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# Solution Spaces of $H$ -Systems and the Ore-Sato Theorem\*

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## Abstract

An  $H$ -system is a system of first-order linear homogeneous difference equations for a single unknown function  $T$ , with coefficients which are polynomials with complex coefficients. We consider solutions of  $H$ -systems which are of the form  $T : \text{dom}(T) \rightarrow \mathbb{C}$  where either  $\text{dom}(T) = \mathbb{Z}^d$ , or  $\text{dom}(T) = \mathbb{Z}^d \setminus S$  and  $S$  is the set of integer singularities of the system. It is shown that any natural number is the dimension of the solution space of some  $H$ -system, and that in the case  $d \geq 2$  there are  $H$ -systems whose solution space is infinite-dimensional. The relationships between dimensions of solution spaces in the two cases  $\text{dom}(T) = \mathbb{Z}^d$  and  $\text{dom}(T) = \mathbb{Z}^d \setminus S$  are investigated. Finally we give an appropriate formulation of the Ore-Sato theorem on possible forms of solutions of  $H$ -systems in this setting.

## Résumé

Par un  $H$ -système nous désignons un système des équations aux différences linéaires homogènes pour une seule fonction inconnue  $T$ , à coefficients polynomiaux sur le corps des nombres complexes. Nous considérons les solutions des  $H$ -systèmes de la forme  $T : \text{dom}(T) \rightarrow \mathbb{C}$  où soit  $\text{dom}(T) = \mathbb{Z}^d$ , soit  $\text{dom}(T) = \mathbb{Z}^d \setminus S$ , et  $S$  est l'ensemble des singularités entières du système. Nous montrons que chaque nombre naturel est égal à la dimension de l'espace des solutions d'un  $H$ -système, et que dans le cas  $d \geq 2$  il y a des  $H$ -systèmes dont la dimension de l'espace des solutions est infinie. Les relations entre les dimensions des espaces des solutions dans les cas  $\text{dom}(T) = \mathbb{Z}^d$  et  $\text{dom}(T) = \mathbb{Z}^d \setminus S$  sont recherchées. Enfin nous présentons une formulation propre du théorème d'Ore-Sato sur les formes possibles des solutions des  $H$ -systèmes.

## 1 Introduction

Linear homogeneous recurrence equations with polynomial coefficients and systems of such equations play a significant role in combinatorics and in the theory of hypergeometric functions; the question of the dimension of the space of solutions of such systems is of great importance for many problems.

Let  $n_1, \dots, n_d$  be variables ranging over the integers and  $E_{n_i}$  the corresponding shift operators, acting on functions (sequences) of  $n_1, \dots, n_d$  by  $E_{n_i} f(n_1, \dots, n_i) = f(n_1, \dots, n_i + 1, \dots, n_d)$ ,  $i = 1, \dots, d$ . We consider  $H$ -systems, i.e., systems of equations of the form  $f_i E_{n_i} T = g_i T$ , where  $f_i, g_i \in \mathbb{C}[n_1, \dots, n_d] \setminus \{0\}$  for  $i = 1, \dots, d$ . The notion of singular points (singularities) of such systems can be defined in the usual way. Such singularities make obstacles (sometimes insuperable) for continuation of partial solutions of the system on all of  $\mathbb{Z}^d$ .

In this paper we consider two spaces of solutions of  $H$ -systems: the space  $V_1$  of solutions defined everywhere on  $\mathbb{Z}^d$ , and the space  $V_2$  of solutions that are defined at all nonsingular points of  $\mathbb{Z}^d$  (more precisely, if  $W$  is the set of all solutions of a given system that are defined at least at all non-singular elements of  $\mathbb{Z}^d$ , then  $V_2$  contains the restrictions of all elements of  $W$  to the set of all non-singular elements of  $\mathbb{Z}^d$ ). In Sections

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3 and 4 we investigate the dimensions of the spaces  $V_1, V_2$ . It is well known [7] that if (in the case  $d = 1$ ) one considers the germs of sequences at infinity (i.e., classes of sequences which agree from some point on), then the dimension of the solution space is 1. However, the situation is different with  $\dim V_1$  and  $\dim V_2$ . In Section 3 we prove for the case  $d = 1$  that if the equation has singularities then  $1 \leq \dim V_1 < \dim V_2 < \infty$ , and for any integers  $s, t$  such that  $1 \leq s < t$  there exists an equation with  $\dim V_1 = s$  and  $\dim V_2 = t$  (the case where there is no singularity is trivial:  $\dim V_1 = \dim V_2 = 1$ ). In turn, in Section 4 we show that in the case  $d > 1$  the possibilities are even richer: for any  $s, t \in \mathbb{Z}_+ \cup \{\infty\}$  there exists an  $H$ -system with  $\dim V_1 = s$  and  $\dim V_2 = t$ .

