University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1 111 Ljubljana, Slovenia

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Katsuya Eda Umed H. Karimov
Dušan Repovš

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A NEW CONSTRUCTION OF NONCONTRACTIBLE SIMPLY CONNECTED CELL-LIKE CONTINUA

KATSUYA EDA, UMED H. KARIMOV, AND DUŠAN REPOVŠ

Abstract. Using the topologist sine curve we present a new functorial construction of cone-like spaces, starting in the category of all path-connected topological spaces with a base point and continuous maps, and ending in the subcategory of all simply connected spaces. If one starts by a noncontractible n-dimensional Peano continuum, then our construction yields a simply connected noncontractible $n+1$-dimensional cell-like Peano continuum. In particular, starting with the circle S^1, one gets a 2-dimensional simply connected noncontractible cell-like Peano continuum.

1. Introduction

It is well known that all cell-like polyhedra are contractible. Griffiths [4] constructed a 2-dimensional nonsimply connected cell-like Peano continuum: Let $\mathbb{H}_1$ be the 1-dimensional *Hawaiian earrings* with the base point θ at which $\mathbb{H}_1$ is not locally simply connected. Let $Y = C(\mathbb{H}_1)$ be the cone on the Hawaiian earrings. Then $\mathbb{H}_1$ can be considered as the base of the cone $C(\mathbb{H}_1)$ and θ as its base point. The Griffiths space is then defined as the bouquet of two copies of Y with respect to the point θ.

A generalization of the Griffiths example is similar – instead of the 1-dimensional one considers the 2-dimensional Hawaiian earrings, i.e. the subspace $\mathbb{H}_2$ of the 3-dimensional Euclidean space, $\mathbb{H}_2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 |\ (x_0 - 1/k)^2 + x_1^2 + x_2^2 = (1/k)^2,\ k \in \mathbb{N}\}$ [3]. It is easy to see that one gets a 3-dimensional noncontractible simply connected cell-like Peano continuum (cf. [1], Example (17.7))

The purpose of the present paper is to construct a functor $SC(-,-)$ from the category of all path connected spaces with a base point and

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continuous mappings, to the subcategory of all simply connected cell-like spaces with a base point. The following are our main results:

Theorem 1.1. *For every path connected space Z, the space $SC(Z)$ is simply-connected.*

Theorem 1.2. *For every noncontractible space Z, the space $SC(Z)$ is noncontractible.*

Let Z be a compact metrizable space. When Z is a Peano continuum, then $SC(Z, z_0)$ is also a Peano continuum, and when Z is an n-dimensional noncontractible space for $n \geq 1$, then the space $SC(Z, z_0)$ is $(n+1)$-dimensional and noncontractible. When Z is finite dimensional, $SC(Z, z_0)$ is cell-like. In particular, for $Z = S^1$ we get a 2-dimensional noncontractible simply connected cell-like Peano continuum $SC(Z, z_0)$.

2. Preliminaries

For any two points a and b in the plane $\mathbb{R}^2$ we denote by $[a, b]$ the linear segment connecting these points. In the case when $a, b \in \mathbb{R}^1 \subset \mathbb{R}^2$ we shall additionally assume that $a < b$. The interval (a, b) is a segment without end points. In a similar way, the half-open intervals $[a, b)$ and $(a, b]$ are defined. The unit segment $[0, 1] \subset \mathbb{R}^1$ will be denoted by $\mathbb{I}$. To avoid confusion between an open interval and an element of the square $\mathbb{I} \times \mathbb{I}$, we shall write $(a; b)$ for the latter, where $a, b \in \mathbb{I}$. Our construction is based on the piecewise linear topologist sine curve T in the plane. Let $A_n = (1/n; 0)$, $B_n = (1/n; 1)$, for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, $A = (0; 0)$, $B = (0; 1)$ be the points of the plane $\mathbb{R}^2$. Let $L_{2n-1} = [A_n, B_n]$ and $L_{2n} = [B_n, A_{n+1}]$. The space T is the subspace of $\mathbb{I}^2$ defined as the union of all segments L_n and $L = \{0\} \times \mathbb{I}$.

