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ON CLOSEDNESS
ASSUMPTIONS IN SELECTION
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ON CLOSEDNESS ASSUMPTIONS IN SELECTION THEOREMS

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ABSTRACT. We begin by a short survey of various attempts in selection theory to avoid the closedness assumption for values of multivalued mappings. We collect special cases when Michael's G_δ -problem admits an affirmative solution and we prove some unified theorems of such type. We also show that in general this problem has a negative solution. In comparison with a recent result of Filippov, we work directly in the Hilbert cube rather than in the space of all probabilistic measures endowed with different topologies.

0. Introduction

One of the basic mathematical constructions arises when one attempts to solve a certain problem by a suitable kind of limits of approximate solutions. Selections of multivalued mappings typically illustrate the point.

A singlevalued mapping $f : X \rightarrow Y$ between sets X and Y is said to be a *selection* of a given multivalued mapping $F : X \rightarrow Y$ if $f(x) \in F(x)$, for each $x \in X$. Note that by the Axiom of Choice selections always exist. In the category of *topological* spaces and *continuous* singlevalued mappings the situation is more complex. There exist many theorems on continuous selections. Practically always the final selection is constructed as a uniform limit of approximate selections. The term "approximate" has in practice two-fold interpretation: we either improve the distance between $f_n(x)$ and $F(x)$ with continuous n -level approximations $f_n : X \rightarrow Y$, or we control the degree of discontinuity of f_n with $f_n(x) \in F(x)$, $n \in \mathbb{N}$. (See [RS2] for an example of simultaneous use of these approaches.)

In both cases a typical difficult situation arises with $\lim_{n \rightarrow \infty} f_n(x)$. Such a limit point can easily end up on the boundary of the set $F(x)$, rather than in the set $F(x)$, if we do not pay more attention to a careful construction of the uniform Cauchy sequence f_n , $n \in \mathbb{N}$. The simplest and the most direct way to overcome this possibility is to consider only closed-valued mappings into a complete range or alternatively, deal only with complete-valued mappings.

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The following is a significant example of the affirmative solution of such a problem. This is the classical Michael selection theorem for convex-valued mappings, whose citation index is by an order of magnitude higher than for any other selection theorem [M1].

Theorem M. *A multivalued mapping $F : X \rightarrow B$ admits a continuous singlevalued selection, provided that the following conditions are satisfied:*

- (1) X is a paracompact space;
- (2) B is a Banach space;
- (3) F is a lower semicontinuous mapping;
- (4) For every $x \in X$, $F(x)$ is a nonempty convex subset of B ; and
- (5) For every $x \in X$, $F(x)$ is a closed subset of B .

A natural question arises concerning the necessity (essentiality) of any of the conditions (1)-(5). We restrict ourselves only to the last one, the closedness assumption. On the one hand, there are lower semicontinuous (LSC) convex-valued mappings $F : X \rightarrow Y$ without any continuous singlevalued selections even for $X = [0; 1]$ (see Example 6.2 of [M1] when values $F(x)$ are unions of finite-dimensional convex sets and Example 6.3 of [M1] when values $F(x)$ are open sets). On the other hand, every convex-valued LSC mapping of a metrizable domain into a separable Banach space admits a selection, provided that all values are finite-dimensional ([M1; special case of Theorem 3.1"']).

Another kind of omission of closedness was suggested in [Ch, M4]. It appears that such omission can be certainly assumed over a σ -discrete subset of the domain.

An alternative to pointwise omission of closedness is to regard some uniform versions of such omission. Namely, we simply consider closedness in a fixed subset $Y \subset B$, instead of closedness in the entire Banach space B . Existence of selections under such assumption implies that Y must be completely metrizable, or in other words, a G_δ -subset of B (see [MPP]). Thus the following was one of the principal problems in selection theory during the last 15 years (note, that the problem was originally stated only for convex Y) [M5, No.396]:

G_δ -problem. *Let Y be a G_δ -subset of a Banach space B . Does then every LSC mapping $F : X \rightarrow Y$ of a paracompact space X with convex closed values into Y have a continuous selection?*

In spite of numerous cases with affirmative answer this problem has in general a negative solution - a counterexample was recently constructed by Filippov [F].

To conclude our introduction we recall that *lower semicontinuity* of a multivalued mapping $F : X \rightarrow Y$ between topological spaces X and Y means that for each $x \in X$ and $y \in F(x)$, and each open neighborhood $U(y)$, there exists an open neighborhood $V(x)$ such that $F(x') \cap U(y) \neq \emptyset$, whenever $x' \in V(x)$. Applying the Axiom of Choice to the family of nonempty intersections $F(x') \cap U(y)$, $x' \in V(x)$, we see that LSC mappings are exactly those, which admit local (noncontinuous)

selections. In other words, the notion of lower semicontinuity is by definition very close to the notion of a selection.

