of color a or b. Every connected component K of $G(a, b)$ is called a *K-component* (short for *Kempe component*). By switching the colors a and b on K, a new coloring is obtained. This operation is called a *K-change* (short for *Kempe change*). Two k-colorings c_1, c_2 are *K-equivalent* (or *K^k-equivalent*), in symbols $c_1 \sim_k c_2$, if c_2 can be obtained from c_1 by a sequence of K-changes, possibly involving more than one pair of colors in successive K-changes.

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Let $\mathcal{C}_k = \mathcal{C}_k(G)$ be the set of all k-colorings of G. The equivalence classes $\mathcal{C}_k/\sim_k$ are called the K^k-*classes* (or just K-*classes*). The number of K^k-classes of G is denoted by $\mathrm{Kc}(G, k)$.

K-changes have been introduced by Kempe in his false proof of the four color theorem. They have proved to be an utmost useful tool in graph coloring theory. It remains one of the basic and most powerful tools. The results of this paper show that some basic theorems about graph colorings can usually be turned into stronger K-equivalence results where it can be proved that all k-colorings are K-equivalent. These results have been one of our motivations to study K-equivalence of colorings. Several such results have been published by Meyniel and Las Vergnas [8, 6] who proved, in particular, that all 5-colorings of a planar graph (respectively, a K_5-minor free graph) are K-equivalent. Fisk [4] proved that all 4-colorings of an Eulerian triangulation of the plane are K-equivalent. We extend these results by showing that in every planar graph G with chromatic number less than k, all k-colorings are K-equivalent (see Corollary 4.5).

The second motivation to study K-equivalence is the possibility to generate colorings either by using K-changes as a heuristic argument [1, 11], or with the goal of obtaining a random coloring by applying random walks and rapidly mixing Markov chains [14]. For instance, Vigoda [14] proved that the Markov chain, whose state space is $\mathcal{C}_k(G)$ and whose transitions correspond to K-changes, quickly converges to the stationary distribution if $k \geq \frac{11}{6}\Delta(G)$. On the other hand [7], there are bipartite graphs for which the Markov chain needs exponentially many steps to come close to the stationary distribution if $k = O(\Delta/\log\Delta)$. Later, Hayes and Vigoda [5] proved rapid mixing for $k > (1 + \varepsilon)\Delta(G)$ for all $\varepsilon > 0$ assuming that G has girth more than 9 and $\Delta = \Omega(\log n)$. Dyer et al. [2] studied the same phenomenon on random graphs with expected average degree d, where d is a constant. Kempe change method has been successfully applied in some experiments [12] leading to new theoretical results.

2 Basic results

The following result shows that the study of K^k-equivalence may be interesting also when k is much larger than the chromatic number of the graph. It also shows that it is possible that $\mathrm{Kc}(G, k-1) = 1$ and $\mathrm{Kc}(G, k) > 1$.

Proposition 2.1 (a) *Let G be a bipartite graph and $k \geq 2$ an integer. Then* $\mathrm{Kc}(G, k) = 1$.

(b) *For any integers $l \geq 3$ and $k > l$, there exists a graph G with chromatic number l such that $\mathrm{Kc}(G, l) = 1$ and $\mathrm{Kc}(G, k) > 1$.*

Proof. (a) Clearly, any two 2-colorings are K^2-equivalent. Hence, it suffices to prove that every k-coloring of G is K-equivalent to a 2-coloring. This is easy to see and is left to the reader.