In Section 5 we revisit the Sato-Ore theorem [5, 6, 8] and show that, contrary to some interpretations in the literature (e.g., [3, 4]), this theorem does not imply that any solution of an  $H$ -system is of the form

$$R(n_1, \dots, n_d) \frac{\prod_{i=1}^p \Gamma(a_{i1}n_1 + \dots + a_{id}n_d + \alpha_i)}{\prod_{j=1}^q \Gamma(b_{j1}n_1 + \dots + b_{jd}n_d + \beta_j)} u_1^{n_1} \dots u_d^{n_d}, \quad (1)$$

where  $R \in \mathbb{C}(x_1, \dots, x_d)$ ,  $a_{ik}, b_{jk} \in \mathbb{Z}$ , and  $\alpha_i, \beta_j \in \mathbb{C}$  (for the case when the solution of the system is holonomic, and  $R$  is required to be a polynomial, we have already noted this in [2]). Finally we give an appropriate corollary of the Ore-Sato theorem on possible forms of solutions of systems under consideration.

We write  $p \perp q$  to indicate that polynomials  $p, q \in \mathbb{C}[x_1, \dots, x_d]$  are relatively prime. We call a set  $A \subseteq \mathbb{Z}^d$  *algebraic* if there is a polynomial  $p \in \mathbb{C}[x_1, \dots, x_d] \setminus \{0\}$  which vanishes on  $A$ . Clearly, if  $A$  is algebraic and  $B$  is not, then  $B \setminus A$  is not algebraic. Also, a finite union of algebraic sets is algebraic. We write  $T =_a T'$  if the set  $\{(n_1, \dots, n_d) \in \mathbb{Z}^d; T(n_1, \dots, n_d) \neq T'(n_1, \dots, n_d)\}$  is algebraic.

**Definition 1** Let  $E$  denote the shift operator corresponding to  $x$ , so that  $Ef(x) = f(x + 1)$  for every  $f \in \mathbb{C}(x)$ . A rational function  $u \in \mathbb{C}(x)$  is *shift-reduced* if there are  $a, b \in \mathbb{C}[x]$  such that  $u = a/b$  and  $a \perp E^k b$  for all  $k \in \mathbb{Z}$ .

**Theorem 1** For every rational function  $F \in \mathbb{C}(x)$  there are rational functions  $u, v \in \mathbb{C}(x)$  such that

- (i)  $F = u \cdot \frac{Ev}{v}$ ,
- (ii)  $u$  is shift-reduced.

**Definition 2** If  $u, v, F$  are as in Theorem 1,  $(u, v)$  is a rational normal form, or RNF, of  $F$ . We denote the set of all RNF's of  $F$  by  $\text{RNF}(F)$ .

**Theorem 2** Let  $(u, v)$  and  $(u_1, v_1)$  be two RNF's of  $F \in \mathbb{C}(x) \setminus \{0\}$ . Write  $u = p/q$  and  $u_1 = p_1/q_1$  where  $p, q, p_1, q_1 \in \mathbb{C}[x]$ ,  $p \perp q$ , and  $p_1 \perp q_1$ . Then  $\deg p = \deg p_1$  and  $\deg q = \deg q_1$ .

For proofs of Theorems 1 and 2, see [1].

## 2 $H$ -systems and their solution spaces

**Definition 3** An  $H$ -system<sup>1</sup> is a system of equations

$$f_i(n_1, \dots, n_d)T(n_1, \dots, n_i + 1, \dots, n_d) = g_i(n_1, \dots, n_d)T(n_1, \dots, n_i, \dots, n_d), \quad i = 1, 2, \dots, d, \quad (2)$$

where  $f_i, g_i \in \mathbb{C}[n_1, \dots, n_d] \setminus \{0\}$  and  $f_i \perp g_i$ . We say that a  $d$ -variate sequence  $T$  (i.e., a function  $T : \text{dom}(T) \rightarrow \mathbb{C}$ ) is a solution of (2) if (2) is satisfied for all  $(n_1, \dots, n_i, \dots, n_d) \in \text{dom}(T)$  such that  $(n_1, \dots, n_i + 1, \dots, n_d) \in \text{dom}(T)$  as well.

**Definition 4** Let  $A$  be an  $H$ -system of the form (2).

A  $d$ -tuple  $(n_1, \dots, n_d) \in \mathbb{Z}^d$  is a *trailing integer singularity* of  $A$  if there exists  $i$ ,  $1 \leq i \leq d$ , such that  $g_i(n_1, \dots, n_d) = 0$ . A  $d$ -tuple  $(n_1, \dots, n_d) \in \mathbb{Z}^d$  is a *leading integer singularity* of  $A$  if there exists  $i$ ,  $1 \leq i \leq d$ , such that  $f_i(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_d) = 0$ . A  $d$ -tuple  $(n_1, \dots, n_d) \in \mathbb{Z}^d$  is an *integer singularity* of  $A$  if it is a leading or a trailing integer singularity of  $A$ .

<sup>1</sup>The prefix “ $H$ ” refers to Jakob Horn and to the adjective “hypergeometric” as well.

Let  $S(A)$  denote the set of all integer singularities of  $A$ . Denote by  $V_1(A)$  the  $\mathbb{C}$ -linear space of all solutions of  $A$  which are defined at all elements of  $\mathbb{Z}^d$ , and by  $V_2(A)$  the  $\mathbb{C}$ -linear space of all solutions of  $A$  which are defined at all elements of  $\mathbb{Z}^d \setminus S(A)$ .