Let Z be any space with a base point z_0. Then the base set of $SC(Z, z_0)$ is the quotient set of $T \times Z \cup \mathbb{I}^2$ obtained by the identification of the points $(s, z_0) \in T \times Z$ with $s \in T \subset \mathbb{I}^2$ and by the identification of each set $\{s\} \times Z$ with a one-point set $\{s\}$ if $s \in L$. There is a natural projection $p : SC(Z, z_0) \to \mathbb{I}^2$. To p there corresponds a pair of functions p_1 and p_2 such that $p(z) = (p_1(z); p_2(z))$. For $z = (x; y) \in T$ with $x > 0$, the set $p^{-1}(z)$ is denoted by Z_z, which is homeomorphic to Z, and for $y \in \mathbb{I}$ the set $p_2^{-1}(\{y\})$ is denoted by M_y. Let $O_\varepsilon(z, Z) = p^{-1}(U_\varepsilon(z))$, where $U_\varepsilon(z)$ is the open ε-ball with the center at $z \in \mathbb{I} \times \mathbb{I}$ with respect to the standard metric.

The topology of $SC(Z, z_0)$ coincides with the quotient topology at each point outside L. A basic neighborhood of a point $z = (0; y) \in L$ is of the form $O_\varepsilon(z, Z)$. Therefore, $SC(Z, z_0)$ has the quotient topology when Z is compact.

Obviously, $SC(-,\ -)$ is a functor from the category of topological spaces with a base point to itself. The space $SC(Z, z_0)$ is path-connected, path-connected and locally connected, finite-dimensional or metrizable if Z is path connected, path-connected and locally connected, finite-dimensional or metrizable, respectively. In particular, $SC(Z, z_0)$ is a Peano continuum if Z is a Peano continuum.

A *path* in X is a continuous mapping of the segment $[a, b] \subset \mathbb{R}^1$ to X. We say that two paths are *homotopic* if they are defined on the same domain and are homotopic with respect to their ends. The *composition* of two paths $f : [a, b] \to X$ and $g : [b, c] \to X$ with $f(b) = g(b)$ is a path $h : [a, c] \to X$ which is defined as follows:

$$h(t) = \begin{cases} f(t) & \text{if } a \leq t \leq b \\ g(t) & \text{if } b \leq t \leq c. \end{cases}$$

Let $f : [a, b] \to X$ and $g : [c, d] \to X$ be paths. We write $f \cong g$ when $f(a + (b - a)t) = g(c + (d - c)t)$ for each $t \in \mathbb{I}$ and define $\overline{f}$ as $\overline{f}(t) = f(a + b - t)$ for $a \leq t \leq b$.

A *loop* with the *base point* x_0 in a space X is a path $f : [a, b] \to X$ for which $f(a) = f(b) = x_0$. The product of two loops is defined in the standard way. The constant mapping to $\{x_0\}$ is denoted by c_{x_0}.

Let $f : [a, b] \to X$ be a path, c any point in $[a, b]$ and α any loop with the base point at $f(c)$. The *modification* of the path along a loop $\alpha : \mathbb{I} \to X$ is the path $g : [a, b] \to X$ which is defined fora segment $[t_1, t_2] \subset [a, b],\ c \in [t_1, t_2]$ as follows:

$$g(s) = \begin{cases} f\left((s - a)(c - a)/(t_1 - a) + a\right) & \text{if } a \leq s \leq t_1 \\ \\ \alpha((s - t_1)/(t_2 - t_1)) & \text{if } t_1 \leq s \leq t_2 \\ \\ f\left((b - s)(b - c)/(b - t_2) + b\right) & \text{if } t_2 \leq s \leq b. \end{cases}$$

The definition of the modification of path depends on the segment $[t_1, t_2]$, however all such paths are homotopy equivalent. (For simplicity of the definition we suppose that the domain of a loop α is $\mathbb{I}$, but we shall use a variant of the modification for loops with arbitrary domains in the sequel.)