(b) Let G be the categorical product $K_l \times K_k$. Its vertices are pairs (i, j), $1 \leq i \leq l$, $1 \leq j \leq k$, and vertices (i, j) and (i', j') are adjacent if and only if $i \neq i'$ and $j \neq j'$. Let c be the l-coloring of G where $c((i, j)) = i$, and let c' be the k-coloring of G where $c'((i, j)) = j$. It is easy to see that c is the unique l-coloring of G, so $\chi(G) = l$ and $\mathrm{Kc}(G, l) = 1$. On the other hand, c' is not K-equivalent to any other k-coloring since all its 2-colored subgraphs $G(a, b)$ are connected. In particular, it is not K-equivalent to c, so $\mathrm{Kc}(G, k) > 1$. $\square$

There are other graphs with the same properties as in Proposition 2.1(b). They can be obtained from $K_l \times K_k$ by replacing every vertex (i, j) by an independent set $U(i, j)$ (of any size) and, for any two adjacent vertices (i, a) and (i', b) of $K_l \times K_k$ adding edges between $U(i, a)$ and $U(i', b)$ so that the subgraph induced on $\cup_{i=1}^{l}(U(i, a) \cup U(i, b))$ is connected. This construction describes the l-colorable graphs with a k-coloring which is not K-equivalent to any other k-coloring. More generally, it would be interesting to characterize l-colorable graphs $(l < k)$ with a k-coloring (k large) which is not K-equivalent to any $(k-1)$-coloring. This problem was considered by Las Vergnas and Meyniel [6] who conjectured that such graphs contain the complete graph K_k as a minor.

Lemma 2.2 *Suppose that c_0 is a $(k-1)$-coloring of a graph G and that U is an independent vertex set of G. Then c_0 is K-equivalent in $\mathcal{C}_k(G)$ to a k-coloring of G, one of whose color classes is U.*

Proof. Let $U = \{u_1, \ldots, u_r\}$. For $i = 1, \ldots, r$, let c_i be the k-coloring of G that is obtained from c_{i-1} by recoloring the vertex u_i with color k. It is clear that the vertex u_i forms a K-component in c_{i-1} for colors k and $c_{i-1}(u_i) = c_0(u_i)$. Therefore, c_i is K-equivalent to c_{i-1}. This shows that c_r is a coloring that is K-equivalent to c_0, and one of its color classes is U. $\square$

Corollary 2.3 *Let k be an integer. Suppose that G is a graph such that every k-coloring of G is K-equivalent to some $(k-1)$-coloring. If U is an independent vertex set of G, then $\mathrm{Kc}(G, k) \leq \mathrm{Kc}(G - U, k - 1)$.*

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Proof. For $i = 1, 2$, let c_i be a k-coloring of G. By assumption, c_i is K-equivalent to a $(k-1)$-coloring c_i'. By Lemma 2.2, c_i' is K-equivalent to a k-coloring c_i'', one of whose color classes is U. The restrictions of c_1'' and c_2'' to $G - U$ are $(k-1)$-colorings of $G - U$. If they are K^{k-1}-equivalent, then $c_1'' \sim_k c_2''$, and hence $c_1 \sim_k c_2$. This completes the proof. $\square$

A graph G is d-*degenerate* if every subgraph of G contains a vertex of degree $\leq d$. Corollary 2.3 can be used to prove the following result of Las Vergnas and Meyniel [6, Proposition 2.1]:

Proposition 2.4 *If G is a d-degenerate graph and $k > d$ is an integer, then* $\mathrm{Kc}(G, k) = 1$.

Proof. Since G is d-degenerate, its vertices can be enumerated as $v_1, \ldots, v_n$ such that the degree of v_i in the subgraph G_i induced on vertices $v_1, \ldots, v_i$ is at most d, $i = 1, \ldots, n$. Let U be the "greedy" independent set of G. The set U can be constructed by the following procedure: start with the empty set, and then consecutively, for $i = 1, \ldots, n$, add v_i into the current set U if no neighbor of v_i has been placed into U in earlier steps. An important property of U is that the graph $G - U$ is $(d-1)$-degenerate. To see this, let G' be a subgraph of $G - U$. Let v_i be the vertex of G' with largest value of i. Vertex v_i has at most d neighbors among $v_1, \ldots, v_{i-1}$. Since $v_i \notin U$, at least one of those neighbors is in U. Therefore, the degree of v_i in G' is at most $d - 1$.