We consider only integer singularities here, therefore we will drop the adjective “integer” in the sequel. Sometimes we will also drop the name of the  $H$ -system, and will write  $V_1, V_2$  instead of  $V_1(A), V_2(A)$ .

**Definition 5** Call the two  $d$ -tuples  $(n_1, \dots, n_d), (n'_1, \dots, n'_d) \in \mathbb{Z}^d$  adjacent if  $\sum_{i=1}^d |n_i - n'_i| = 1$ . Call a finite sequence  $t_1, \dots, t_k \in \mathbb{Z}^d$  a path from  $t_1$  to  $t_k$  if  $t_i$  is adjacent to  $t_{i+1}$  for all  $i = 1, \dots, k-1$ . Given an  $H$ -system  $A$ , we define components induced by  $A$  on  $\mathbb{Z}^d$  as the equivalence classes of the following equivalence relation  $\sim$  in  $\mathbb{Z}^d$ :  $t' \sim t''$  iff there exists a path from  $t'$  to  $t''$  which contains no singularity of  $A$ . If  $T$  is a solution of an  $H$ -system  $A$ , then its constituent is the sequence that is the restriction of  $T$  on a component induced by  $A$ .

**Definition 6** Rational functions  $F_1, \dots, F_d \in \mathbb{C}(n_1, \dots, n_d)$  are consistent if

$$(E_{n_j} F_i) F_j = F_i (E_{n_i} F_j)$$

for all  $1 \leq i \leq j \leq d$ .

Note that a single rational function (corresponding to the case  $d = 1$ ) is always consistent.

**Proposition 1** Let  $A$  be an  $H$ -system of the form (2) where  $g_1/f_1, \dots, g_d/f_d$  are consistent rational functions. Then  $\dim V_2$  is equal to the number of components induced by  $A$ .

**Proof:** To each component  $C_i$  induced by  $A$  on  $\mathbb{Z}^d$  we assign a solution  $T_i$  of (2) which is 1 at a selected point  $p_i \in C_i$ , and 0 on all the remaining components. The values of  $T_i$  on the remaining points of  $C_i$  are uniquely determined by (2). It is clear that the set of all  $T_i$  is a basis for  $V_2$ .  $\square$

### 3 Dimensions of solution spaces: The univariate case

When  $d = 1$  the system (2) is of the form

$$f(n)T(n+1) = g(n)T(n) \tag{3}$$

where  $f(n), g(n) \in \mathbb{C}[n] \setminus \{0\}$  and  $f(n) \perp g(n)$ .

**Example 1** ( $\dim V_1 = 1, \dim V_2 = k$ ) Consider the recurrence

$$T(n+1) = p_k(n) T(n) \tag{4}$$

where  $k \geq 1$  and  $p_k(n) = \prod_{i=0}^{k-2} (n - 2i + 1)$ . Here we use the convention that a product is 1 if its lower limit exceeds its upper limit. Clearly the set of singularities of (4) is  $\{2i - 1; i = 0, 1, \dots, k - 2\}$ , so  $\dim V_2 = k$ . To compute  $\dim V_1$ , note that any solution  $T(n)$  of (4) defined for all  $n \in \mathbb{Z}$  is a constant multiple of

$$F_k(n) = \begin{cases} (-1)^{(k-1)n} / \prod_{i=0}^{k-2} (2i - n - 1)!, & n < 0, \\ 0, & n \geq 0. \end{cases}$$

Therefore  $\dim V_1 = 1$ .

**Example 2** ( $\dim V_1 = m, \dim V_2 = m + 1$ ) Now consider the recurrence

$$q_m(n+1) T(n+1) = q_m(n) T(n) \tag{5}$$

where  $m \geq 1$  and  $q_m(n) = \prod_{i=1}^m (n + 2i + 1)$ . The set of singularities is  $\{-(2i + 1); i = 1, 2, \dots, m\}$ , so  $\dim V_2 = m + 1$ . Let  $T(n)$  be a solution of (5) defined for all  $n \in \mathbb{Z}$ . By substituting  $n = -2(i + 1)$  for  $i = 1, 2, \dots, m$  into (5), we see that  $T(n) = 0$  for these values of  $n$ . Likewise, by substituting  $n = -3$  into

(5), we find that  $T(-2) = 0$ . Using (5) it follows by induction on  $n$  that  $T(n) = 0$  for all  $n \leq -2(m+1)$  and for all  $n \geq -2$  as well. On the other hand, it is easy to check that

$$G_m^{(i)}(n) = \delta_{n, -(2i+1)}$$

(where  $\delta$  is the Kronecker delta) is a solution of (5) for  $i = 1, 2, \dots, m$ . Therefore  $\dim V_1 = m$ .

Before describing the general situation we need a definition and a lemma.

**Definition 7** Let  $A$  be an  $H$ -system of the form (3). An interval of integers

$$I = \{k, k+1, \dots, k+m\}, \quad m \geq 0, \tag{6}$$

is a segment of singularities of  $A$  if  $I \subseteq S(A)$  while  $k-1, k+m+1 \notin S(A)$ .

