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SHAPE INJECTIVITY IS NOT IMPLIED BY BEING STRONGLY HOMOTOPICALLY HAUSDORFF

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Abstract. Given a topological space X and the natural homomorphism $\pi_1(X) \xrightarrow{\psi} \check{\pi}_1(X)$ where $\check{\pi}_1(X)$ is the first shape group, we say that X satisfies the shape injectivity provided that ψ is one–to–one. A space X is (strongly) homotopically Hausdorff, if in every (free) non-zerohomotopic homotopy class there exists a minimal diameter for every path representing this class. Both properties and hence the question to investigate their mutual relationship have appeared in the theory of generalized covering spaces. The implication from the title is ruled out by constructing two spaces (the second of which will be a Peano continuum) which has the second, but not the first property.

§ 0. Introduction

This paper is about constructing a space which has the property of shape injectivity but is not strongly homotopically Hausdorff. More precisely, two spaces are constructed, one of which is fairly easy to construct, and one, which requires more work, but whose existence shows that even for Peano continua such relationship does not hold. Both properties mentioned above play a role in the construction of generalized universal covering spaces. The idea to generalize the theory of covering spaces is old, and authors like Lubkin ([Lbk]), Fox in his theory of overlays ([Fox1], [Fox2]) and Mardešić and Matijević ([MaMa], [Matj]) have considered such questions. More recent work in this direction has been done by Zastrow ([Za95]), Bogley and Sieradski ([BoSi]), Conner and Lamoreaux ([CoLa]), Biss ([Biss]) and Fischer and Zastrow ([FiZa2]). The property "homotopically Hausdorff" was introduced by Cannon and Conner in ([CaCo; §1 before Lemma 5.6, cf. our Def.1.1(ii)]) and played also an essential role in [CoLa]. However, it appeared under a different name as a necessary criterion for the existence of covering spaces already in [Za95; 1.1(ii)]. [FiZa1; Thm.2] introduced the shape injectivity (cf. 1.1(i)) and proved that it is satisfied by all planar sets, and such sets are by [Za95; the last criterion of Thm.1.2] known to have generalized covering spaces. It was then proved in [FiZa2] that shape injectivity is a sufficient condition for the existence of such generalized covering spaces, and none of the previous papers could offer a purely algebraic

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criterion for the existence of generalized covering spaces. Hence shape injectivity implies the property "homotopically Hausdorff" (although this can also seen more directly). The converse implication is false, and the corresponding space which demonstrated this appeared already in [FiZa1; Expl.9, cf. here 1.7]. However, the question whether the property "strongly homotopically Hausdorff" is strong enough to imply the shape injectivity, at least for Peano continua, was addressed by Greg Connerto the second author during the Park City Conference 2003 where [FiZa1] was presented. This paper is devoted to answering it – as already stated in the title. This will be done according to the following structure:

(i) In the first section we shall construct our two spaces and prove those of their relevant properties which are easy to verify.

(ii) In the second section we shall then adapt a method from [Za20; §1] to associate infinite combinatorial objects with all homotopy classes of the Peano continuum which we constructed before; and

(iii) In the third section we shall finally prove the last missing property, that our Peano continuum is also strongly homotopically Hausdorff. This property will to some extent require understanding the variety of homotopy classes of paths in that space, and that is why the argument will have to be based on the results of the second section.

§ 1. Basic Constructions and Easy Proofs

Definition 1.1.

(i) Let X be a topological space. Then X satisfies the *shape-injectivity,* if the natural homomorphism from the fundamental group into the first shape-group, $\pi_1(X) \to \check{\pi}(X)$, is injective. See [FiZa1; end of Sect.2] for more information on how this homomorphism is defined.

(ii) A topological space X is said to be *"(weakly) homotopically Hausdorff",* if for all points $P \in X$ and for any non-trivial homotopy class $\alpha \in \pi_1(X, P)$ there exists a neighbourhood U of P such that α cannot be represented by any path inside U (cf. [CaCo; §1 before Lemma 5.6], [CoLa; Def.1.1] and [Za95; 1.1(ii)]).

(iii) A metric space is said to be *"strongly homotopically Hausdorff",* if for any non-trivial homotopy class $\alpha \in \pi_1(X)$ there exists $\varepsilon > 0$ such that no path with diameter $< \varepsilon$ can lie in the same free homotopy class as α.

We start by constructing a space V that will be the first example to disprove the implication (iii) ⇒ (i). However, this space will not be a Peano continuum.