The proof proceeds by induction on n. If $n = 1$, the statement is clear. Otherwise, let $G' = G - v_n$ and let c_1' be the restriction of c_0 to G'. Let c' be a $(k-1)$-coloring of G'. By the induction hypothesis, c_1' is K-equivalent to c'. There is a sequence of K-changes, $c_1' \sim_k c_2' \sim_k \cdots \sim_k c_r' = c'$. Let c_i be an extension of c_i' to G, and let c be a $(k-1)$-coloring of G extending c'.

There are two colors, say a_i and b_i, that are involved in the K-change yielding c_{i+1}' from c_i'. We assume that $a_i \neq k$. Since v_n has degree at most $d < k$, there were at least two candidates α_i, β_i for $c_i(v_n)$ (when we have made our choice what c_i would be). If $\{a_i, b_i\} = \{\alpha_i, \beta_i\}$, then we take any one of them to be $c_i(v_n)$. Otherwise, we take one that is not in $\{a_i, b_i\}$. Such a choice implies that $c_i \sim_k c_{i+1}''$, where c_{i+1}'' is an extension of c_{i+1}' to G. Now we define c_{i+1} by obeying the condition imposed above, i.e. that $c_{i+1}(v_n)$ is not in $\{a_{i+1}, b_{i+1}\}$ whenever possible. Clearly, $c_{i+1}'' \sim_k c_{i+1}$ since the two colorings differ only at v_n which is a K-component for colors α_{i+1} and β_{i+1}. This shows that c_0 is K-equivalent to c_r'', an extension of c_r'. If $c_r''(v_n) = k$, another K-change can replace this color by another one, thus yielding a $(k-1)$-coloring.

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The above proof shows that every k-coloring of G is K-equivalent to some $(k-1)$-coloring. By Corollary 2.3, $\mathrm{Kc}(G, k) \leq \mathrm{Kc}(G - U, k - 1)$, where U is the vertex set selected at the beginning of the proof. Now, the proof is easily completed by applying induction on d. □

An immediate corollary of Proposition 2.4 is:

Corollary 2.5 *Let Δ be the maximum degree of a graph G and let $k \geq \Delta + 1$ be an integer. Then $\mathrm{Kc}(G, k) = 1$. If G is connected and contains a vertex of degree $< \Delta$, then also $\mathrm{Kc}(G, \Delta) = 1$.*

We conjecture that the last statement of Corollary 2.5 can be extended to include all connected Δ-regular graphs with the exception of odd cycles and complete graphs.

The following proposition is left as an exercise.

Proposition 2.6 *Let G be a graph of order n, let α be the cardinality of a largest independent vertex set in G, and let $k \geq n - \alpha + 1$ be an integer. Then every k-coloring of G is K^k-equivalent to the k-coloring in which a fixed maximum independent set is a color class and every other color class is a single vertex. In particular, $\mathrm{Kc}(G, k) = 1$.*

3   Edge-colorings

Coloring the edges of a graph G is the same as coloring the vertices of its line graph $L(G)$. Vizing's Theorem states that the edges of a graph with maximum degree Δ can be colored with $\Delta + 1$ colors, i.e., $\chi'(G) = \chi(L(G)) \leq \Delta + 1$. We prove:

Theorem 3.1 *Let Δ be the maximum degree of a graph G. If $k \geq \chi'(G) + 2$ is an integer, then $\mathrm{Kc}(L(G), k) = 1$.*

Proof.   The proof is by induction on $\chi'(G)$. The case when $\chi'(G) \leq 2$ follows by Proposition 2.1(a), so assume that $\chi'(G) \geq 3$.