A homotopy connecting injective mapping with the constant one is called a *contraction*. We shall denote a mapping and its restrictions by the same symbol. Whenever possible we shall use the symbol $SC(Z)$ instead of $SC(Z, z_0)$.

In the remaining part of this section we show that the shape type of $SC(Z)$ is that of the one-point space. To see this let $\mathcal{U}$ be an open cover of $SC(Z)$. Then there exist $U_1, \cdots, U_n \in \mathcal{U}$ such that $\{0\} \times \mathbb{I} \subseteq \bigcup_{i=1}^{n} U_i$

and hence we have $\varepsilon > 0$ such that $p^{-1}([0,\varepsilon] \times \mathbb{I}) \subseteq \bigcup_{i=1}^n U_i$. On the other hand we have a strong deformation retraction from $SC(Z)$ to $p^{-1}([0,\varepsilon] \times \mathbb{I})$, since $p^{-1}([\varepsilon, 1] \times \mathbb{I} \cap T)$ is a strong deformation retract of $p^{-1}([\varepsilon, 1] \times \mathbb{I})$ and $p^{-1}([\varepsilon, 1] \times \mathbb{I} \cap T)$ is homeomorphic to $\mathbb{I} \times Z$. These imply the conclusion.

The preservations of compactness and finite-dimensionality by the functor SC are proved similarly and those of path-connectivity and local path-connectivity are proved easily. Hence $SC(Z)$ is cell-like when Z is a finite dimensional compact metric space [6, 7].

3. Proof of Theorem 1.1

Lemma 3.1. *Let A be a strong deformation retract of X and let $\alpha : [0,1] \to X$ be a path with the end points $\alpha(0)$ and $\alpha(1)$ in A. Then there exists a path $\alpha' : [0,1] \to A \subset X$ which is homotopic to α.*

Proof. The assertion of the lemma follows directly from the definition of the strong deformation retraction. □

Lemma 3.2. *Let X be any space and α any path in $X \times \mathbb{I}$ with the end points $\alpha(0) = (\alpha_1(0), \alpha_2(0)) \in X \times \{0\}$ and $\alpha(1) \in X \times \{1\}$. Then there exists a special path α' in $X \times \mathbb{I}$ homotopic to α and such that $\mathrm{Im}(\alpha') \subset \{\alpha_1(0)\} \times \mathbb{I} \cup X \times \{1\}$.*

Proof. Let $H : \mathbb{I} \times \mathbb{I} \to X \times \mathbb{I}$ be the homotopy which is defined by the following formula:

$$H(s,t) = \begin{cases} (\alpha_1(0), 2s) & \text{if } 0 \le s \le t/2 \\ (\alpha_1(\dfrac{2s-t}{2-t}),\ (1-t)\alpha_2(\dfrac{2s-t}{2-t}) + t) & \text{if } t/2 \le s \le 1. \end{cases}$$

Obviously, $H(s,0) = \alpha(s)$ and $\mathrm{Im}(H(-,1)) \subset \{\alpha_1(0)\} \times \mathbb{I} \cup X \times \{1\}$ so H is the desired homotopy connecting α and $\alpha' = H(-,1)$. □

The path $\alpha'|_{[0,\ 1/2]}$ is called the *linear part* and $\alpha'|_{[1/2,\ 1]}$ is called the *residual part* of the path α'.

In the following lemmata we use the symbols of B_n, Z_{B_n}, $U_\varepsilon(B_n)$, $O_\delta(B_n, Z)$ and M_0 which were defined in Section 2.