Construction 1.2. *(The "First Example")* First take in the (x, z)-plane of the (x, y, z)-space the graph of the function $z = \sin(\frac{1}{|x| - \pi/2})$, $-\pi/2 < x < +\pi/2$. Then make it an open surface with circular waves *("inner waved surface")* by rotating it around the z-axis in (x, y, z)-space. The fact that it has a corner at $(x, y) = (0, 0)$ is not relevant for a topologist. The waves of this surface accumulate as $\|(x, y)\| \to \pi/2$. Therefore we need to metrically complete this space by an *"outer cylinder"* $\{(x, y, z) \mid \|(x, y)\| = \frac{\pi}{2}, -1 \le z \le +1\}$, where $\{(x, y, 0) \mid \|(x, y)\| = \frac{\pi}{2}\}$ will be called the *"center line (of the outer cylinder)".* In order to establish path-connectedness we should connect the outer cylinder with the mid-point of the inner surface by an arc (which is exempted from being rotated). This at least guarantees a unique fundamental group, but this arc will play no role in the forthcoming discussions. We can think of this arc as having the form of an *"arch"* and we call it accordingly. However, we finally let

$$V := \textit{“inner waved surface”} \cup \textit{“outer cylinder”} \cup \textit{“arch”}.$$

In order to reach the goal of this paper we improve our space V as constructed in Constr. 1.1 by turning it into a Peano continuum W.

Construction 1.3. *(The “Main Example”)* We will use the same inner waved surface as in 1.2, and the same outer cylinder. With these we let

$$W := \textit{“inner waved surface”} \cup \textit{“outer cylinder”} \cup \textit{system of line segments,}$$

which will be constructed in the following.

(i) Each line segment is horizontal, i.e. has constant z-coordinate.

(ii) Each line segment is radial, i.e. when using cylinder coordinates (ρ, ϕ, z) it has also constant ϕ-coordinate.

Thus is just remains to give the initial and the end-value of the ρ-coordinate, and to select the (ϕ, z)-pairs for which we want to have such line-segments. Determining those is the subject of the subsequent items:

(iii) The initial value for each line segment will be $\frac{\pi}{2}$. I.e. we will think of each line segment as starting at the outer cylinder and ending somewhere in the interior.

(iv) First we choose some finite rectangular grid in $[0, 2\pi] \times [-1, +1]$, and segments with these (ϕ, z)-coordinates will be called “of the first generation”. They will end where they intersect our waved surface between the innermost minimal circle (at $\rho = \frac{\pi}{2} - \frac{2}{\pi}$) and the innermost maximal circle (at $\rho = \frac{\pi}{2} - \frac{2}{3\pi}$).

(v) Then we refine the grid by halving the mesh. I.e. we add new points at the midpoints between two neighbouring grid-points, and also at the midpoints in the interior of each square of four adjacent grid-points. The segments that we will construct between these newly added points will be called “of the second generation”. They will end where they intersect the waved surface between the innermost maximal circle and the penultimate minimal circle.

(vi) In this way we continue our construction: In each new step we halve the mesh of our grid, add line-segments with a new generation index at the newly created grid-points and let them end at the intersection points of the waved surface, where the waved surface passes between a minimal circle and a maximal circle, before the passage where the segments of the prior generation ended.

(vii) This construction step is repeated infinitely many times, and in this way infinitely many segments are defined and included into our Main Example. Note that they are only becoming dense as we approach the outer cylinder.

A subsegment of each of these segments which is bounded by two adjacent intersections with the inner waved surface will be called a tunnel. We are associating *to each of these tunnels “generation indices”,* which are not automatically the same generation indices as for the segments. In principle the last (innermost) tunnel of a segment has the same generation index as the segment, and if we run on a segment towards the outer cylinder, then with each intersection of the inner waved surface the generation index increases by one. Note that this definition makes sure that all tunnels which are bounded by the same two passages of the inner waved surface between two adjacent minimal circles and the maximal circle in between (or vice versa) have the same generation index.

Lemma 1.4.