Let c be an arbitrary k-edge-coloring of G. First, we claim that c is K-equivalent to a $(k-1)$-edge-coloring. To prove this, we may assume that c has $m > 0$ edges of color k and that every k-edge-coloring which is K-equivalent to c has at least m edges of color k. The standard "fan" arguments of Vizing show that there is a sequence of K-changes which transforms c into an edge-coloring with $m - 1$ edges of color k, a contradiction. (Cf., e.g., [3]

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for details.) However, in order to make the proof self-contained, we repeat those arguments.

Let c and $m > 0$ be as above. We say that a color a is *missing at* a vertex v of G if no edge incident with v is colored a. Since $k \geq \Delta + 2$, at least two colors are missing at each vertex, and at least one of them is different from k. Clearly, if the same color a is missing at adjacent vertices u and v, then the change of the color of the edge uv to a represents a K-change.

Let $e = uv$ be an edge of color k. Suppose that color a_1 is missing at u and that a_0 is missing at v. If $a_0 = a_1$, then changing the color of e to a_0 is a K-change, yielding a coloring with $m - 1$ edges of color k, a contradiction. So, there is an edge vv_1 of color a_1. There is a color $a_2 \neq k$ which is missing at v_1. If a_2 is missing at v, we recolor vv_1 with a_2 and, as mentioned above, are henceforth able to get rid of the color k at e. Therefore, there is an edge vv_2 of color a_2.

Consider the K-component $K \subseteq G(a_0, a_2)$ at the vertex v_1. After the corresponding K-change, a_0 becomes missing at v_1. If a_0 is still missing at v, then we get a contradiction as above. Therefore, the corresponding K-change has changed the color a_2 at v to a_0, so $vv_2 \in E(K)$.

We shall now repeat the above procedure and henceforth have distinct edges $vv_1, \ldots, vv_r$ whose colors are $a_1, \ldots, a_r$ (respectively), and such that color a_i is missing at v_{i-1} for $i = 2, \ldots, r$. Moreover, the K-component in $G(a_0, a_i)$ at v_{i-1} is a path from v_{i-1} to v, whose last edge is $v_i v$. Having this situation, there is a color $a_{r+1} \neq k$ that is missing at v_r. If $a_{r+1} \notin \{a_0, \ldots, a_r\}$, then we consider the K-component $K \subseteq G(a_0, a_{r+1})$ at v_r. If this is a path ending at v, its last edge, call it $v_{r+1}v$, has color a_{r+1}, and we proceed with the next step. If K does not contain v, then after the K-change at K, a_0 is missing at v_r and at v. Now, we recolor vv_r with a_0, then we recolor vv_{r-1} with a_r, vv_{r-2} with a_{r-1}, $\ldots$, vv_1 with a_2. Finally, recolor e with a_1. All these recolorings were K-changes, so we have a coloring with $m - 1$ edges of color k, a contradiction.

From now on, we may assume that $a_{r+1} = a_j$, where $0 \leq j < r$. Let us consider $K \subseteq G(a_0, a_j)$ at v_r. The K-change at K makes a_0 missing at v_r. Since the component of $G(a_0, a_j)$ containing vv_j is a path from v_{j-1} to v_j and v, K does not contain vv_j. Therefore, a_0 is still missing at v. Now we conclude as above. This completes the proof of the claim.

By repeating the above arguments again if necessary, we conclude that c is K-equivalent to a $(\Delta + 1)$-edge-coloring.