Lemma 3.3. *Let $f : \mathbb{I} \to SC(Z)$ be a path. Then for every $n \in \mathbb{N}$ and every $\varepsilon > 0$ there exist a path $f_{n,\varepsilon} : \mathbb{I} \to SC(Z)$ and a homotopy $H_{n,\varepsilon} : \mathbb{I}^2 \to SC(Z)$ such that:*

(1) $H_{n,\varepsilon}(s,0) = f(s)$, $\quad H_{n,\varepsilon}(s,1) = f_{n,\varepsilon}(s)$;
(2) $\mathrm{Im}(f_{n,\varepsilon}) \cap Z_{B_n} = \emptyset$; *and*
(3) $H_{n,\varepsilon}(s,t) = f(s)$ *if* $f(s) \notin O_\varepsilon(B_n, Z)$.

Proof. Let δ be a number such that $0 < \delta < \varepsilon$ and $U_\delta(B_n) \cap T = U_\delta(B_n) \cap (L_n \cup L_{n+1})$.

Since $f^{-1}(Z_{B_n})$ is a compact subset of $f^{-1}(O_\delta(B_n, Z))$, there exists a finite set of pairwise disjoint segments $\{[a_k, b_k] : k \in K_n\}$ which cover $f^{-1}(Z_{B_n})$ in $f^{-1}(O_\delta(B_n, Z))$ and whose end points lie outside $f^{-1}(Z_{B_n})$. Using the modifications of paths along loops we can assume without loss of generality that end points of all paths lie on T. For given $k \in K_n$ consider the path $f : [a_k, b_k] \to O_\delta(B_n, Z)$. Since $p^{-1}(U_\delta(B_n) \cap T)$ is a strong deformation retract of $O_\delta(B_n, Z)$, the path f is homotopic to the path $f_{n,k} : [a_k, b_k] \to p^{-1}(U_\delta(B_n) \cap T) \subset O_\delta(B_n, Z)$ due to Lemma 3.1.

The space $p^{-1}(U_\delta(B_n) \cap T)$ is naturally homeomorphic to the product of space Z and the interval. The product $Z \times ([f(a_k), B_n] \cup [B_n, f(b_k)])$ is a strong deformation retract of $p^{-1}(U_\delta(B_n) \cap T)$. Therefore the path $f : [a_k, b_k] \to O_\delta(B_n, Z)$ is homotopic to a path in $Z \times ([f(a_k), B_n] \cup [B_n, f(b_k)])$, again by Lemma 3.1. By Lemma 3.2, the path $f : [a_k, b_k] \to O_\delta(B_n, Z)$ is homotopic to a special path the linear part of which lies in $L_n \cup L_{n+1}$ and the residual part of which does not intersect Z_{B_n}. The linear part can be slightly deformed in $\mathbb{I} \times \mathbb{I}$ to $[f(a_k), f(b_k)]$, which does not contain the point B_n. Since the index k is arbitrary and the number of the segments $\{[a_k, b_k] : k \in K_n\}$ is finite we get the desired mapping $f_{n,\varepsilon}$. □

Next lemma is a direct consequence of Lemma 3.3:

Lemma 3.4. *A loop in $SC(Z)$ with the base point in M_0 is homotopic to a loop in $SC(Z) \setminus \bigcup_{n\in\mathbb{N}} Z_{B_n}$.*

Lemma 3.5. *M_0 is a strong deformation retract of $SC(Z) \setminus \bigcup_{n\in\mathbb{N}} Z_{B_n}$.*

Proof. The deformation $D : (SC(Z) \setminus \bigcup_{n\in\mathbb{N}} Z_{B_n}) \times \mathbb{I} \to SC(Z) \setminus \bigcup_{n\in\mathbb{N}} Z_{B_n}$ is defined by the piecewise linear mapping (linear on every triangle $A_nB_nA_{n+1}$ and $A_{n+1}B_{n+1}B_n$) which maps half-open intervals $[A_n,\ B_n)$ and $[A_{n+1},\ B_n)$ to the points A_n and A_{n+1}, respectively (see Figure 1).

Since the spaces Z_{B_n} have been deleted, D is well-defined and continuous. □

We get the following from Lemmata 3.4 and 3.5 :

Lemma 3.6. *Let f be a loop in $SC(Z)$ whose base point is in M_0. Then f is homotopic to a loop in M_0.*

Before we show the simple connectivity of $SC(Z)$, we exhibit a homotopy from the canonical winding to the constant, in case when Z is the circle in Figure 2. To generalize this simple procedure I, II, III, we need to describe it more precisely.