**Lemma 1** Each segments of singularities (6) of equation (3) is of one of the following types:

- (i) all elements of the segment are trailing singularities;
- (ii) all elements of the segment are leading singularities;
- (iii) there exists  $j$ ,  $0 \leq j < m$ , such that  $k, k+1, \dots, k+j$  are leading singularities, while  $k+j+1, \dots, k+m$  are trailing singularities.

**Proof:** If  $u \in \mathbb{Z}$  is a trailing singularity and  $u+1$  a leading singularity of (3) then  $f(u) = g(u) = 0$ , contrary to the assumption  $f \perp g$ . So any segment of singularities of (3) consists of a (possibly empty) interval of leading singularities followed by a (possibly empty) interval of trailing singularities.  $\square$

**Theorem 3** Let  $S$  denote the set of singularities of equation (3).

- a) If  $S = \emptyset$  then  $\dim V_1 = \dim V_2 = 1$ .
- b) If  $S \neq \emptyset$  then  $1 \leq \dim V_1 < \dim V_2 < \infty$ , and for any integers  $s, t$  such that  $1 \leq s < t$  there exists an equation of the form (3) such that  $\dim V_1 = s$  and  $\dim V_2 = t$ .

**Proof:** a) This is clear.

b) There is only a finite set of components induced on  $\mathbb{Z}$  by (3), therefore  $\dim V_2 < \infty$ .

Next we prove that  $\dim V_1 < \dim V_2$ . First we show that if (6) is a segment of singularities of (3), then the restriction of  $V_1$  to this segment has dimension  $\leq 1$ , while the analogous restriction of  $V_2$  has dimension 2. Indeed, if  $u$  is a trailing singularity, then any sequence from  $V_1$  vanishes at  $u+1$ ; and if  $u$  is a leading singularity, then any sequence from  $V_1$  vanishes at  $u-1$ . By Lemma 1 we have three possibilities (i), (ii), (iii) for (6). In case (i) we have  $T(k+1) = T(k+2) = \dots = T(k+m+1) = 0$ , in case (ii)  $T(k-1) = T(k) = \dots = T(k+m-1) = 0$ , in case (iii)  $T(k-1) = T(k) = \dots = T(k+j-1) = 0$  and  $T(k+j+1) = T(k+j+2) = \dots = T(k+m+1) = 0$ ; in each case  $T(n)$  can be nonzero at most in two points, however the value at one of them is uniquely determined by the value at the other one. Therefore the dimension of the restricted  $V_1$  is  $\leq 1$ . The same holds for dimension of the restriction of  $V_1$  to the set

$$\{k-v, k-v+1, \dots, k, k+1, \dots, k+m, k+m+1, \dots, k+w\},$$

where  $k, k+1, \dots, k+m$  are singularities, while  $k-v, \dots, k-1$  and  $k+m+1, \dots, k+w$  are not. Gluing together two such restrictions with coinciding, say,  $k+m+1, \dots, k+w$ , and non-intersecting singular parts, we get the dimension  $\leq 2$ , while the dimension of the corresponding restriction of  $V_2$  is 3 and so on. This proves that  $\dim V_1 < \dim V_2$ .

Finally we prove that if the set of singularities is not empty, then  $\dim V_1 > 1$ . If there is a segment of singularities of type (iii) then  $V_1$  contains a solution  $T(n)$  such that  $T(n+j) = 1$  and  $T(n) = 0$  when  $n \neq k+j$  and  $n \neq k+j+1$ . If all the segments are of types (i) and (ii), then either the segment with the minimal  $k$  is of type (i), or the segment with the maximal  $k$  is of type (ii), or there is a segment of type (ii) immediately followed by a segment of type (i). In each of the cases  $V_1$  contains a nonzero solution.  $\square$

**Theorem 4** For any integers  $s, t$  such that  $1 \leq s < t$  there exists an equation of the form (3) such that  $\dim V_1 = s$  and  $\dim V_2 = t$ .

**Proof:** Consider the recurrence

$$p_m(n) T(n+1) = r_k(n) q_m(n) T(n) \quad (7)$$

where  $k, m \geq 1$ ,  $r_k(n)$  is as in Example 1, and  $p_m(n), q_m(n)$  are as in Example 2. Here the set of singularities is  $\{2i-1; i=0, 1, \dots, k-2\} \cup \{-(2i+1); i=1, 2, \dots, m\}$ , so  $\dim V_2 = k+m$ . Let  $T(n)$  be a solution of (7) defined for all  $n \in \mathbb{Z}$ . In exactly the same way as in Example 2 we can see that  $T(n) = 0$  for  $n = -2, -4, \dots, -2(m+1)$ ,  $n \leq -2(m+1)$  or  $n \geq -2$ , and that  $G_m^{(i)}(n) = \delta_{n, -(2i+1)}$  is a solution of (7) for  $i = 1, 2, \dots, m$ . Therefore  $\dim V_1 = m$ .