Fix an edge-coloring c_0 with the color partition $E(G) = M_1 \cup \cdots \cup M_r$, $r = \chi'(G)$. It suffices to prove that any $(\Delta + 1)$-edge-coloring c of G is K-equivalent with c_0 in $\mathcal{C}_{\Delta+2}(L(G))$. By Lemma 2.2, c is K-equivalent to a

$(\Delta + 2)$-coloring c' whose first color class is M_r. Now, the proof is complete by applying induction on the graph $G - M_r$. □

It would be interesting to extend Theorem 3.1 to include $(\Delta+1)$-colorings as well. It is possible that $\mathrm{Kc}(L(G), \chi'(G) + 1)$, $\mathrm{Kc}(L(G), \Delta + 2)$, or even $\mathrm{Kc}(L(G), \Delta+1)$ are always 1. Let us remark, however, that there are graphs for which $\mathrm{Kc}(L(G), \chi'(G)) > 1$. Such examples are given after Theorem 3.3 below. We emphasize, specifically, the following interesting special case of the above speculations:

Conjecture 3.2 *If G is a graph with $\Delta(G) \leq 3$, then all its 4-edge-colorings are K-equivalent.*

We can say more if G is bipartite.

Theorem 3.3 *Let Δ be the maximum degree of a bipartite graph G. If $k \geq \Delta + 1$ is an integer, then $\mathrm{Kc}(L(G), k) = 1$.*

Proof. The proof is the same as for Theorem 3.1 except that we need to show that every k-edge-coloring of G is K^k-equivalent to a Δ-edge-coloring. This is a standard exercise and is left to the reader. □

The complete bipartite graph $K_{p,p}$ (where p is a prime) has a p-edge-coloring in which any two color classes form a Hamiltonian cycle. This example shows that Theorem 3.3 cannot be extended to Δ-colorings, not even for complete bipartite graphs.

Problem 3.4 *For which cubic bipartite graphs is $\mathrm{Kc}(L(G), 3) = 1$?*

A special case of this problem, when G is planar and 3-connected has been solved by Fisk. Let G be a 3-connected cubic planar bipartite graph. Its dual graph T is a 3-colorable triangulation of the plane. Fisk [4] proved (see Theorem 4.1 below) that any two 4-colorings of T are K-equivalent. If c_1, c_2 are 3-edge-colorings of G, they determine 4-colorings c_1^*, c_2^* (respectively) of T. It is easy to see that a K-change on 4-colorings of T corresponds to a sequence of one or more K-changes among the corresponding 3-edge-colorings in G. This implies that c_1 and c_2 are K-equivalent, and hence $\mathrm{Kc}(L(G), 3) = 1$.

Let us observe that planarity is essential for the above examples since the graph $K_{3,3}$ has non-equivalent edge-colorings, $\mathrm{Kc}(L(K_{3,3}), 3) = 2$.

4 Planar graphs

In [10], the author described an infinite class of "almost Eulerian" triangulations of the plane that have a special 4-coloring which is not K-equivalent to any other 4-coloring (and other 4-colorings exist). This shows that there are planar triangulations for which $\mathrm{Kc}(G,4) \geq 2$. By taking 3-sums of such graphs, we get planar triangulations with arbitrarily many equivalence classes of 4-colorings.

Meyniel [8] proved that $\mathrm{Kc}(G,5) = 1$ for every planar graph (and also $\mathrm{Kc}(G,k) = 1$ if $k \geq 6$, which follows by 5-degeneracy of planar graphs). In this section we prove a similar result for 4-colorings in the case when G is 3-colorable (cf. Theorem 4.4). A special case of this result, when G is a 3-colorable triangulation of the plane was proved by Fisk [4].

Theorem 4.1 (Fisk [4]) *Let G be a 3-colorable triangulation of the plane. Then $\mathrm{Kc}(G,4) = 1$.*

In order to extend Theorem 4.1, we shall need two auxiliary results.

Lemma 4.2 *Suppose that G is a subgraph of a graph $\tilde{G}$. Let $\tilde{c}_1, \tilde{c}_2$ be r-colorings of $\tilde{G}$. Denote by c_i the restriction of $\tilde{c}_i$ to G, $i = 1, 2$. If $\tilde{c}_1$ and $\tilde{c}_2$ are K^r-equivalent, then c_1 and c_2 are K^r-equivalent colorings of G.*

Proof. Any K-component in $\tilde{G}$ gives rise to one or more K-components in G, with respect to the induced coloring of G. This implies the lemma. $\square$

A *near-triangulation* of the plane is a plane graph such that all its faces except the outer face are triangles.

