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GOSPER'S ALGORITHM AND ACCURATE SUMMATION AS DEFINITE SUMMATION TOOLS

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Gosper's Algorithm and Accurate Summation as Definite Summation Tools *

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Abstract

Sufficient conditions are given for validity of the discrete Newton-Leibniz formula when the indefinite sum is obtained either by Gosper's algorithm or by Accurate Summation algorithm. It is shown that sometimes a polynomial can be factored from the summand in such a way that the safe summation range is inreased.

1 Introduction

Let K be a field of characteristic zero. A discrete function $t : I \mapsto \mathbb{Z}$ defined on an infinite interval of integers $I \subseteq \mathbb{Z}$ is a

- *hypergeometric term* if there are nonzero polynomials $a_0(n), a_1(n) \in K[n]$ such that $a_1(n)t(n+1) + a_0(n)t(n) = 0$ for all n such that $n, n+1 \in I$;
- *P-recursive sequence* if there are polynomials $a_0(n), \ldots, a_\rho(n) \in K[n]$ such that $a_0(n), a_\rho(n)$ are nonzero and $a_\rho(n)t(n+\rho) + \cdots + a_0(n)t(n) = 0$ for all n such that $n, \ldots, n+\rho \in I$.

Each hypergeometric term is, of course, a P-recursive sequence.

If $t(n)$ is a hypergeometric term, one can use the well-known Gosper's algorithm [4] to find another hypergeometric term $u(n)$ which satisfies

$$u(n+1) - u(n) \;=\; t(n) \tag{1}$$

for all $n \in I \setminus S$ where S is a finite set. Summing this equation on n from v to w we get the discrete analog of the Newton-Leibniz formula

$$\sum_{n=v}^{w} t(n) \;=\; u(w+1) - u(v) \tag{2}$$

provided that $[v, w] \cap S = \emptyset$.

In many existing implementations of Gosper's algorithm, however, indiscriminate use of equation (1) sometimes results in wrong answers. Here is a case in point.

Example 1 *Consider the sequence*

$$t(n) = \frac{\binom{2n-3}{n}}{4^n}, \tag{3}$$

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which is defined for all $n \in \mathbb{Z}$. This is a hypergeometric term which satisfies

$$2(n+1)(n-2)t(n+1) = (2n-1)(n-1)t(n)$$

for all $n \in \mathbb{Z}$. Gosper's algorithm succeeds with input $t(n)$ and returns

$$u(n) = \frac{2n(n+1)\binom{2n-3}{n}}{(n-2)4^n}.$$

Summing equation (1) on n from 0 to m the left-hand side telescopes, and we obtain

$$\sum_{n=0}^{m} t(n) \ =(?) \ \ u(m+1) - u(0) \ = \ \frac{(m+1)(m+2)\binom{2m-1}{m+1}}{2(m-1)4^m}. \tag{4}$$

But the expression on the right gives the true value of the sum only at $m = 0$. At $m = 1$ it is undefined, while at each $m \geq 2$ its value is 3/8 less than the actual value of the sum. The problem here is that $u(n)$ is undefined at $n = 2$, hence equation (1) does not hold for $n \in S = \{1, 2\}$, and summing it over a range including 1 or 2 may give a wrong result.

This is not an isolated example: a similar phenomenon seems to occur with the sum

$$\sum_{n=0}^{m} \frac{\binom{2n-p}{n}}{4^n}$$

for each positive integer p.

If t is a P-recursive sequence, then one can use Accurate Summation algorithm [3] to solve equation (1) (we discuss this algorithm in Section 4). Problems similar to those arising in Example 1 are possible when one uses Accurate Summation algorithm for definite summation. Notice that one can apply Accurate Summation algorithm in the case $\rho = 1$ as an alternative to Gosper's algorithm; then the incorrect formula (4) will appear again.