In the sequel we shall often use the compact-valued selection theorem [M2] which in particular, guarantees the existence of a LSC mapping $H : X \rightarrow Y$ whose values $H(x)$ are compact nonempty subsets of $F(x)$, $x \in X$, whenever $F : X \rightarrow Y$ is a LSC mapping from a paracompact space X into a completely metrizable space Y with closed nonempty values. Also, we shall call by *localization principle* any statement that a convex-valued mapping on a paracompact domain admits a global selection, provided that it has local selections, i.e. selections over each member of some open covering of the domain.

1. Affirmative results

For finite-dimensional domains X , the G_δ -problem has an affirmative solution simply because the family of convex closed subsets of a Banach space is C^n and ELC^n for every $n \in \mathbb{N}$. Hence the finite-dimensional selection theorem can be applied. For the finite-dimensional range B and moreover, for all finite-dimensional values closed in $Y \subset B$, the problem is also trivial, because one can use the compact-valued selection theorem and the fact that the closed convex hull of a finite-dimensional compactum coincides with its convex hull.

Gutsev [G] proved that the answer to this the problem is affirmative when domain X is either a countably dimensional metric space or a strongly countably dimensional paracompact space. In both cases he in fact proved that under the hypothesis of the problem the existence of a singlevalued continuous selection is equivalent to the existence of a compact-valued upper semicontinuous (USC) selection. The last statement is true, because each domain of such type can be represented as the image of some zero-dimensional paracompact space under some closed surjection whose preimages of points are all finite.

Recently, Gutsev and Valov [GV] obtained an affirmative answer for domains with the so-called C -property. They introduced a certain swelling of the original mapping F . Roughly speaking, they defined $W_n(x)$ as the set of all $y \in Y = \bigcap_{n \in \mathbb{N}} G_n$ which are closer to $F(x)$ than to $B \setminus G_n$. It turns out that each of the mappings W_n has an open graph and all of its values are contractible. Applying the selection theorem of Uspenskii [U] for C -domains, one can find selections for each W_n and hence for the pointwise intersections $\bigcap W_n(x)$. Unfortunately, it seems that their method does not work outside the class of C -domains, because Uspenskii's theorem gives the *characterization* of the C -property.

We begin with a relatively simple (folklore) fact which was communicated to us by Vesko Valov around 1990. It can be proved in several different ways: via outside (or inside) approximations, by using integrals and Milyutin mappings [RS1], or by exploiting the compact-valued selection theorem (see Lemma 1.5 below). Let us consider perhaps the simplest approach.

Theorem 1.1. *For any paracompact space X and any open subset G of a Banach space B , any LSC mapping $F : X \rightarrow G$ with nonempty convex values admits a singlevalued continuous selection whenever all values $F(x)$ are closed in G .*

Proof. For $x \in X$ pick $y \in F(x) \subset G$ and fix an arbitrary open ball D , centered at y such that the closure $Cl(D)$ is a subset of G . For every z from the open neighborhood $U = F^{-1}(D)$ of the point x , the intersection $F(z) \cap D$ is nonempty and convex. Its closure in the entire Banach space $Cl_B(F(z) \cap D)$ is a subset of $F(z)$, because $F(z)$ is closed in G and $Cl(D) \subset G$. Therefore Theorem M applies to the mapping $z \mapsto Cl_B(F(z) \cap D) \subset F(z)$, $z \in W$, where W is some closed subneighborhood of U . So by to the localization principle, the convex-valued mapping $F : X \rightarrow G$ has a continuous selection. \square

We present a slight generalization of this fact in which values of the convex-valued mapping can be nonclosed even in G (see Fig.1).

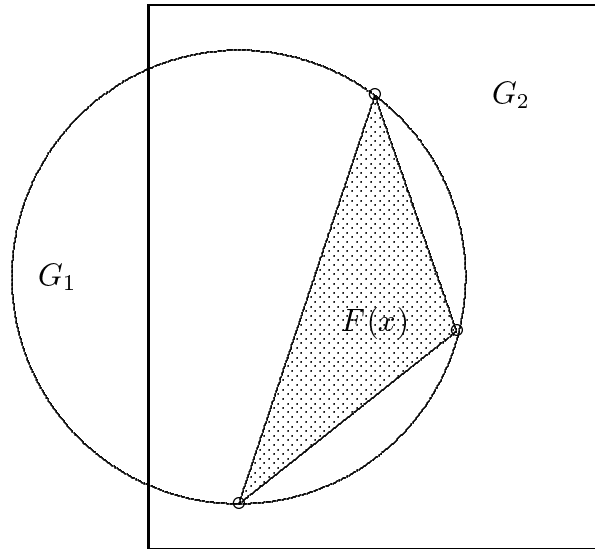


Figure 1: $F(x)$ is closed in G_1 but not closed in $G_1 \cup G_2$.