Proposition 4.3 *Suppose that G is a planar graph with a facial cycle C. If c_1, c_2 are 4-colorings of G, then there is a near-triangulation T of the plane with the outer cycle C such that $T \cap G = C$ and there are 4-colorings c_1', c_2' of G which are K-equivalent to c_1 and c_2, respectively, such that they both can be extended to 4-colorings of $G \cup T$. Moreover, if the restriction of c_1 to C is a 3-coloring, then $c_1' = c_1$, and c_1 can be extended to a 3-coloring of T.*

Proof. Let $C = v_1 v_2 \ldots v_k v_1$. The proof is by induction on k. If $k = 3$, then $T = C$, $c_1' = c_1$, and $c_2' = c_2$. Suppose now that $k \geq 4$. If there are indices i, j ($1 \leq i < j \leq k$) such that v_i and v_j are not consecutive vertices of C and such that $c_1(v_i) \neq c_1(v_j)$ and $c_2(v_i) \neq c_2(v_j)$, then we add the edge $v_i v_j$ inside C and apply induction on $C_1 = v_i v_{i+1} \ldots v_j v_i$ and $C_2 = v_j v_{j+1} \ldots v_k v_1 \ldots v_i v_j$.

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More precisely, let $G_1 = G + v_i v_j$. By the induction hypothesis for the cycle C_1, there are sequences of K-changes in G_1 (and hence also in G, by Lemma 4.2) transforming c_1 into c_{11}, and transforming c_2 into c_{12}, respectively, and there is a near-triangulation T_1 with outer cycle C_1 such that c_{11} and c_{12} can be extended to colorings $\bar{c}_{11}$ and $\bar{c}_{12}$ of $G_1 \cup T_1$. Next, apply the induction hypothesis to $G_1 \cup T_1$ for the facial cycle C_2 and colorings $\bar{c}_{11}$ and $\bar{c}_{12}$. Let T_2 be the corresponding near-triangulation, and c'_{11}, c'_{12} the corresponding colorings of $G_1 \cup T_1$ that can be extended to $G_1 \cup T_1 \cup T_2$. By Lemma 4.2, the K-changes which produce c'_{11} and c'_{12} from $\bar{c}_{11}$ and $\bar{c}_{12}$, respectively, can be made in G. All together, the restriction c'_l of the coloring c'_{1l} to G is K-equivalent to c_l in G ($l = 1, 2$). Clearly, c'_l has an extension to $G \cup T$, where $T = T_1 \cup T_2$. Therefore, T can be taken as the required near-triangulation for C.

If the restriction of c_1 to C is a 3-coloring, then $c_{11} = c_1$ and the restriction of c_1 to C_1 can be extended to a 3-coloring of T_1. Similarly, $c'_{11} = c_1$, and c_1 can be extended to a 3-coloring of T. This proves the "moreover" part of the proposition.
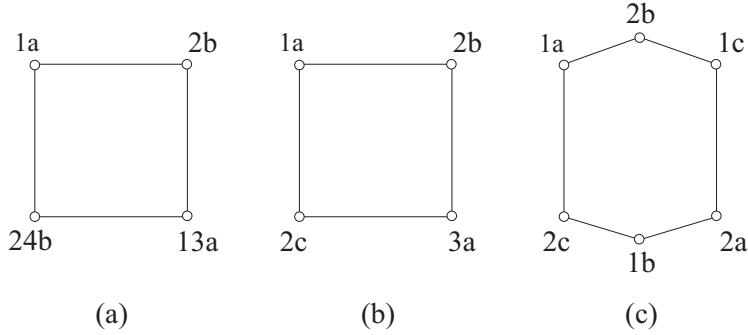