This common error is the discrete analogon of a well-known error in definite integration committed by some of the early symbolic integrators: when attempting to evaluate $I = \int_a^b f(x)dx$ by computing first an antiderivative $F(x)$ such that $F'(x) = f(x)$, and then using the Newton-Leibniz formula $I = F(b) - F(a)$, we may obtain an incorrect answer unless $F(x)$ is continuous on $[a, b]$. For example, the actual value of

$$\int_{-1}^{1} \frac{x^2+1}{x^4+x^2-1}\, dx$$

is π, but using the antiderivative $\arctan(x - 1/x)$ in the Newton-Leibniz formula gives 0.

The obvious solution is to split the summation interval into several subintervals that do not contain the exceptional points from S. In this paper we analyze the exceptional set S that appears in Gosper's algorithm when summing hypergeometric terms, and more generally, in the Accurate Summation algorithm [3] when summing P-recursive sequences.

Section 3 provides sufficient conditions for the Newton-Lebniz formula (2) to hold when the indefinite sum $u(n)$ is obtained by Gosper's algorithm, and Section 4 does the same for Accurate Summation. As it turns out, the exceptional set S is contained in an interval determined by the minimal and maximal integer singularities of the leading and trailing coefficients of the operator annihilating the summand. A deeper analysis in Section 5 shows that sometimes a polynomial can be factored from the summand in such a way that the size of the bounding interval is decreased. Computation of such polynomial factors is described in Section 6 for Gosper's algorithm, and in Section 7 for Accurate Summation.

2 Terminology and notation

We define the rising factorial power $(\alpha)_n$ for all $\alpha \in K$ and $n \in \mathbb{Z}$ by

$$(\alpha)_n = \begin{cases} \prod_{k=0}^{n-1} (\alpha + k), & n \geq 0, \\ \prod_{k=1}^{-n} \frac{1}{\alpha - k}, & n < 0 \text{ and } \alpha \neq 1, 2, \ldots, -n, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

We use E to denote the shift operator w.r.t. n, so that $Et(n) = t(n+1)$. Composition of operators is denoted by $\circ$.

Definition 1 *For a linear difference operator*

$$L = a_\rho(n)E^\rho + \cdots + a_0(n) \tag{5}$$

where $a_\rho(n), \ldots, a_0(n) \in K[n]$, $a_\rho(n)a_0(n) \not\equiv 0$ *and* $gcd(a_0(n), \ldots, a_\rho(n)) = 1$, *we define the sets* S_{Ll} *of* leading, S_{Lt} *of* trailing, *and* S_L *of* all integer singularities *by*

$$\begin{aligned} S_{Ll} &= \{x \in \mathbb{Z};\ a_\rho(x - \rho) = 0\}, \\ S_{Lt} &= \{x \in \mathbb{Z};\ a_0(x) = 0\}, \\ S_L &= S_{Ll} \cup S_{Lt}. \end{aligned}$$

We call

- $m_{Ll} = \min(S_{Ll} \cup \{+\infty\})$ *the* minimal leading *singularity of* L,
- $M_{Ll} = \max(S_{Ll} \cup \{-\infty\})$ *the* maximal leading *singularity of* L,
- $m_{Lt} = \min(S_{Lt} \cup \{+\infty\})$ *the* minimal trailing *singularity of* L,
- $M_{Lt} = \max(S_{Lt} \cup \{-\infty\})$ *the* maximal trailing *singularity of* L,
- $m_L = \min(S_L \cup \{+\infty\})$ *the* minimal *singularity of* L,
- $M_L = \max(S_L \cup \{-\infty\})$ *the* maximal *singularity of* L.

Proposition 1 *([1]) Let* L *be as in (5) and* $b(n) \in K[n]$. *If a rational function* $y(n) \in K(n)$ *satisfies*

$$a_\rho(n)y(n+\rho) + \cdots + a_0(n)y(n) = b(n), \tag{6}$$

then $y(n)$ *has no integer poles outside the interval (possibly empty)* $[m_{Ll}, M_{Lt}]$.

As a consequence, rational-function solutions of equation (6) have no integer poles outside $[m_L, M_L]$ (obviously $m_L \leq m_{Ll}$ and $M_{Lt} \leq M_L$).