For a loop α in Z with base point z_0 and a point $u \in T$, let α_u be a loop in Z_u induced naturally from the homeomorphism between Z and Z_u, i.e. $\alpha_u(t) = (u, \alpha(t))$ and particularly base point of α_u is u.

We call $\beta : [a, b] \to SC(Z)$ as a *basic loop* at A_n, if there exists a loop α in Z with base point z_0 such that

(a) $\beta(a) = \beta(b) = A$, $\beta((2a+b)/3) = \beta((a+b)/2) = A_n$;
(b) $\beta|_{[a,(2a+b)/3]}$ and $\beta|_{[(a+b)/2,b]}$ are linear mappings;
(c) $\beta|_{[(2a+b)/3,(a+b)/2]} \cong \alpha_{A_n}$.

Lemma 3.7. *A basic loop $\beta : [0, 1] \to SC(Z)$ at A_n is homotopic to the constant mapping B in the subspace $p^{-1}([A, A_n] \times \mathbb{I})$.*

Proof. We modify β to γ_0 so that

(1) $\gamma_0(0) = \beta(0) = A$, $\gamma_0|_{[1/3,1]} = \beta|_{[1/3,1]}$;
(2) $\gamma_0|_{[1/(4k+1),1/(4k)]} \cong \overline{\alpha}_{A_{n+k}}$ and $\gamma_0|_{[1/(4k+3),1/(4k+2)]} \cong \alpha_{A_{n+k}}$ for $k \geq 1$;
(3) $\gamma_0|_{[1/4k,1/(4k-1)]}$ is a linear mapping and $\gamma_0|_{[1/(4k+2),1/(4k+1)]}$ is constant for $k \geq 1$.

It is easy to see that γ_0 is homotopic to β in $p^{-1}([0, 1/n] \times \{0\})$. This homotopy corresponds to the procedure I in Figure 2. Next we describe the homotopy corresponding to the procedure II in Figure 2 according to the above classification $(1) - (3)$. Let $E_{n,t}$ be a point $((t+n)/((n+1)n); t)$ on L_{2n} and $F_{n,t}$ be a point $(1/n; t)$ on L_{2n-1}. We define $H : \mathbb{I} \times \mathbb{I} \to p^{-1}([0, 1/n] \times \mathbb{I})$ so that $H(s, 0) = \gamma_0(s)$ and the following kold:

(1) Let $H(0, t) = (0; t)$, $H(s, t) = (2(1 - s)/n; t)$ for $s \in [1/2, 1]$ and $H(-, t)|_{[1/3,1/2]} \cong \alpha_{(1/n;t)}$.
(2) $H(-, t)|_{[1/(4k+1),1/4k]} \cong \overline{\alpha}_{E_{n+k-1,t}}$ and $H(-, t)|_{[1/(4k+3),1/(4k+2)]} \cong \alpha_{F_{n+k,t}}$.
(3) $H(1/(4k+2), t) = F_{n+k,t}$, $H(1/(4k+1), t) = H(1/(4k), t) = E_{n+k-1,t}$ and $H(-, t)|_{[1/4k,1/(4k-1)]}$ and $H(-, t)|_{[1/(4k+2),1/(4k+1)]}$ are linear mappings.

Then H is continuous and is a homotopy. Let $\gamma_1 = H(-, 1)$. Notice that $\gamma_1|_{[1/(4k+1),1/(4k-2)]} \cong \overline{\alpha}_{B_{n+k-1}} c_{B_{n+k-1}} \alpha_{B_{n+k-1}}$ and $\gamma_1|_{[1/(4k+2),1/(4k+1)]}$ is a linear mapping onto $[B_{n+k}, B_{n+k-1}]$ and $\gamma_1|_{[1/2,1]}$ is a linear mapping onto $[B_n, B]$.