Figure 1: The special cases

Next, we show that vertices v_i, v_j exist unless one of the cases in Figure 1 occurs (where c_1 is represented by colors 1–4 and c_2 by colors a–d), up to permutations of colors, dihedral symmetries of C and up to changing the roles of c_1 and c_2. This is easy to see if $k = 4$. The details are left to the reader. If $k \geq 5$, we argue as follows. Suppose that v_i, v_j do not exist. We may assume that $c_1(v_1) = c_1(v_3) = 1$. Then $c_1(v_1) \neq c_1(v_4)$, so $c_2(v_1) = c_2(v_4) = a$ may be assumed. Suppose that $c_1(v_2) = 2$ and $c_2(v_2) = b$. Since $b = c_2(v_2) \neq c_2(v_4) = a$, we have $c_1(v_4) = c_1(v_2) = 2$. Now, $c_1(v_2) \neq c_1(v_5)$, so $c_2(v_5) = b$. Next, $c_2(v_5) \neq c_2(v_3)$ implies that

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$c_1(v_5) = c_1(v_3) = 1$. It follows, in particular, that $k \geq 6$. Similar conclusions as before imply that $c_1(v_6) = 2$ and $c_2(v_3) = c_2(v_6) = c$. If $k = 6$, this is the exceptional case of Figure 1(c). If $k \geq 7$, then we see that either v_2, v_7 or v_4, v_7 is the required pair v_i, v_j.

Let us consider the exceptional case shown in Figure 1(c). Let v_1 be the vertex with $c_1(v_1) = 1$ and $c_2(v_1) = a$ (upper-left). Try first a K-change of colors 1 and 3 at v_1. If this change gives rise to the same exception, there is a (1,3)-colored path P joining v_1 and v_3. Now, a K-change of colors 2 and 4 at v_2 changes c_1 into a coloring which does not fit Figure 1(c). None of these K-changes affects c_2, and we are done unless we do not want to change c_1 because of the "moreover" part. In that case we are allowed to use the fourth color in the extension of c_2, and we take T to be the near-triangulation with one interior point joined to all vertices on C.

Consider now the case of Figure 1(b). By symmetry, we may assume that c_1 is not allowed to be changed according to the "moreover" part of the proposition. Let v_1 be the vertex in the upper-left corner. By a K-change of colors a and d at v_1, or of b and c at v_2, we replace c_2 either by a 4-coloring which uses on C all four or only two of the colors. In each case, we can triangulate C by adding two adjacent vertices p, q such that p is adjacent to v_1, v_2, v_4, and q is adjacent to v_2, v_3, v_4.

The final case is the one shown in Figure 1(a). If we want $c_2' = c_2$, then we let T be the near-triangulation consisting of C and a vertex of degree 4 inside. Then c_2 extends to a 3-coloring of T, and c_1 also extends to a 4-coloring with the exception of the case when all vertices on C have distinct colors. In the latter case, we can either K-change colors 1 and 3 at v_1, or change 2 and 4 at v_2, without affecting the colors at v_3 and v_4. The new coloring c_1' extends to T.

Suppose, finally, that we want $c_1' = c_1$. In this case, c_1 is a 3-coloring on C. Up to symmetries, we may assume either $c_1(v_3) = 1$ and $c_1(v_4) = 2$, or $c_1(v_3) = 3$ and $c_1(v_4) = 2$. In the first case we can take the same near-triangulation T as above (one interior vertex of degree 4). In the latter case, we take two interior vertices u_1, u_2 in T, where u_1 is adjacent to u_2, v_4, v_1, v_2 and u_2 is adjacent to u_1, v_2, v_3, v_4. This completes the proof. $\square$

Theorem 4.4 *Let G be a 3-colorable planar graph. Then $\mathrm{Kc}(G, 4) = 1$.*

Proof. In order to be able to assume that G is 2-connected, we apply induction on the number of blocks of G. If $G = G_1 \cup G_2$, where $G_1 \cap G_2$ is either empty or a cutvertex v, we apply induction hypotheses on G_1 and G_2

but making sure that we never use a K-change on a K-component containing v. This is possible since a K-change using component K of $G_i(a, b)$ is the same as making K-changes on all components of $G_i(a, b)$ distinct from K.