3 When can Gosper's algorithm be used to sum hypergeometric terms?

We denote Gosper's algorithm hereafter by $\mathcal{GA}$. Consider the case when (6) has the form

$$a_1(n)t(n+1) + a_0(n)t(n) = 0 \tag{7}$$

and set $L = a_1(n)E + a_0(n)$. Let a hypergeometric term $t(n)$ satisfy equation (7). Given $a_0(n)$, $a_1(n)$ as input, $\mathcal{GA}$ tries to construct $r(n) \in K(n)$ such that

$$a_0(n)r(n+1) + a_1(n)r(n) = -a_1(n) \tag{8}$$

(this can also be done by the algorithm from [1] or the algorithm from [2]). If such $r(n)$ is found, we can use it to compute definite sums of $t(n)$ in the following way:

Theorem 1 *Assume that*

- $L = a_1(n)E + a_0(n)$ *is an operator of type (5) with* $\rho = 1$,
- v, w *are integers such that* $v \le w$ *and* $[v, w-1] \cap [m_L, M_L] = \emptyset$,
- $t(n)$ *is a* K*-sequence which is defined for all* $n \in [v, w] \cap \mathbb{Z}$ *and satisfies (7) for all* $n \in [v, w-1] \cap \mathbb{Z}$,
- $r(n) \in K(n)$ *is a rational function which satisfies (8) as an equation in* $K(n)$,
- $u(n)$ *is a* K*-sequence such that* $u(n) = r(n)t(n)$ *whenever both* $r(n)$ *and* $t(n)$ *are defined.*

Then the "Newton-Leibniz" or "telescoping" formula

$$\sum_{k=v}^{w} t(k) \;=\; t(w) + u(w) - u(v) \tag{9}$$

is valid.

Proof: Denote $I = [v, w-1] \cap \mathbb{Z}$. By assumption, (7) is valid for all $n \in I$. Since $r(n)$ satisfies (8), Proposition 1 implies that $r(n)$ has no integer poles outside the interval $[\alpha, \beta]$ where

$$\begin{array}{rclcl} \alpha & = & \min(\{x \in \mathbb{Z};\ a_0(x-1) = 0\} \cup \{+\infty\}) & = & m_{Lt} + 1, \\ \beta & = & \max(\{x \in \mathbb{Z};\ a_1(x) = 0\} \cup \{-\infty\}) & = & M_{Ll} - 1. \end{array}$$

It follows that $r(n)$ is defined for all $n \notin [\alpha, \beta] \subseteq [m_L, M_L]$, $r(n+1)$ is defined for all $n \notin [\alpha - 1, \beta - 1] \subseteq [m_L, M_L]$, and (8) is valid for all $n \notin [\alpha - 1, \beta] \subseteq [m_L, M_L]$. But I is disjoint from $[m_L, M_L]$, so $r(n)$ and $r(n+1)$ are defined for all $n \in I$, (8) is valid for all $n \in I$, and $u(n)$, $u(n+1)$ are defined for all $n \in I$. Therefore, for all $n \in I$,

$$\begin{array}{rcl} a_1(n-1)a_0(n)u(n+1) & = & a_1(n-1)a_0(n)r(n+1)t(n+1) \\ & = & -a_1(n-1)a_1(n)(1 + r(n))t(n+1) \quad \text{(by (8))} \\ & = & a_1(n-1)a_0(n)(1 + r(n))t(n) \quad \text{(by (7))} \\ & = & a_1(n-1)a_0(n)u(n) + a_1(n-1)a_0(n)t(n), \end{array}$$

or equivalently,

$$a_1(n-1)a_0(n)(u(n+1) - u(n) - t(n)) \;=\; 0.$$

As $a_1(n-1)a_0(n) \neq 0$ for $n \notin [m_L, M_L]$, it follows that

$$t(n) \;=\; u(n+1) - u(n) \tag{10}$$

for all $n \in I$. Summing (10) over I yields (9). □

4 When can Accurate Summation algorithm be used to sum P-recursive sequences?

By Accurate Summation algorithm (hereafter denoted by $\mathcal{AS}$) we mean a specific version of the general Accurate Integration algorithm given in [3] for integration/summation of solutions of Ore equations. This version, which is adapted for sequences that satisfy equations of the form (6) with $b(n) = 0$, solves the following problem: Let the minimal annihilator L of the form (5) be known for a K-valued sequence $t(n)$. Determine whether there exists a sequence $u(n)$ that has the minimal annihilator $\tilde{L}$ of order ρ and such that