Then it is easy to see that γ_1 is null-homotopic in $p^{-1}([0, 1/n] \times \{1\})$, which corresponds to the procedure III in Figure 2. □

We now proceed with the proof of Theorem 1.1: by Lemma 3.6, we start from a loop $f : [0, 1] \to M_0$ with base point A. Moreover, since A_m's are isolated points, are connected by intervals in $\mathbb{I} \times \{0\}$ and converge to A, we may assume having a disjoint family of open intervals (a_n, b_n) $(n < \nu)$, where $\nu \leq \omega$, such that each $f|_{[a_n,b_n]}$ is a basic loop at some A_m and $\bigcup_{n<\nu}(a_n, b_n)$ is dense in $\mathbb{I}$. Then $f(s) = A$ for $s \notin \bigcup_{n<\nu}(a_n, b_n)$.

We observe that the procedure II in the proof of Lemma 3.7 can be performed uniformly for $f|_{[a_n,b_n]}$'s. Then we obtain a homotopy from f to a loop in M_1 which consists of possibly infinitely many null-homotopic loops. Since the homotopies which correspond to the procedure III converge to B, we have a homotopy from f to the constant mapping B. By a standard method, i.e. extending the domain $[0, 1]$ into both directions and adding a path from A to B, we have a homotopy from f to the constant mapping A with respect to their ends. □

4. Proof of Theorem 1.2

We shall show that $SC(Z)$ is noncontractible for every noncontractible space Z. Let $SC_n(Z)$ be the subspace of $SC(Z)$ defined as $p^{-1}(L_{2n-1} \cup L_{2n} \cup L_{2n+1})$.

Definition 4.1. *A mapping $f : SC_n(Z) \to SC(Z)$ is said to be flat if $p_2(f(z_1)) = p_2(f(z_2))$, whenever $p_2(z_1) = p_2(z_2)$ for $z_1, z_2 \in SC_n(Z)$. A homotopy $H : SC_n(Z) \times \mathbb{I} \to SC(Z)$ is said to be flat if for every t, the mapping $H(-, t)$ is flat.*

Lemma 4.2. *Let $n \in \mathbb{N}$ and $H : SC_n(Z) \times \mathbb{I} \to SC(Z)$ be a mapping such that for every $y \in \mathbb{I}$ and $t \in \mathbb{I}$, the set $p_2(H(M_y, t))$ does not simultaneously contain both points 0 and 1 and that both $H(-, 0)$ and $H(-, 1)$ are flat mappings. Then there exists a flat homotopy from $H(-, 0)$ to $H(-, 1)$.*

Proof. Fix numbers y and t. Let $A(y, t)$ and $B(y, t)$ be the minimum and the maximum of the function $p_2(H(-, t)) : M_y \to \mathbb{I}$, respectively. Let

$$C(y, t) = \frac{A(y, t)}{1 + A(y, t) - B(y, t)}.$$

Consider the subset $\mathbb{I} \times [A(y, t), B(y, t)]$. Let φ be its piecewise linear retraction to the interval $\mathbb{I} \times \{C(y, t)\}$, which is defined by the mappings of vertices (see Figure 1): $\varphi(B_{n,0}) = C_{2n-1}, \varphi(B_{n1}) = C_{2n}, \varphi(A_{n+1\,0}) = C_{2n}, \varphi(A_{n,1}) = C_{2n-1}$. If $A(y, t) = 0$ or $B(y, t) = 1$, then $C(y, t) = 0$ or $C(y, t) = 1$ and $A_{n,0} = A_{n,1} = A_n$ or $B_{n,0} = B_{n,1} = B_n$, respectively and the mapping φ is correctly defined.