(i) *Neither the center line itself, nor any of its non-trivial multiples can be nullhomotopic, neither in V, nor in W.*

(ii) *In V no continous path can connect the outer cylinder with the inner waved surface without having to run through the arch.*

(iii) *In W no continuous path can connect the outer cylinder with the inner waved surface without having run through tunnels.*

Proof.
Ad (i): The axis of the cylinder as an infinite line in $\mathbb{R}^3$ has to be crossed by any nullhomotopy in $\mathbb{R}^3$, and therefore the only intersection point with the inner waved surface has to be taken on by any nullhomotopy in V or in W. The same argument holds for any parallel line, which still passes through the interior of the outside cylinder. By taking out lines that would also intersect tunnels or the arch, there remains a dense set of points of the inner waved surface, which can be proven to be included into the image by any contracting homotopy. However, since a homotopy map having all points of the inner waved surface in its image cannot continuously be extended to the closure of its domain of definition, such a homotopy does not exist.
Ad (ii) and (iii): Represent the inner waved surface in cylindrical coordinates (r, ϕ, z) and then project to the (r, z)–plane. The result is a topologists' sine–curve. Hence any path which avoids the arch or the tunnels and which connects the inner waved surface to the outer cylinder, would give a continuous image in this topologists' sine–curve which connects the accumulation line with the sine–curve. This gives the desired contradiction.

Observation 1.5. *V is a metric separable space which satisfies 1.1(iii) but does not satisfy 1.1(i).*

Proof. Metric separability is given, since our space lies in $\mathbb{R}^3$. It does not satisfy the shape injectivity, since by 1.4(i) the center line of the outer cylinder is not nullhomotopic. However, every polyhedral neighbourhood of the space engulfs almost all waves of the inner waved surface, so that this curve is nullhomotopic in each polyhedral neighbourhood, and hence in the inverse limit of the fundamental groups of these neighbourhoods, which gives by standard shape theory the shape-group.

Now, the fact that the space is homotopically Hausdorff can be seen as follows: By 1.4(ii) a sufficiently general picture of an arbitrary path is one who alters several times between the outer cylinder and the inner waved surface by passing through the arch. Since the inner waved surface is a homeomorphic image of an open disk, those parts of our path can be contracted, and that way the whole path can be homotoped via the arch to a path which only takes on points on the outer cylinder. Hence paths in that cylinder suffice to represent all homotopy classes. Homotopically each path is a multiple of the center line, where the zeroth multiple is nullhomotopic, and the other multiples are by 1.4(i) non-nullhomotopic, not even in V. Thus the minimal diameter of any of those paths is the diameter of the outer cylinder, since any path with a smaller diameter cannot encircle the entire cylinder and would therefore automatically be nullhomotopic. These observations guarantee the strong homotopical Hausdorff property.

We are now able to state our main result:

Theorem 1.6. *W is a Peano Continuum which satisfies 1.1(iii) but does not satisfy 1.1(i).*

Outline of proof. It follows immediately from the construction that the space is locally pathwise connected at each of its points: The violation of the local pathwise connectedness of V at the points of the outside cylinder has been undone by inserting the tunnels (similarly as it has been performed in [KaRe; §6, after Conj.6.3]), and, since the tunnels are only inserted densely at the outside cylinder, also the points that W has in addition to the points of V do not violate the local pathwise connectivity. Metric separability and the fact that shape injectivity does not hold can be seen analogously as for V in 1.5.

However, the proof that it is strongly homotopically Hausdorff cannot be that straightforward. For, with having added all these tunnels, we have created an uncountable family of new homotopy classes. There are definitely uncountably many, since a path can pass through infinitely many tunnels, and we have at least as many possiblities to combine such passages as we have for the Hawaiian Earrings. We need to understand to some extent all these homotopy classes, since the property of being strongly homotopically Hausdorff requires that we can find a minimal diameter for each of these classes. So we need to understand this variety of paths to some degree in order to be able to construct all these diameters.

Therefore we construct in Sect.2 some combinatorial descriptors for these homotopy classes, which then will allow the crucial distinction of cases by which we can complete this proof by demonstrating the property strongly homotopically Hausdorff for W in Section 3.

Comments 1.7. Example 9 from [FiZa1] is, roughly described, our first Example 1.2, but turned inside-out. I.e. in [FiZa1] the waved surface does not accumulate to an outside cylinder, but just to the central axis of the figure. Therefore metric separability and the reason why shape-injectivity is violated, follow analogously as they have been proven in 1.5 for our Example 1.2. It also follows by an analogous projection argument as in 1.4(ii) that in order to be non-nullhomotopic a path has to encircle the central axis — but a path with non-zero winding number will also be non-nullhomotopic. Hence the distance between the base point on the waved surface and the central axis is a minimal diameter for each non-nullhomotopic path in the sense of the weak homotopic Hausdorff-property, but if we go to free homotopy classes there is no minimal diameter any more, since they can be homotoped to arbitrary small paths encircling the central axis. Hence this space is not strongly homotopically Hausdorff.