$$(E - 1)u = t. \tag{11}$$

It is shown in [3] that if such a u exists then it can be expressed as Rt where R is an operator of order $\rho - 1$ with rational-function coefficients. $\mathcal{AS}$ constructs R if it exists. ($\mathcal{GA}$ solves this problem when $\rho = 1$.) The notion of adjoint operator for the difference case is used:

Definition 2 *The* adjoint operator *of L in the form (5) is*

$$L^* = a_\rho(n-\rho)E^{-\rho} + \cdots + a_1(n-1)E^{-1} + a_0(n).$$

Note that $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$.

$\mathcal{AS}$ is as follows:

Algorithm $\mathcal{AS}$
Input: $L \in K[n, E]$.
Output: The summation operator $R \in K(n)[E]$, if it exists.
Compute rational-function solutions of $L^* y = 1$;
if there exists a rational-function solution **then**
let r be a rational-function solution;
$R :=$ LeftQuotient$(1 - rL,\ E - 1)$
(this means that $(E-1) \circ R = 1 - rL$);
else
the summation operator R does not exist.

We can find a rational-function solution $r(n)$ using, e.g., the algorithm from [1] or the algorithm from [2]. Let $\mathcal{AS}$ succeed with L. Then

- if L is the minimal annihilator for t, then the minimal annihilator for $u = Rt$ is $\tilde{L} = 1 - R \circ (E-1)$ ($\tilde{L}$ has the same order ρ as L);
- the sequence $u = Rt$ satisfies (11) for any sequence t such that $Lt = 0$ (not only for the t's whose minimal annihilator is L).

Proposition 2 *Rational-function solutions $y(n)$ of equation $L^* y = 1$ where L is of the form (5) have no integer poles outside the (possibly empty) interval $[m_{Lt}, M_{Ll} - \rho]$ and, as a consequence, outside of $[m_L, M_L - \rho]$.*

Proof: Rewrite $L^* y = 1$ in the equivalent form $L' y = 1$ where $L' = E^\rho \circ L^*$. By Proposition 1, no rational-function solution of equation $L' y = 1$, and hence of $L^* y = 1$, can have integer poles outside $[m_{L'l}, M_{L't}]$. But $S_{L'l} = S_{Lt}$ and $S_{L't} = S_{Ll} - \rho$, therefore

$$\begin{aligned} m_{L'l} &= \min(S_{Lt} \cup \{+\infty\}) = m_{Lt} \geq m_L, \\ M_{L't} &= \max((S_{Ll} - \rho) \cup \{-\infty\}) = M_{Ll} - \rho \leq M_L - \rho, \end{aligned}$$

which implies that $[m_{L'l}, M_{L't}] \subseteq [m_{Lt}, M_{Ll} - \rho] \subseteq [m_L, M_L - \rho]$. □

Proposition 3 *Let $P = \sum_{k=0}^{\rho} b_k(n) E^k$ and $R = \sum_{k=0}^{\rho-1} c_k(n) E^k$ be operators from $K(n)[E]$ such that $P - (E-1) \circ R \in K(n)$ (i.e., R is the left quotient of P by $E-1$). If the coefficients $b_k(n)$ of P have no integer poles outside an interval $[\alpha, \beta]$, then the coefficients $c_k(n)$ of R have no integer poles outside of $[\alpha+1, \beta+\rho]$.*

Proof: A direct check shows that

$$c_k(n) = \sum_{j=1}^{\rho-k} b_{k+j}(n-j).$$

By assumption, $c_k(n)$ has no integer poles outside $\bigcup_{j=1}^{\rho-k} [\alpha+j, \beta+j] = [\alpha+1, \beta+\rho-k]$. Hence the coefficients of R have no integer poles outside $\bigcup_{k=0}^{\rho-1} [\alpha+1, \beta+\rho-k] = [\alpha+1, \beta+\rho]$. □