Let $\psi_{y,t}$ be the natural retraction of $p^{-1}(\mathbb{I} \times [A(y, t), B(y, t)])$ to $p^{-1}(\mathbb{I} \times \{C(y, t)\})$ generated by φ. Define now the homotopy $H' : SC_n(Z) \times \mathbb{I} \to SC(Z)$ by $H'(z, t) = \psi_{(p_2(z),t)}(H(z, t))$. It is easy to check that $p_2(H'(z, t)) = C(p_2(z), t)$ and H' is a flat homotopy which is a contraction. □

To prove the following Lemma 4.4 we introduce a notion to investigate flat homotopy.

For $s \in (0,1)$ and $t \in \mathbb{I}$, we define a property $P(s,t)$ of H as follows:

> $H(M_s \cap SC_n(Z), t) \subseteq p^{-1}(\mathbb{I} \times (0,1))$ and the restriction of $H(-,t)$ to $M_s \cap SC_n(Z)$ is homotopic to the identity mapping on $M_s \cap SC_n(Z)$ in $p^{-1}(\mathbb{I} \times (0,1))$.

We remark that by the flatness of H, if $H(M_s \cap SC_n(Z), t) \subseteq p^{-1}(\mathbb{I} \times (0,1))$, then there is a neighborhood U of $(s;t)$ such that $H(M_{s'} \cap SC_n(Z), t') \subseteq p^{-1}(\mathbb{I} \times (0,1))$ for any $(s';t') \in U$.

Lemma 4.3. *Let Z be a non-contractible space and $H : SC_n(Z) \times \mathbb{I} \to SC(Z)$ a flat homotopy. If $H(M_0 \cap SC_n(Z), t_0) < 1$, then there exists a neighborhood U of $(0, t_0)$ such that H does not satisfy $P(s,t)$ for any $(s;t) \in U$ with $s > 0$. A similar statement holds for $H(M_1 \cap SC_n(Z), t_0)$.*

Proof. We have a neighborhood U of $(0,t_0)$ such that $H(M_s \cap SC_n(Z), t) \subseteq p^{-1}(\mathbb{I} \times (0,1))$ for any $(s;t) \in U$. We fix $(s;t) \in U$ with $s > 0$. Let $P_{2n+1}, P_{2n}, P_{2n-1}$ be the intersection of $\mathbb{I} \times \{s\}$ and $L_{2n+1}, L_{2n}, L_{2n-1}$ respectively and $I_{2n} = [P_{2n+1}, P_{2n}]$ and $I_{2n-1} = [P_{2n}, P_{2n-1}]$. Then we have $M_s \cap SC_n(Z) = Z_{P_{2n+1}} \cup I_{2n} \cup Z_{P_{2n}} \cup I_{2n-1} \cup Z_{P_{2n-1}}$. Since $H(-,t)$ maps $\bigcup_{u \in [P_{2n+1}, A_{n+1}] \cup [A_{n+1}, P_{2n}]} Z_u$ into $p^{-1}(\mathbb{I} \times (0,1))$, the restriction of $H(-,t)$ to $Z_{P_{2n+1}} \cup I_{2n} \cup Z_{P_{2n}}$ is homotopic to a map $f : Z_{P_{2n+1}} \cup I_{2n} \cup Z_{P_{2n}} \to I_{2n} \cup Z_{P_{2n}}$ in $p^{-1}(\mathbb{I} \times (0,1))$.

Since M_s is a strong deformation retract of $p^{-1}(\mathbb{I} \times (0,1))$ similarly as in Lemma 3.5, $Z_{P_{2n+1}} \cup I_{2n} \cup Z_{P_{2n}}$ is a retract of $p^{-1}(\mathbb{I} \times (0,1))$. Since Z is not contractible, the identity mapping on $Z_{P_{2n+1}} \cup I_{2n} \cup Z_{P_{2n}}$ is not homotopic to any map $f : Z_{P_{2n+1}} \cup I_{2n} \cup Z_{P_{2n}} \to I_{2n} \cup Z_{P_{2n}}$ in $p^{-1}(\mathbb{I} \times (0,1))$, which implies the conclusion.