If  $1 \leq s < t$ , let  $m = s$  and  $k = t - s$ . Then for equation (7),  $\dim V_1 = m = s$  and  $\dim V_2 = k + m = t$ .  $\square$

We conclude this section by remarks on computation of  $\dim V_1$  and  $\dim V_2$ . Let  $A$  denote equation (3). According to Proposition 1,  $\dim V_2(A)$  is the number of components induced on  $\mathbb{Z}$  by  $A$  and is thus easy to compute. We claim that  $\dim V_1(A)$  equals the dimension of the kernel of a bidiagonal matrix  $B$  defined as follows. Let  $\alpha$  be the maximum and  $\beta$  the minimum of the integer roots of  $f(x)g(x)$ ; if  $A$  has no integer singularities then we can take  $\alpha = \beta = 1$ . Let  $B$  be the  $(\alpha - \beta + 1) \times (\alpha - \beta + 2)$  matrix with entries

$$b_{i,j} = \begin{cases} f(\alpha - i + 1), & j = i, \\ -g(\alpha - i + 1), & j = i + 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $1 \leq i \leq \alpha - \beta + 1$  and  $1 \leq j \leq \alpha - \beta + 2$ . Indeed, any vector  $v$  such that  $Bv = 0$  can be extended to a solution of  $A$  in a unique way. This mapping is an isomorphism between the kernel of  $B$  and  $V_1(A)$ .

Incidentally, this gives an alternative proof of the inequality  $\dim V_1 \geq 1$ :  $B$  has more columns than rows, hence its kernel is nontrivial.

## 4 Dimensions of solution spaces: The multivariate case

If  $d \geq 2$  in (2) then the dimensions of  $V_1$  and/or  $V_2$  can be infinite as shown by the following examples.

**Example 3** ( $\dim V_1 = \infty$ ,  $\dim V_2 = 1$ ) Let  $A$  be the system

$$\begin{aligned} (n_1 - 4n_2 + 1)T(n_1 + 1, n_2) &= (n_1 - 4n_2)T(n_1, n_2), \\ (n_1 - 4n_2 - 4)T(n_1, n_2 + 1) &= (n_1 - 4n_2)T(n_1, n_2). \end{aligned}$$

It is easy to check that

$$T_i(n_1, n_2) = \delta_{n_1, 4i} \delta_{n_2, i}, \quad \text{for } i \in \mathbb{Z},$$

are linearly independent solutions of  $A$  on all of  $\mathbb{Z}^2$ , hence  $\dim V_1 = \infty$ . On the other hand,  $S(A) = \{(n_1, n_2); n_1 = 4n_2\}$ , so  $A$  induces a single component on  $\mathbb{Z}^2$ , and  $\dim V_2 = 1$ .

**Example 4** ( $\dim V_1 = 1$ ,  $\dim V_2 = \infty$ ) Let  $B$  be the system

$$\begin{aligned} (n_1 - 4n_2)T(n_1 + 1, n_2) &= (n_1 - 4n_2 + 1)T(n_1, n_2), \\ (n_1 - 4n_2)T(n_1, n_2 + 1) &= (n_1 - 4n_2 - 4)T(n_1, n_2). \end{aligned}$$

It can be shown that any solution of  $B$  defined on all  $\mathbb{Z}^2$  is a constant multiple of  $n_1 - 4n_2$ , so  $\dim V_1 = 1$ . On the other hand,  $S(B) = \{(n_1, n_2); n_1 - 4n_2 \in \{-4, -1, 1, 4\}\}$ , so each of the points  $(4i, i)$  for  $i \in \mathbb{Z}$  is a separate component of  $\mathbb{Z}^2$  induced by  $B$ , hence  $\dim V_2 = \infty$ .

**Example 5** ( $\dim V_1 = \dim V_2 = \infty$ ) Let  $C$  be the system

$$\begin{aligned}(n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1 + 1, n_2) &= (n_1 - n_2)(n_1 - n_2 + 2)T(n_1, n_2), \\ (n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1, n_2 + 1) &= (n_1 - n_2)(n_1 - n_2 - 2)T(n_1, n_2).\end{aligned}$$

It is easy to check that

$$T_i(n_1, n_2) = \delta_{n_1, i} \delta_{n_2, i}, \quad \text{for } i \in \mathbb{Z}, \quad (8)$$

are linearly independent solutions of  $C$  on all of  $\mathbb{Z}^2$ , hence  $\dim V_1 = \infty$ . As  $S(C) = \{(n_1, n_2); n_1 - n_2 \in \{-2, 0, 2\}\}$ , each of the points  $(i, i - 1)$  and  $(i, i + 1)$  for  $i \in \mathbb{Z}$  is a separate component of  $\mathbb{Z}^2$  induced by  $C$ , so  $\dim V_2 = \infty$  as well.

The following theorem describes the general situation.

**Theorem 5** Let  $1 \leq s, t \leq \infty$ . Then there exists an  $H$ -system such that  $\dim V_1 = s$  and  $\dim V_2 = t$ .