Theorem 2 *Let $t(n)$ be a sequence such that $Lt = 0$ and $t(n)$ is defined for $n \notin [m_L, M_L]$, where L is of the form (5). If $\mathcal{AS}$ succeeds with input L and returns an operator $R \in K(n)[E]$, then the sequence $u = Rt$ is defined everywhere outside the interval*

$$[m_L - \rho + 1, M_L]. \tag{12}$$

If $v < w < m_L - \rho + 1$ or $M_L < v < w$ then the telescoping formula (9) is valid. If the rational-function solution $r(n)$ of $L^* y = 1$ computed by $\mathcal{AS}$ has no integer poles then (9) is valid if $v < w < m_{Ll} - \rho + 1$ or $M_{Lt} < v < w$.

Proof: Let $r(n) \in K(n)$ and $R \in K(n)[E]$ be such that $L^* r = 1$ and $(E - 1) \circ R = 1 - rL$. Write $R = \sum_{k=0}^{\rho-1} c_k(n) E^k$. By Proposition 2, $r(n)$ has no integer poles outside of $[m_L, M_L - \rho]$. Hence by Proposition 3, the coefficients $c_k(n)$ of R have no integer poles outside of $[m_L + 1, M_L]$. By assumption, $t(n)$ is defined for $n \notin [m_L, M_L]$, hence $t(n + k)$ is defined for $n \notin [m_L - k, M_L - k]$. It follows that $u(n) = \sum_{k=0}^{\rho-1} c_k(n) t(n + k)$ is defined everywhere outside of (12). □

5 Taking advantage of polynomial factors

One can suspect that the statements of Theorems 1, 2 could be made stronger: e.g., if the sequences under consideration are polynomial, then the integer singularities of the corresponding equations present no obstacle to validity of the telescoping formula (9). It is natural to expect that if $t(n)$ has a "polynomial factor" then the telescoping formula has a wider application than that described in Theorems 1, 2.

Theorem 3 *Assume that*

- *$L = a_1(n)E + a_0(n)$ and $\bar{L} = \bar{a}_1(n)E + \bar{a}_0(n)$ are operators of type (5) with $\rho = 1$ and $[m_{\bar{L}}, M_{\bar{L}}] \subseteq [m_L, M_L]$,*
- *v, w are integers such that $v \leq w$ and $[v, w - 1] \cap [m_{\bar{L}}, M_{\bar{L}}] = \emptyset$,*
- *$t(n)$, $\bar{t}(n)$ are K-sequences with infinitely many nonzero values defined for all $n \notin [m_{\bar{L}} + 1, M_{\bar{L}}]$,*
- *$t(n)$, $\bar{t}(n)$ satisfy $Lt(n) = \bar{L}\bar{t}(n) = 0$ and $t(n) = p(n)\bar{t}(n)$ for all $n \notin [m_{\bar{L}}, M_{\bar{L}}]$,*
- *$r(n) \in K(n)$ is a rational function which satisfies*

$$a_0(n) r(n + 1) + a_1(n) r(n) = -a_1(n) \tag{13}$$

  as an equation in $K(n)$,

- *$u(n)$ is a K-sequence such that $u(n) = r(n)p(n)\bar{t}(n)$ whenever both $r(n)$ and $\bar{t}(n)$ are defined.*

Then the telescoping formula (9) is valid.