§ 2. A word sequence calculus for our Main Example

The word-sequence calculus of this space is modelled on the word-sequence-calculus that was built for the Hawaiian Earrings in [Za20; §1] (cf. also [Za98; 2.3–2.9] and [Za03; 3.1–3.3]). In this preliminary version we restrict ourself to some brief description that is based on the analogy of this construction: The HE-concept ("HE" as abbreviation for "Hawaiian Earrings") was based on acknowledging by a letter that a path runs through a loop of the Hawaiian Earrings. Every such passage is acknowledged by a letter, the passages of the very small loops are inserted into the description rather lately, but each passage is inserted into the description according to the number of the loop that is passed. The HE-Concept (in difference to the concept for partially filled modified Sierpiński Carpets of [Za97; A.4.6, §B.1(Fig. g_0–g_4, B.1.4)] – cf. also the appendix[1] of [FiZa1]) is a mere insertion concept ([Za98; 2.4(ii)]), i.e. every word repeats just the letters from its predecesors and inserts new letters which in addition state whether, in what relative order to the other passages and in what direction our path runs also through the loop that is considered on that level. We are using the same idea here: Instead of the loops of the Hawaiian Earrings we are considering here the tunnels. The generation index which was introduced at the end of Constr. 1.3 is here our measure at what level of our sequence a letter must be inserted into the words, if it is to be contained in the entire sequence. Hence one of the differences with the HE-concept is, that we have more than one new letter on each level.

Another difference is that we cannot attempt to set up word sequences as a combinatorial concept that describes homotopy classes of paths, therefore we are not able to require the reducedness-axiom [Za98; 2.4(iii)] for our word sequences, but finiteness [Za98; 2.4(i)] and insertion compatibility [Za98; 2.4(ii)] will be of course required. Instead of working with homotopy classes we have to work with *"generic paths"*, i.e. with a class of paths that is big enough for obviously containing a homotopic representative of each path. The important genericity-condition requires: If $w : [0,1] \to W$ is a path which at some parameter value t takes with $w(t)$ a value in the interior of a tunnel, then there exist ε_1 and ε_2 such that $w(t-\varepsilon_1)$ and $w(t+\varepsilon_2)$ are the points at the opposite ends of the tunnel, and that $w_{|[t-\varepsilon_1,t+\varepsilon_2]}$ is a smooth monotone passage through this tunnel. Obviously every path once entering a tunnel can be either completely homotoped out of this tunnel, or be homotoped in such a way, that it smoothly and monotonically passes through this tunnel. Since only finitely many tunnels have big size, it is then a standard exercise in topology of such wild spaces, to perform continuously the possibly infinitely many of these homotopies for one path simultaneously.

In this way we can associate with our path an infinite combinatorial object that reports which tunnels our path passes through and in which order – although this might be infinitely many with accumulation of passages also possibly occurring in the interior.

However, there are also essential differences with the situation for Hawaiian Earrings:

(i) This concept of word sequence does not even attempt to capture the movements of our paths outside of the tunnels. It does not see which way on the waved surface our path takes (e.g., when going to a diametral point there might be a choice between going straight in the (x, y) coordinates but over the waves, or in a semicircle, but staying on the same z–level), and it does not see whether a segment of the path on the outside cylinder is or is not nullhomotopic on this segment.

(ii) Therefore, unlike as the concept of word sequences from [Za98] and [Za20] we do not have a chance to rebuilt a path or its homotopy class from a word sequence.

(iii) Also, unlike as the concept of word sequences from [Za98] and [Za20], there is no chance to tell by rules that only apply to one word or to two consecutive words, whether or not this word sequence is the word sequence of a continuous path. In particular, every finite word sequence (and hence every finite initial part of a word sequence) can be realized by a continuous path. However, it is not difficult

[1]The appendix is only in the revised version which is not yet circulated

to find arragements of tunnels, where with each new insertion a path has to run additionally between entrance and exit points of tunnels that are comparatively far apart, so that these demands accumulate to a continuity obstruction and no path can realize the entire word sequence. But such an obstruction can never be determined from any finite part of a word sequence.

Hence this concept of word sequences is weaker than the HE-concepts from [Za98], [Za20] and [Za03], but it is sufficient to achieve the result that was claimed in the title.