From now on, we assume that G is 2-connected. Let c_1 be a 3-coloring of G. It suffices to see that every 4-coloring of G is K-equivalent to c_1.

Let c_2 be a 4-coloring of G. Let $C_1, \ldots, C_m$ be the facial cycles of G. Let $G_0 = G$, $c_1^0 = c_1$ and $c_2^0 = c_2$. For $i = 1, \ldots, m$, we apply Proposition 4.3 to the graph G_{i-1}, its facial cycle C_i and the colorings c_1^{i-1} and c_2^{i-1}. We conclude that there is a near-triangulation T_i with outer cycle C_i such that $G_{i-1} \cap T_i = C_i$. Let $G_i = G_{i-1} \cup T_i$. By Proposition 4.3, c_1^{i-1} can be extended to a 3-coloring c_1^i of G_i, and c_2^{i-1} is K-equivalent in G_{i-1} to a 4-coloring that has an extension c_2^i to G_i.

The final graph G_m is a triangulation with the 3-coloring c_1^m. By Theorem 4.1, c_2^m is K-equivalent to c_1^m. By successfully applying Lemma 4.2 to $G_{m-1} \subseteq G_m$, etc. up until $G_0 \subseteq G_1$, we conclude that $c_2^{m-1} \sim_k c_1^{m-1}$ in G_{m-1}, etc., until finally concluding that $c_1 = c_1^0 \sim_k c_2^0 = c_2$ in $G_0 = G$. □

Theorem 4.4 combined with Proposition 2.1(a) and the aforementioned result of Meyniel [8] yield:

Corollary 4.5 *Let G be a planar graph and $k > \chi(G)$ and integer. Then* $\mathrm{Kc}(G, k) = 1$.

A planar graph G may have 4-colorings which are not K-equivalent. However, if G is "almost 3-colorable", this is not likely to happen.

Problem 4.6 *Suppose that G is a 4-critical planar graph. Is it possible that G has two 4-colorings that are not K-equivalent to each other?*

5   Some further open problems

In the preceding sections, we have exposed several open problems about K-equivalence of graph colorings. Two further questions are presented below.

Meyniel [9] proved that a graph, in which every odd cycle of length 5 or more has at least two chords, is perfect. He conjectured that for every such graph G and every integer $k \geq \chi(G)$, $\mathrm{Kc}(G, k) = 1$. He proved that every k-coloring of G is K-equivalent to some $\chi(G)$-coloring. The last property does not hold for arbitrary perfect graphs.

Let G be a triangulation of some orientable surface, and let c be a 4-coloring of G. Let t_1^+ be the number of facial 3-cycles whose coloring (in the

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clockwise order around the face) is 234, and let t_1^- be the number of facial 3-cycles whose coloring is 432. Let $d(c) = |t_1^+ - t_1^-|$. The number $d(c)$ turns out to be invariant on permutations of the colors, and it is called the *degree* of the coloring c [4]. The degree, in particular its parity has been studied by Tutte who also observed that this is Kempe invariant, i.e., all colorings within the same K-class have the same parity.

One can also define two colorings to be *close* if they have at least one color class in common. Two colorings are *similar* if there is a sequence of colorings, starting with one and ending with the other, such that any two consecutive colorings in this sequence are close. The parity of the degree is constant on close colorings. It is not difficult to find triangulations of the plane with two similarity classes of 4-colorings. However, in all such examples known, colorings in different similarity classes have different parity of the degree. Tutte [13] asked if it is possible to have non-similar 4-colorings of a planar triangulation whose degrees have the same parity.

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