Proof: We have

$$\bar{a}_1(n)\bar{t}(n + 1) + \bar{a}_0(n)\bar{t}(n) = 0, \tag{14}$$

$$a_1(n)p(n + 1)\bar{t}(n + 1) + a_0(n)p(n)\bar{t}(n) = 0 \tag{15}$$

for all $n \notin [m_{\bar{L}}, M_{\bar{L}}]$. Multiplying (14) by $a_1(n)p(n + 1)$, (15) by $\bar{a}_1(n)$ and subtracting, we obtain $\bar{a}_0(n)\bar{t}(n)a_1(n)p(n + 1) = a_0(n)p(n)\bar{t}(n)\bar{a}_1(n)$. Since $\bar{t}(n)$ has infinitely many nonzero values, this implies that $\bar{a}_0(n)a_1(n)p(n + 1) = a_0(n)p(n)\bar{a}_1(n)$ holds infinitely often, hence

$$a_0(n)p(n)\bar{a}_1(n) = \bar{a}_0(n)a_1(n)p(n + 1) \tag{16}$$

as an equation in $K[n]$. Multiplying (13) by $\bar{a}_1(n)p(n)$, (16) by $r(n + 1)$, subtracting and cancelling $a_1(n)$ we obtain

$$\bar{a}_0(n)p(n + 1)r(n + 1) + \bar{a}_1(n)p(n)r(n) = -\bar{a}_1(n)p(n). \tag{17}$$

It follows from Proposition 1 that the rational function $p(n)r(n)$ has no integer poles outside the interval $[m_{\bar{L}t}+1, M_{\bar{L}l}-1]$. Similarly as in Theorem 1 we now establish that $p(n)r(n)$, $p(n+1)r(n+1)$, $u(n)$, $u(n+1)$ are defined for all $n \notin [m_{\bar{L}}, M_{\bar{L}}]$, that (17) is valid for all $n \notin [m_{\bar{L}}, M_{\bar{L}}]$, and from (14) and (17) we deduce that

$$u(n+1) - u(n) \;=\; t(n)$$

for all $n \notin [m_{\bar{L}}, M_{\bar{L}}]$. The claim now follows by summing this equation on n from v to $w-1$. □

Theorem 4 *Let $t(n)$ and $\bar{t}(n)$ be sequences such that $Lt = \bar{L}\bar{t} = 0$ ($L, \bar{L}$ are both of the form (5)) and such that $[m_{\bar{L}}, M_{\bar{L}}] \subseteq [m_L, M_L]$. Let polynomials $p(n), q(n) \in K[n]$ be such that*

$$L \circ p = q\bar{L}. \tag{18}$$

If $\mathcal{AS}$ succeeds with input L and returns an operator R, then to sum the sequence $t(n) = p(n)\bar{t}(n)$ one can apply the telescoping formula (9) with $u = R(p\,\bar{t})$ and $v < w < m_{\bar{L}} - \rho + 1$ or $M_{\bar{L}} < v < w$ (if the rational-function solution $r(n)$ of $L^*y = 1$ computed by $\mathcal{AS}$ is such that $r(n)q(n)$ has no integer poles outside $[m_{\bar{L}l}, M_{\bar{L}t}]$ then (9) can be applied to $t(n) = p(n)\bar{t}(n)$ with $v < w < m_{\bar{L}l} - \rho + 1$ or $M_{\bar{L}t} < v < w$).

Proof: $L \circ p = q\bar{L}$ implies $pL^* = \bar{L}^* \circ q$. Let $y = r$ be the rational-function solution of $L^*y = 1$ computed by $\mathcal{AS}$. Then $\bar{L}^*(qr) = (\bar{L}^* \circ q)r = pL^*r = p$. By Proposition 2 the rational-function $r(n)q(n)$ has no integer poles outside $[m_{\bar{L}}, M_{\bar{L}} - \rho]$.

We have $p - rq\bar{L} = p - rL \circ p = (1 - rL) \circ p = (E-1) \circ R \circ p$. Hence the operator $R \circ p$ is the left quotient of $p - rq\bar{L}$ by $E - 1$. By Proposition 3, the coefficients of $R \circ p$ have no integer poles outside $[m_{\bar{L}} + 1, M_{\bar{L}}]$. By assumption, $\bar{t}(n+k)$ is defined outside $[m_{\bar{L}} - k, M_{\bar{L}} - k]$. Thus the sequence $R(p\,\bar{t}) = (R \circ p)\bar{t}$ is defined everywhere outside $[m_{\bar{L}} - \rho + 1, M_{\bar{L}}]$. □

Theorems 2 and 4 show that if a polynomial factor is known this can help to extend the applicability of (9). However the problem is how to compute a polynomial factor for a given operator L.