§ 3. The Main Example is strongly homotopically Hausdorff

Recall that in order to complete the proof of our Theorem (cf. 1.6), it is only necessary to show that W is strongly homotopically Hausdorff. This follows by associating to each generic path a word sequence according to Sect.2, and by distinguishing cases according to how these word sequences look like. The following lemmata together will accordingly give a proof of the desired statement. Recall that for any path we either have to show that it is nullhomotopic, or we must find a minimal radius that each representative in the free homotopy class has to adopt.

Lemma 3.1. *Let $v : [0,1] \to W$ be a closed path and ω be its associated word sequence. Assume that ω has at least one word that according to the calculus of freely presented finitely generated free groups cannot be reduced to the empty word. Then v is not nullhomotopic and has a positive minimal diameter.*

Proof. Let i be the index of a word which satisfies the hypothesis of this lemma. Construct a topological space W_i by thickening the outside cylinder of W in such a way that the thickening engulfs all tunnels and the inner waved surfaces in the area where the tunnels are lying that have a generation index bigger than i, but that the remainder of the inner waved surfaces and the appropriate tunnels stay as the are. Thay way our space has become a finite simplicial complex. The fundamental group is the free group generated by all the remaining tunnels, and from the theory of free groups is follows, that the i^{th} word of our word sequence can be interpreted so as to describe the representation of our path in this free group. Hence the non–cancelability of this means that even in the space W_i which has W as a subspace our path is not nullhomotopic.

The homotopy class of our path can then be described by a non-empty reduced word, and any word describing the same homotopy class is obtained by inserting additional letters into this word. Thus any path that lies in this homotopy class has to pass through at least those tunnels that occur in the reduced word, and from this fact we obtain a minimal radius that all paths in our homotopy class have to adopt.

Lemma 3.2. *Let v and ω be as in 3.1. If, vice versa, each word can be cancelled through to the empty word, there also exists a compatible cancellation pattern.*

Next we define what this assertion means before we prove Lemma 3.2.

Definition 3.3. *("compatible cancellation pattern")*
Graphical visulizations of cancellation patterns have already appeared in [Za98; Fig. 2.1] or [Za20; Fig. 2.]. Since here we only have non–commuting variables, a cancellation pattern for a single word is a system of non–intersecting arches,

each of which connects a letter to its inverse such that they are visualizing by their nesting which letters might be cancelled against which in the order of a consecutively performed cancellation process. A cancellation pattern for a word sequence is such a cancellation pattern for each of the words, and it is called *"compatible",* if it is compatible with the insertion structure, i.e. if any two letters that are connected by an arch on the i^{th} level will also be connected by an arch on any higher level on which they must reappear (cf. §2).

Proof of 3.2. Let us arrange all cancellation patters for each word of such a completely cancelling word sequence in a graph. Each possible cancellation pattern of one word gives a vertex, and the vertices are naturally arranged in "lines", so that the vertices representing cancellation patters of the i^{th} word sit on the i^{th} line. Only vertices of adjacent lines may be connected by a edge, and they are to be connected by an edge if and only if they are compatible in the sense this has been described in 3.3. That way each vertex from the i^{th} line will be connected to precisely one vertex of the $(i-1)^{\text{st}}$ line, since deleting all letters $\alpha_i^{\pm 1}$ from a completely cancelable word leave us with a word which can in the same way be cancelled as before. Thus each vertex is by a monotone path be connected to the root vertex (i.e. the empty cancellation pattern of the empty word in the zeroth line), and that way it follows that the whole graph will be a connected tree. Since there are only finitely many possibilities to cancel a finite word, this tree will be finitely branching. It then follows from König's lemma [Kö; VI.§2 Satz 6], that such a tree must have an infinite ray. By construction of such a tree, each infinite ray describes a compatible cancellation pattern.

The construction scheme 3.4 of a *"Kenyon diagramme"***.** Following a suggestion of R. Kenyon [Keny] to use hyperbolic geometry in order handle phenomena like infinite cancellation processes elegantly, we can try to turn the information that comes from a compatible cancellation pattern in the sense of 3.2/3.3 into a map that is defined on a disk $\mathbb{D}$ and hopefully will give a nullhomotopy. This item is for only describing the procedure, whether or not it will under the conditions of Lemma 3.2 have the desired result, will be discussed in the subsequent paragraphs.