Theorem 1.2. *Let $\{G_\alpha\}$, $\alpha \in A$, be any family of open subsets of a Banach space B and let $F : X \rightarrow \bigcup_{\alpha \in A} G_\alpha$ be a LSC mapping of a paracompact space X with nonempty convex values. Then F admits a singlevalued continuous selection whenever for each $\alpha \in A$ and each $x \in F^{-1}(G_\alpha)$ the values $F(x)$ are closed in G_α .*

Proof. We begin by shrinking the open covering $\{F^{-1}(G_\alpha)\}_{\alpha \in A}$ to an open covering $\{V_\alpha\}_{\alpha \in A}$ such that $Cl(V_\alpha) \subset F^{-1}(G_\alpha)$. Next, apply Theorem 1.1 to each $Cl(V_\alpha)$. It now suffices to once more use the localization principle. \square

We shall explain a different approach for open sets G that are also *convex*, which yields additional useful information.

Lemma 1.3. *For any subcompactum C of an open convex subset G of a Banach space B , the closed convex hull $\overline{\text{conv}}C$ also lies in G .*

Proof. One can assume that the origin O of B belongs to G . So let $\mu : B \rightarrow [0, +\infty)$ be the Minkowsky functional of G . By definition and the continuity of $\mu(\cdot)$, we can conclude that $\max\{\mu(C)\} = m < 1$. Moreover, $\mu(\cdot)$ is a seminorm, therefore $\max\{\mu(\text{conv}C)\} = m < 1$ and $\max\{\mu(\overline{\text{conv}}C)\} = m < 1$. Hence, $\overline{\text{conv}}C \subset G$. \square

As a corollary we obtain the following:

Proposition 1.4. [M5] *Let $\{G_n\}, n \in \mathbb{N}$, be a sequence of open convex subsets of a Banach space and let $F : X \rightarrow Y = \bigcap_n G_n$ be a LSC mapping of a paracompact space X with nonempty convex values. Then F admits a singlevalued continuous selection whenever all values $F(x)$ are closed in Y .*

Proof. The complete metrizable of Y guarantees the existence of a compact-valued LSC selection $H : X \rightarrow Y$ of the mapping F . By Lemma 1.3, the multivalued mapping $\overline{\text{conv}} H : x \mapsto \overline{\text{conv}} H(x)$ is a selection of the given mapping F . It remains to apply Theorem M to the LSC mapping $\overline{\text{conv}} H$. \square

In fact, an analogue of Lemma 1.3 holds for an arbitrary, not necessarily convex open set (this was stated without proof in [RS1, p.117]).

Lemma 1.5. *For any subcompactum C of a convex closed subset F of an open convex subset G of a Banach space B , the closed convex hull $\overline{\text{conv}}C$ lies in F .*

Proof. Choose a finite covering $C \subset D_1 \cup D_2 \cup \dots \cup D_n$ of the compactum C by closed balls $D_i \subset G$. Then the sets $F \cap D_i$ are convex and closed in G . Denote by C_i the intersection $C \cap (F \cap D_i)$ and by K_i the closed convex hull of C_i . Clearly, all C_i as well as all $K_i \subset F$ and $\text{conv}\{K_1 \cup K_2 \cup \dots \cup K_n\} \subset F$ are compacta.

Therefore if

$$x \in \text{conv}C, \quad x = \sum_{k=1}^N \lambda_k c_k, \quad \lambda_k > 0, \quad \sum_{k=1}^N \lambda_k = 1$$

then by the well-known partition

$$x = \mu_1 \sum_{k \in A_1} \frac{\lambda_k}{\mu_1} c_k + \mu_2 \sum_{k \in A_2} \frac{\lambda_k}{\mu_2} c_k + \dots,$$

where

$$A_1 = \{k : c_k \in C_1\}, \quad A_j = \{k : c_k \in C_j \setminus (C_1 \cup \dots \cup C_{j-1})\} \quad \text{and} \quad \mu_j = \sum_{k \in A_j} \lambda_k.$$