We will describe two approaches to this problem. The first one is applicable in Gosper's case ($\rho = 1$) and is based on a special form of representation of hypergeometric terms. The second one concerns the general case ($\rho \geq 1$) and is based on the notion of the "universal denominator" of rational-function solutions of a given linear difference equation with polynomial coefficients.

6 Computing polynomial factors: Gosper's case ($\rho = 1$)

Theorem 5 *Let q, r, v, w be nonnegative integers such that $v \leq w$. Let $p(n) \in K[n]$ be a polynomial and let z, $\alpha_1, \dots, \alpha_r$, $\beta_1, \dots, \beta_q \in K$ be constants such that $\beta_j \neq 0, -1, \dots, -w+1$ for $1 \leq j \leq q$ and $\alpha_i - \beta_j$ is not a nonnegative integer for $1 \leq i \leq r$, $1 \leq j \leq q$.*

Denote $a(n) = z\prod_{i=1}^{r}(n + \alpha_i)$, $b(n) = \prod_{j=1}^{q}(n + \beta_j)$, and

$$t(n) \;=\; p(n)z^n \frac{\prod_{i=1}^{r}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n} \qquad \text{for} \;\; v \leq n \leq w. \tag{19}$$

If a polynomial $y(n) \in K[n]$ satisfies $a(n)y(n+1) - b(n-1)y(n) \;=\; p(n)$ and if

$$u(n) \;=\; y(n)z^n \frac{\prod_{i=1}^{r}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_{n-1}} \qquad \text{for} \;\; v \leq n \leq w,$$

then

$$u(n+1) - u(n) \;=\; t(n) \qquad \text{for} \;\; v \leq n \leq w. \tag{20}$$

Proof: By assumption,

$$a(n)y(n+1) - b(n-1)y(n) \;=\; p(n) \tag{21}$$

for all integer n. Now let $v \le n \le w$. Multiplying (21) by $z^n \prod_{i=1}^r (\alpha_i)_n / \prod_{j=1}^q (\beta_j)_n$ yields

$$z^n \frac{\prod_{i=1}^r (\alpha_i)_n}{\prod_{j=1}^q (\beta_j)_n} a(n) y(n+1) - z^n \frac{\prod_{i=1}^r (\alpha_i)_n}{\prod_{j=1}^q (\beta_j)_n} b(n-1) y(n) \ = \ t(n).$$

As $a(n) \prod_{i=1}^r (\alpha_i)_n = z \prod_{i=1}^r (\alpha_i)_{n+1}$ and $b(n-1)/\prod_{j=1}^q (\beta_j)_n = 1/\prod_{j=1}^q (\beta_j)_{n-1}$, this is the same as (20). □

Note that $p(n)$ in Theorem 5 is a polynomial factor of $t(n)$ in the sense of Section 5.

Example 2 *The sequence (3) from Example 1 can be written as*

$$t(n) = \begin{cases} 2s(n), & 0 \le n \le 2, \\ s(n), & n \ge 2, \end{cases}$$

where

$$s(n) = (2-n) \frac{(-1/2)_n}{4(1)_n}$$

is of the form assumed by Theorem 5. For $m \ge 1$, one should first split summation range in two

$$\sum_{n=0}^{m} t(n) = \frac{3}{4} + \sum_{n=2}^{m} s(n),$$

then Gosper's algorithm and the discrete Newton-Leibniz formula can be safely used on the sum on the right.

7 Computing polynomial factors: The general case ($\rho \ge 1$)

Here we show how to construct p, q and $\bar{L}$ satisfying (18) and such that

(i) $m_{\bar{L}l} \ge m_L$, $M_{\bar{L}t} \le M_L$,

(ii) qr has no poles and, as a consequence, no integer poles for any $r \in K(n)$ with $L^* r \in K[n]$.