**Proof:** Let  $t \geq 2$  and  $p_t(n_1, n_2) = \prod_{i=0}^{t-2} (n_1 - n_2 + 3i)$ . Then the set of singularities of

$$\begin{aligned}p_t(n_1 + 1, n_2)T(n_1 + 1, n_2) &= p_t(n_1, n_2)T(n_1, n_2), \\ p_t(n_1, n_2 + 1)T(n_1, n_2 + 1) &= p_t(n_1, n_2)T(n_1, n_2)\end{aligned}$$

is  $S = \{(n_1, n_2); n_1 - n_2 \in \{-3i; 0 \leq i \leq t - 2\}\}$ . As in Example 5, the functions (8) are linearly independent solutions of this system on all of  $\mathbb{Z}^2$ , hence  $\dim V_1 = \infty$ . On the other hand, the number of components induced on  $\mathbb{Z}^2$  is  $t$ , so  $\dim V_2 = t$ .

Let  $s \geq 2$  and

$$q_s(n_1, n_2) = \prod_{i=1}^{s-1} ((n_1 - 2i)^2 + n_2^2). \quad (9)$$

Then the set of singularities of

$$\begin{aligned}(n_1 - 4n_2)q_{s+1}(n_1 + 1, n_2)T(n_1 + 1, n_2) &= (n_1 - 4n_2 + 1)q_{s+1}(n_1, n_2)T(n_1, n_2), \\ (n_1 - 4n_2)q_{s+1}(n_1, n_2 + 1)T(n_1, n_2 + 1) &= (n_1 - 4n_2 - 4)q_{s+1}(n_1, n_2)T(n_1, n_2)\end{aligned}$$

is  $S = \{(n_1, n_2); n_1 - 4n_2 \in \{-4, -1, 1, 4\}\} \cup \{(2i, 0); 1 \leq i \leq s\}$ . Each of the points  $(4i, i)$  for  $i \in \mathbb{Z}$  is a separate component, so  $\dim V_2 = \infty$ . It can be shown that any solution  $T(n_1, n_2)$  defined on all  $\mathbb{Z}^2$  vanishes everywhere except at the points  $(2i, 0)$  where  $1 \leq i \leq s$ , and that

$$T_i(n_1, n_2) = \delta_{n_1, 2i} \delta_{n_2, 0}, \quad (10)$$

for  $i = 1, 2, \dots, s$ , are linearly independent solutions of this system defined on all  $\mathbb{Z}^2$ . Hence  $\dim V_1 = \infty$ .

Together with Examples 3 – 5 this proves the assertion in the case when at least one of  $s, t$  is infinite.

Now assume that  $s, t$  are natural numbers, and let  $r_t(n_1, n_2) = \prod_{i=1}^{t-1} (n_1 + 2i + 1)$ . Consider the system

$$\begin{aligned}q_s(n_1 + 1, n_2)T(n_1 + 1, n_2) &= q_s(n_1, n_2)r_t(n_1, n_2)T(n_1, n_2), \\ q_s(n_1, n_2 + 1)T(n_1, n_2 + 1) &= q_s(n_1, n_2)T(n_1, n_2),\end{aligned}$$

where  $q_s$  is as in (9). It can be shown that any solution  $T(n_1, n_2)$  defined on all  $\mathbb{Z}^2$  vanishes for all  $(n_1, n_2)$  such that  $n_1 > -(2t - 1)$  and  $(n_1, n_2)$  is not of the form  $(2i, 0)$  with  $1 \leq i \leq s - 1$ . Further, a basis of  $V_1$  is given by the  $s$  functions  $T_i(n_1, n_2)$  for  $i = 0, 1, \dots, s - 1$  where

$$T_0(n_1, n_2) = \begin{cases} \frac{(-1)^{(t-1)n_1}}{\prod_{i=1}^{s-1} ((n_1 - 2i)^2 + n_2^2) \prod_{i=1}^{t-1} (-n_1 - 2i - 1)!}, & n_1 \leq -(2t - 1), \\ 0, & \text{otherwise,} \end{cases}$$

and  $T_i(n_1, n_2)$  are as in (10) for  $i = 1, 2, \dots, s - 1$ . It follows that  $\dim V_1 = s$ . The set of singularities of this system is  $S = \{(2i, 0); 1 \leq i \leq s - 1\} \cup \{(-2i + 1, j); 1 \leq i \leq t - 1, j \in \mathbb{Z}\}$ , and the number of components induced on  $\mathbb{Z}^2$  is  $t$ , so  $\dim V_2 = t$  as desired.  $\square$

We considered the case  $d = 2$  here. The corresponding  $H$ -systems for the case of an arbitrary  $d > 1$  can be obtained by adding equations  $E_{n_i}T = T$ ,  $i = 3, \dots, d$ , to the systems with  $d = 2$ .

## 5 The Ore-Sato theorem and its consequences

The well-known Ore-Sato theorem (see [5], [6], [8]) is commonly believed to imply that any solution of an  $H$ -system (2) is of the form (1). We show that this is not so, and give an appropriate corollary of the Ore-Sato theorem on possible forms of solutions of  $H$ -systems in our setting.