Similar constructions have been carried out in the appendix of the revised version of the [FiZa1]–paper and in Proposition 3.6 in the [MRRZ]-paper. However, observe that each of these constructions has to be adapted to the precise geometry of the space. In the present construction the points where the path enters or leaves the tunnel have to be connected by lines of the one–skeleton of the cellulation of the inner disk of the Kenyon diagram, and the map has to be defined constant on such lines. In that way we connect the entrace points and the exit points of those inverse passages of the same tunnels where the corresponding letters are associated by an arch in the compatible cancellation pattern. Since this gives always two parallel lines coming from the entrance and the exit of the same tunnel, we will call the type of cell that sits between such lines a "corridor". Apart from such corridors we will have one other type of cells, namely those which sit between such corridors, and which might have a complicated boundary consisting of up to infinitely many segments. However, since the path which is defined on such a boundary cannot pass through tunnels (since this only happens for the boundary paths of corridors), we know by 1.4(iii) that these boundary paths are well–defined paths either on the inner waved surface or on the outer cylinder.

Lemma 3.5. *There is one and only one reason, why under the hypotheses of Lemma 3.2 the resulting Kenyon diagram might not give a contracting homotopy for our path, namely if there is at least one cell in the cellulation of* $\mathbb{D}^2$ *which has a non–nullhomotopic boundary-path in the outside cylinder.*

Sketch of Proof. There are two possible obstructions why the filling of a Kenyon diagram might fail: either there might be one cell for which there is no filling, or the independent fillings of infinitely many cells by continuous mappings might give a non–continuous mapping at an accumulation point of those cells:

The corridors connecting the tunnel–passages can be always filled, and also the cells between these corridors where the boundary path lies on the inner waved surface. Observe that by construction of the filling of the Kenyon diagram, no boundary path of a cell between the corridors can take on values on both, the inner waved surface and the outside cylinder. Since already the formulation of the lemma acknowledges that for the remaining type of cells there might occur non–nullhotopic boundary paths as an obstruction for filling a Kenyon diagram, we can skip to discussing whether there might be an obstruction as it was described as second possibility of the first sentence of the sketch of this proof. Such an obstruction can only occur if the contracting homotopies of the boundary paths of the cells that are used to fill the cells are considerably bigger than their boundary paths. However, when constructing the contracting homotopies sensibly, the geometry of our space gives only little chances why this should happen: The only source to construct such paths which cannot avoid to have a contracting homotopy that is consirably bigger than the path itself are such paths that encircle the central axis of the inner waved surface almost concentric at a fairly big radius, so that not the path, but the contracting homotopy has to go over various different maxima or minima. But since such curves have to have a fairly big radius itself, they can never play a role in any accumulation of the Kenyon diagram. Hence there remains only one possible obstruction for filling a Kenyon diagram – as claimed.

This proof was only a sketch, since in this preliminary version for the whole construction we neglected the effect that the corridors inside a Kenyon-diagram might accumulate. Hence there will be an additional "closure–step" of this set of corridors, before we can achieve a cellulation that can be used for the above described filling construction.

Completion of the proof of Theorem 1.6. According to the remarks at the beginning of this section, we only need to find for each path in W either a minimal diamater of each homotopic representative, or a contracting homotopy. We therefore distingiush the following two cases:

(i) The minimal diameter of our path is at least π, i.e. the diameter at the centerline.
(ii) The minimal diameter is smaller.

Since we are automatically done in Case(i), we only need to worry about (ii). We take a homotopic representative with an according diameter, and then we search on the outside cylinder (in polar coordinates: $\{(\frac{\pi}{2}, \phi, z) \mid 0 \leq \phi \leq 2\pi\ ,\ -1 \leq z \leq 1\}$) for some ϕ_0–value, such that the vertical line $\{(\frac{\pi}{2}, \phi_0, z) \mid -1 \leq z \leq +1\}$ does not intersect with the representative of our path. Such a ϕ_0–value has to exist, since otherwise the diameter of this represantative would be at least π. Let ω be the word–sequence that is associated with our representative. If at least one word of ω cannot be cancelled to the empty word, then we are done by Lemma 3.1. If every word cancels through, then we use Lemma 3.2 to first find a compatible cancellation pattern and then use 3.4 to build and try to complete a Kenyon diagram. In our case Lemma 3.5 tells, that the diagramme can be completed, since the boundary path of every cell of the cellulation of the Kenyon diagram takes only values that are also taken on by our path. Since this means that it cannot cross the vertical line at ϕ_0, such a path in the outside cylinder must be nullhomotopic and the obstruction from 3.5 vanishes. However, a completed Kenyon diagram can be interpretated as a nullhomotopy for our path, so that we are done in this case, too.

Hence our Main Example has all the desired properties asserted in the title of our paper.

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