For any sequence $\bar{t}$, satisfying $\bar{L}\bar{t} = 0$ everywhere outside of $[m_{\bar{L}l}, M_{\bar{L}t}]$, the sequence $t = p\bar{t}$ satisfies $Lt = 0$, and if $M_{\bar{L}t} < v < w$ or $v < w < m_{\bar{L}l} - \rho + 1$ then formula (9) can be applied to t with $u = R(p\bar{t})$.

Recall that using the algorithm [2] for constructing a universal denominator we can, given $A \in K[n, E]$, construct $q \in K[n]$ such that if an equation $Ay = \psi$, $\psi \in K[n]$, has a solution $r \in K(n)$, then qr has no poles (in particular, no integer poles). Additionally, if $d \in K[n]$ is a polynomial of minimal degree such that

$$A \circ \frac{1}{q} = \frac{1}{d} B, \quad B \in K[n, E],$$

then

$$m_{Bl} \ge m_{Al}, \quad M_{Bt} \le M_{At}.$$

The algorithm (hereafter denoted by $\mathcal{UD}$) for computation auch universal denominator is as follows:

Algorithm $\mathcal{UD}$
Input: $L \in K[n, E]$ of the form (5).
Output: The universal denominator $q(n) \in K[n]$.
  $F(n) := a_\rho(n-\rho)$; $G(n) := a_0(n)$; $q(n) := 1$;
  $z(x) := \mathrm{Res}_n(F(n), G(n+x))$;
  Compute nonnegative integer roots of $z(x)$;
  if there exist nonnegative integer roots **then**
    let N be the maximal nonnegative integer root;
    for $i = N, N-1, \ldots, 0$ **do**
      $g(n) := \gcd(F(n), G(n+i))$;
      $F(n) := F(n)/g(n)$; $G(n) := G(n)/g(n-i)$;
      $q(n) := q(n)g(n)g(n-1)\cdots g(n-i)$.

Consider $A = E^\rho \circ L^*$. Then

$$E^\rho \circ L^* \circ \frac{1}{q} = \frac{1}{d} B$$

and

$$\frac{1}{q} L \circ E^{-\rho} = B^* \circ \frac{1}{d}.$$

Therefore $L \circ E^{-\rho} \circ d = qB^* \circ \frac{1}{d} \circ d = qB^*$. Multiplying by E^ρ on the right gives $L \circ E^{-\rho} \circ d \circ E^\rho = L \circ d(n-\rho) = qB^* \circ E^\rho$. We set $p(n) = d(n-\rho)$, $\bar{L} = B^* \circ E^\rho$ and get decomposition (18) with properties (i) and (ii).

Example 3 *Take*

$$L = 2(n+1)(n-2)E - (2n-1)(n-1).$$

By Theorem 2 we can apply formula (9) for $v > M_L = 3$. *Using the algorithm from [2], we compute* $q(n) = (n-1)(n-2)$ *and*

$$A \circ \frac{1}{q} = E \circ L^* \circ \frac{1}{(n-2)(n-1)} = \frac{1}{n-1}(-(2n+1)E + 2(n+1)).$$

We have

$$\begin{aligned} d(n) &= n-1, \\ B &= -(2n+1)E + 2(n+1), \\ \bar{L} &= B^* \circ E = 2(n+1)E - (2n-1), \\ p(n) &= n-2. \end{aligned} \tag{22}$$

Finally we get

$$L \circ (n-2) = (n-1)(n-2)\bar{L}$$

with $M_{\bar{L}} = 0$.

Using the algorithm from [1] or the algorithm from [2] we compute $r(n) = -(n+2)/((n-1)(n-2))$. *Therefore* $R = 2n(n+1)/(n-2)$ *and* $u(n) = 2n(n+1)\bar{t}(n)$. *If* $w > v > 0$ *we have*

$$\sum_{n=v}^{w-1} t(n) = 2w(w+1)\bar{t}(w) - 2v(v+1)\bar{t}(v)$$

for any sequence $t(n)$ *annihilated by* L *which has the form* $(n-2)\bar{t}(n)$ *with* $\bar{t}(n)$ *annihilated by (22).*

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