In case of $H(M_1, t_0)$ we use $Z_{P_{2n}} \cup I_{2n-1} \cup Z_{P_{2n-1}}$ and argue at a neighborhood of B_n and obtain a similar conclusion. □

Lemma 4.4. *Let Z be a noncontractible space. If $H : SC_n(Z) \times \mathbb{I} \to SC(Z)$ is a flat homotopy such that $H(u,0) = u$ for every $u \in SC(Z)$, then $H(-,1)$ is not a constant mapping.*

Proof. To show this by contradiction, suppose that $H(-,1)$ is a constant mapping. Consider the line $L_n \cup L_{n+1} \cup L_{n+2}$ and the circle with some base point $s_0 \in S^1$. Let $d : [0,1] \to S^1$ be a winding with base point s_0, i.e. both $d|_{[0,1)}$ and $d|_{(0,1]}$ are bijective continuous mappings with $d(0) = d(1) = s_0$.

We define a homotopy $H^* : S^1 \times \mathbb{I} \to S^1$ as follows:

$$H^*(u,t) = \begin{cases} d^{-1}(u), & \text{if } u \neq s_0 \text{ and } P(d^{-1}(u),t) \text{ holds;} \\ s_0, & \text{otherwise.} \end{cases}$$

We have a contradiction with the fact $H^*(s,0) = s$ and that S^1 is not contractible, if H^* is a homotopy (compare with [2]). Hence it suffices to verify the continuity of H^*.

If $u \neq s_0$ and $P(d^{-1}(u),t)$ holds, the continuity at (u,t) is clear. Otherwise, $u \neq s_0$ but $P(d^{-1}(u),t)$ does not hold, or $u = s_0$.

(Case 1) $u \neq s_0$: If $p_2 \circ H(M_{d^{-1}(u)},t) = 0$ or 1, then the continuity at (u,t) follows from that of H. Otherwise, since $H(-,t)$ maps $M_{d^{-1}(u)} \cap SC_n(Z)$ continuously with respect to u and t, the restriction of $H(-,t)$ to $M_{d^{-1}(u)} \cap SC_n(Z)$ is not homotopic to the identity on $M_{d^{-1}(u)} \cap SC_n(Z)$ in $p^{-1}(\mathbb{I} \times (0,1))$, i.e. H^* takes the value s_0 in a neighborhood of (u,t).

(Case 2) $u = s_0$: If both of $p_2 \circ H(M_0,t)$ and $p_2 \circ H(M_1,t)$ are equal to $\{0\}$ or $\{1\}$, the continuity at (u,t) follows from that of H. The remaining case is when $0 < p_2 \circ H(M_0,t) < 1$ or $0 < p_2 \circ H(M_1,t) < 1$. In this case the continuity follows from Lemma 4.3 and that of H. □

We now proceed with proof of Theorem 1.2: suppose that $SC(Y)$ were a contractible space. Then there would exist a contraction $H : SC(Z) \times \mathbb{I} \to SC(Z)$. By the compactness of $\{0\} \times \mathbb{I}$ and since every continuous mapping is uniformly continuous on a compactset, there would exist $\varepsilon > 0$ such that the diameter of the image of any $\varepsilon-$set (set with diameter less then ε) would be less than 1. Let n be a number such that $1/n < \varepsilon$. By Lemma 4.2 we may then assume that H is a flat contraction. However, this contradicts Lemma 4.4. □

The space $SC(S^1)$ is simply connected and it follows by the Mayer-Vietoris exact sequence for the singular homology that $H_n(SC(S^1)) = 0$ for $n \neq 2$. The following question remains open:

Question 4.5. *Is* $H_2(SC(S^1)) = 0$*?*

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School of Science and Engineering, Waseda University, Tokyo 169-8555, Japan
E-mail address: eda@logic.info.waseda.ac.jp

Institute of Mathematics, Academy of Sciences of Tajikistan, Ul. Ainy 299^A, Dushanbe 734063, Tajikistan
E-mail address: umed-karimov@mail.ru

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, P.O.Box 2964, Ljubljana 1001, Slovenia
E-mail address: dusan.repovs@guest.arnes.si