Hence $x = \sum_{j=1}^n \mu_j d_j$ with $d_j \in K_j$. The set

$$K = \left\{ \sum_{j=1}^n \mu_j d_j : d_j \in K_j, \mu_j \geq 0, \sum_{j=1}^n \mu_j = 1 \right\}$$

is the compact being the image of $\Delta_n \times K_1 \times \dots \times K_n$ under the continuous mapping, where Δ_n is the standard simplex with n vertexes. Hence,

$$\text{conv}C \subset K \subset \text{conv}\{K_1 \cup K_2 \cup \dots \cup K_n\} \subset F, \quad \overline{\text{conv}C} \subset K \subset F.$$

□

Recently, Dobrowolski and Van Mill [DM] have shown that for every non-locally convex, completely metrizable linear range E , a LSC mapping $F : X \rightarrow E$ with complete convex values admits a selection provided that $\dim X < \infty$ or, that $\max\{\dim F(z) : z \in U(x)\} < \infty$ (local versions are also proved). Moreover, in the nonlocal assumption, each of these restrictions practically has no weaker version of an affirmative answer (see Proposition 4.1 and Example 5.3 of [DM]). Note that for such range spaces compactness is not preserved under the convex closed hull operation and the intersections of convex subsets with balls are in general, non-convex. We prove that Theorem 5.4. [DM] admits an extension for not necessarily complete values of F .

Theorem 1.6.

- (1) Let $F : X \rightarrow Y$ be a LSC convex-valued mapping of a paracompact space X into a G_δ -subset Y of a completely metrizable linear space E . Then F admits a singlevalued continuous selection provided that the values $F(x)$ are closed in Y and that for every $x \in X$ there exists a neighborhood $U(x)$ such that $\max\{\dim F(x) : x \in U(x)\} < \infty$.
- (2) Let $F : X \rightarrow E$ be a LSC convex-valued mapping of a metrizable space X into a completely metrizable linear space E . Then F admits a singlevalued continuous selection provided that for every $x \in X$ there exists a neighborhood $U(x)$ such that $\max\{\dim F(x) : x \in U(x)\} < \infty$.

Proof (1). It suffices to consider only the case of the global restriction $\max\{\dim F(x) : x \in U(x)\} < \infty$ and then use the localization principle. Apply once more the compact-valued selection theorem: F has a LSC compact-valued selection, say $H : X \rightarrow E$, $H(x) \subset F(x)$. However, for every $x \in X$ we have that $\overline{\text{conv}H(x)} = \text{conv}H(x) \subset F(x)$, due to the finite dimensionality of the convex sets $H(x)$ and $F(x)$. Therefore Theorem 5.4 of [DM] properly works for the mapping $x \mapsto \text{conv}H(x)$, $x \in X$. □

Proof (2). Recall, that Theorem 5.1 of [M3] in particular asserts that for every closed-valued LSC mapping $\Phi : X \rightarrow Y$ into a completely metrizable space Y there is a countable, pointwise dense family of selections, provided that all values $\Phi(x)$ are separable and two special properties (a) and (b) are satisfied.

So, if $\Phi(x) = Cl(F(x))$, $x \in X$, then, having such selections, say $\varphi_1, \dots, \varphi_n, \dots$ for Φ , one gets the desired selection of F by the following formula (see [M3, p.176]):

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \left(\varphi_1(x) + \frac{\varphi_n(x) - \varphi_1(x)}{1 + \|\varphi_n(x) - \varphi_1(x)\|} \right).$$

We start by checking (b):

(b). *If $A \subset X$ is closed, then every selection s of $\Phi|_A$ can be extended to a selection of Φ .*

According to selection standards it suffices to introduce a LSC mapping $\Psi : X \rightarrow Y$ by setting $\Psi(a) = \{s(a)\}$, $a \in A$, and $\Psi(x) = \Phi(x)$, otherwise. Then Theorem 5.4 directly works for Ψ and each of its selections will be the desired extension of the partial selection s .

Let us now verify (a):

(a). *For every $y \in Y$ and every neighborhood U of y there exists a neighborhood $V \subset U$ of y such that for each closed $A \subset X$ with all intersections $\Phi(a) \cap V$, $a \in A$, nonempty there is a selection $\varphi : A \rightarrow U$ of $\Phi|_A$.*

Let ρ be a translation invariant complete metric on E . Then

$$\rho(a + b, 0) \leq \rho(a + b, b) + \rho(b, 0) = \rho(a, 0) + \rho(b, 0)$$

and hence

$$\rho\left(\sum_{i=1}^N \lambda_i y_i, 0\right) \leq N \max\{\rho(y_i, 0)\}$$

for arbitrary $0 \leq \lambda_i \leq 1$. In particular, if all elements of a set $M \subset E$ are ε -close to $y \in E$, then all points of the N -convex hull

$$\text{conv}_N M = \left\{ \sum_{i=1}^N \lambda_i y_i : y_i \in M, \lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1 \right\}$$

are $N\varepsilon$ -close to y .