**Definition 8** *Let  $T$  be a solution of (2). We write  $\text{supp } T$  for the support of  $T$ , i.e., for the set of points in  $\mathbb{Z}^d$  where  $T$  is defined and does not vanish.*

If (2) has a solution with non-algebraic support, then the rational functions  $f_i/g_i$ ,  $i = 1, \dots, d$ , are consistent, and uniquely determined by this solution (see [2]).

**Definition 9** *A polynomial  $p \in \mathbb{C}[x_1, \dots, x_d]$  is integer-linear if  $p(x_1, \dots, x_d) = a_0 + a_1x_1 + \dots + a_dx_d$  where  $a_1, \dots, a_d \in \mathbb{Z}$ .*

The Ore-Sato theorem states (in the case  $d = 2$ ) that for any consistent rational functions  $F_1(x, y)$  and  $F_2(x, y)$  there are consistent rational functions  $G_1(x, y)$  and  $G_2(x, y)$  which factor into integer-linear factors, and a rational function  $R(x, y)$  such that  $F_1(x, y) = G_1(x, y)R(x+1, y)/R(x, y)$  and  $F_2(x, y) = G_2(x, y)R(x, y+1)/R(x, y)$ . The full statement gives a precise description of the integer-linear factors.

In the literature one often encounters the claim that as a corollary of this theorem, any solution of an  $H$ -system (2) is of the form (1). For example, in [3, p. 223] one can read: “From Ore’s result it can be deduced that the most general form of  $A_{mn}$  is of the form

$$A_{mn} = R(m, n)\gamma_{mn}a^mb^n$$

where  $R$  is a fixed rational function of  $m$  and  $n$ ,  $a$  and  $b$  are constants, and  $\gamma_{mn}$  is a gamma product (...) that is to say it is of the form

$$\gamma_{mn} = \prod_i \Gamma(a_i + u_im + v_in)/\Gamma(a_i)$$

where the  $a_i$  are arbitrary (real or complex) constants, and the  $u_i$  and  $v_i$  are arbitrary integers which may be positive, negative, or zero.” A similar quote can be found in [4, p. 5].

It may be the case that in the literature referred to above the term  $A_{mn}$  is implicitly assumed to be nonzero for all  $m, n \in \mathbb{Z}$ . This possibility is supported by the fact that, e.g., in [3] the corresponding  $H$ -system is given in terms of the two quotients  $A_{m+1, n}/A_{mn}$  and  $A_{m, n+1}/A_{mn}$ . But such a severe restriction would preclude many important functions from being hypergeometric, such as the binomial coefficient  $T(n_1, n_2) = \binom{n_1}{n_2}$ , and all polynomials with integer roots.

However if we do not adopt this restriction, then there are hypergeometric terms which cannot be written in the form (1), as illustrated by the following examples.

**Example 6** *Take the  $H$ -system*

$$\begin{aligned} p(n_1, n_2)T(n_1 + 1, n_2) &= p(n_1 + 1, n_2)T(n_1, n_2), \\ p(n_1, n_2)T(n_1, n_2 + 1) &= p(n_1, n_2 + 1)T(n_1, n_2), \end{aligned} \tag{11}$$

where  $p(n_1, n_2) = (n_1 - n_2 - 1)(n_1 - n_2 + 1)$ . It can be checked that any sequence  $T$  which satisfies  $T(n_1, n_2) = 0$  unless  $n_1 = n_2$  is a solution of (11). In particular, the sequence

$$T(n_1, n_2) = \begin{cases} 2^{n_1^2}, & n_1 = n_2, \\ 0, & \text{otherwise,} \end{cases}$$

is a solution of (11), even though it does not have the form (1) because it grows too fast along the diagonal.

There are examples which look less artificial and where the solution has a non-algebraic support.

**Example 7** Let  $T(n_1, n_2) = |n_1 - n_2|$ . Then

$$\begin{aligned} (n_1 - n_2)T(n_1 + 1, n_2) &= (n_1 - n_2 + 1)T(n_1, n_2), \\ (n_1 - n_2)T(n_1, n_2 + 1) &= (n_1 - n_2 - 1)T(n_1, n_2) \end{aligned} \quad (12)$$

for all  $n_1, n_2 \in \mathbb{Z}$ , so  $T(n_1, n_2)$  is a hypergeometric term. It is also holonomic as its generating function is rational

$$\sum_{n_1, n_2 \geq 0} |n_1 - n_2| z_1^{n_1} z_2^{n_2} = \left( \frac{z_1}{(1 - z_1)^2} + \frac{z_2}{(1 - z_2)^2} \right) \frac{1}{1 - z_1 z_2}. \quad (13)$$

We claim that  $|n_1 - n_2|$  is not equal to any term of the form (1), not even modulo an algebraic set. To prove this, assume on the contrary that  $|n_1 - n_2| =_a T'(n_1, n_2)$  where  $T'(n_1, n_2)$  is of the form (1). Then there is a nonzero polynomial  $P \in \mathbb{C}[x, y]$  such that  $|n_1 - n_2| P(n_1, n_2) = T'(n_1, n_2) P(n_1, n_2)$  for all  $n_1, n_2 \in \mathbb{Z}$ . Write

$$T'(n_1, n_2) = R(n_1, n_2) u^{n_1} v^{n_2} \frac{\prod_{i=1}^p \Gamma(a_i n_1 + b_i n_2 + \alpha_i)}{\prod_{j=1}^q \Gamma(c_j n_1 + d_j n_2 + \beta_j)}$$

where  $R \in \mathbb{C}(x, y)$ ,  $u, v, \alpha_i, \beta_j \in \mathbb{C}$ , and  $a_i, b_i, c_j, d_j \in \mathbb{Z}$ .

Pick  $k_1, k_2 \in \mathbb{Z}$  such that  $k_1 < k_2$  and  $P(k_1, k_2) \neq 0$ . Such  $k_1, k_2$  certainly exist, for otherwise the univariate polynomials  $p_{k_1}(n_2) = P(k_1, n_2)$  would be identically zero for each  $k_1$ , as they would vanish for all  $n_2 > k_1$ , and hence  $P$  itself would be the zero polynomial. Let  $t(n_1) = T'(n_1, k_2) P(n_1, k_2) = |n_1 - k_2| P(n_1, k_2)$ . It can be verified that for  $n \in \mathbb{Z}$ ,  $a \in \mathbb{Z} \setminus \{0\}$  and  $z \in \mathbb{C}$  such that  $an + z$  is not a negative integer,

$$\Gamma(an + z) = \begin{cases} C a^{an} \prod_{i=0}^{a-1} \Gamma(n + (z + i)/a), & a \in \mathbb{Z}, a > 0, \\ C a^{an} / \prod_{i=1}^{|a|} \Gamma(n + (z - i)/a), & a \in \mathbb{Z}, a < 0, \end{cases}$$

where  $C \in \mathbb{C}$  is independent of  $n$ . Therefore  $t$  can be written in the form

$$t(n_1) = r(n_1) w^{n_1} \frac{\prod_{i=1}^{p'} \Gamma(n_1 + \gamma_i)}{\prod_{j=1}^{q'} \Gamma(n_1 + \delta_j)}, \quad \text{for all } n_1 \in \mathbb{Z}, \quad (14)$$

where  $r \in \mathbb{C}(x)$  and  $w, \gamma_i, \delta_j \in \mathbb{C}$ . If  $\gamma_i - \delta_j \in \mathbb{Z}$  then  $\Gamma(n_1 + \gamma_i)/\Gamma(n_1 + \delta_j)$  is a rational function of  $n_1$ , hence we can rewrite (14) as

$$t(n_1) = s(n_1) w^{n_1} t'(n_1), \quad \text{for all } n_1 \in \mathbb{Z},$$

where  $s \in \mathbb{C}(x)$  and  $f'(n_1) := t'(n_1 + 1)/t'(n_1)$  is a shift-reduced rational function. Let

$$\begin{aligned} f(n_1) &=_a \frac{t(n_1 + 1)}{t(n_1)} =_a \frac{|n_1 + 1 - k_2|}{|n_1 - k_2|} \frac{P(n_1 + 1, k_2)}{P(n_1, k_2)} \\ &=_a \frac{n_1 + 1 - k_2}{n_1 - k_2} \frac{P(n_1 + 1, k_2)}{P(n_1, k_2)}. \end{aligned}$$

Then both  $(w f'(n_1), r(n_1))$  and  $(1, (n_1 - k_2) P(n_1, k_2))$  belong to  $\text{RNF}(f)$ . It follows from Theorem 2 that  $w f'(n_1) = 1$ , hence  $t'(n_1) = c/w^{n_1}$  for all  $n_1 \in \mathbb{Z}$ , where  $c \in \mathbb{C} \setminus \{0\}$  is a constant, so  $t(n_1) = c s(n_1)$  for all  $n_1 \in \mathbb{Z}$ . But  $t(n_1) = |n_1 - k_2| P(n_1, k_2) = (n_1 - k_2) P(n_1, k_2)$  for all  $n_1 \geq k_2$ , therefore the two rational functions  $c s(n_1)$  and  $(n_1 - k_2) P(n_1, k_2)$  agree infinitely often and so must be identical. Thus  $t(n_1) = (n_1 - k_2) P(n_1, k_2) = |n_1 - k_2| P(n_1, k_2)$  for all  $n_1 \in \mathbb{Z}$ . In particular,  $|k_1 - k_2| P(k_1, k_2) = (k_1 - k_2) P(k_1, k_2)$ . As  $P(k_1, k_2) \neq 0$ , it follows that  $|k_1 - k_2| = k_1 - k_2$ , contrary to our choice of  $k_1 < k_2$ . This contradiction proves our claim.

In the theory of multivariate hypergeometric series  $H$ -systems are used to specify coefficients for such series. The simple rational function in the right-hand side of (13) has series expansion whose coefficients satisfy the  $H$ -system (12), however are not of the form (1).

The following statement is a corollary of the Ore-Sato theorem.

**Corollary 1** Any constituent (see Definition 5) of a solution with non-algebraic support of an  $H$ -system (2) is of the form (1).

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