So, for $U = D(y, \varepsilon)$ put $V = D(y, \delta)$ with $N\delta < \varepsilon$, where

$$N = \max\{\dim F(x) : x \in X\} = \max\{\dim \Phi(x) : x \in X\},$$

and let $\Psi : A \rightarrow E$ be defined by setting

$$\Psi(a) = \text{Cl}(\text{conv}_N(\Phi(a) \cap V)) \subset \Phi(a), \quad a \in A.$$

It easy to see that Theorem 5.4 of [DM] applies to Ψ . Finally,

$$\text{conv}_N(\Phi(a) \cap V) \subset \text{conv}_N V \subset D(y, N\delta),$$

i.e. $\Psi(A) \subset \text{Cl}(D(y, N\delta)) \subset D(y, \varepsilon) = U$. So, each selection ψ of Ψ will automatically be a mapping into U . This completes the proof. \square

As for the case of finite-dimensional domains, we introduce some changes into the method of outside approximations.

Theorem 1.7. *For any finite-dimensional paracompact space X and any open subset G of a completely metrizable linear space E , every LSC mapping $F : X \rightarrow G$ with nonempty convex values admits a singlevalued continuous selection whenever all values $F(x)$ are closed in G .*

Proof. Let ρ be a translation invariant complete metric on E . Fix $x_0 \in X, y_0 \in F(x_0) \subset G$ and pick $\varepsilon_0 > 0$ so small that $3N\varepsilon_0 < \text{dist}(y_0, E \setminus G)$. For the open ball $D_0 = D(y_0, \varepsilon_0)$ choose a closed subneighborhood $W = W(x_0)$ of $F^{-1}(D_0)$.

If we construct a continuous selection of the multivalued mapping F over W , $\dim W \leq \dim X < N < \infty$, then we have proved the existence of selections of F locally and thus the existence of global continuous selection of F . Note, that the identity mapping $x \mapsto y_0, x \in W$, is an ε_0 -selection of the mapping $F_0, F_0 : x \mapsto F(x) \cap D_0$ which is simultaneously ε_0 -separated from $E \setminus G$. Observe also that the mapping $x \mapsto \text{conv}_N \Phi(x)$ is LSC provided that $x \mapsto \Phi(x)$ is LSC.

As in the standard proof of Theorem M, one can choose for an arbitrary $\varepsilon_1 > 0$ a partition of unity $e^1 = \{e_\alpha^1\}_{\alpha \in A}$ and a family $\{y_\alpha\}_{\alpha \in A}$ of points $y_\alpha \in D_0$ such that the supports $\text{supp}(e_\alpha^1)$ are subsets of $F^{-1}(D(y_\alpha; \varepsilon_1))$, $\alpha \in A$, and the family $\{\text{supp}(e_\alpha^1)\}_{\alpha \in A}$ constitutes a covering of W of order $\leq N$.

Thus

$$x \mapsto f_1(x) = \sum_{\alpha \in A} e_\alpha^1(x) y_\alpha(x), \quad x \in W,$$

is a continuous $(N\varepsilon_1)$ -selection of the mapping $x \mapsto \text{conv}_N(F_0(x))$. Choosing $z_1(x) \in \text{conv}_N(F_0(x))$ in an arbitrary fashion, but so that $\rho(f_1(x), z_1(x)) < N\varepsilon_1$, we find a selection (not necessarily continuous) of the mapping $x \mapsto \text{conv}_N(F_0(x)) \subset F(x)$.

Suppose two finite functional sequences f_1, f_2, \dots, f_n and z_1, z_2, \dots, z_n have been constructed for arbitrary chosen positive decreasing $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$. Define the LSC mapping F_{n+1} with nonempty values by setting $F_{n+1}(x) = F(x) \cap D(f_n(x), N\varepsilon_n)$. Repeating the above procedure, find for an arbitrary $0 < \varepsilon_{n+1} < \varepsilon_n$ a continuous $(N\varepsilon_{n+1})$ -selection $f_{n+1} : W \rightarrow G$ of the mapping $\text{conv}_N F_n$ and a selection $z_{n+1} : W \rightarrow G$ of the same mapping such that

$$\rho(f_{n+1}(x), z_{n+1}(x)) < N\varepsilon_{n+1}, \quad \rho(f_{n+1}(x), f_n(x)) < N(N\varepsilon_n + \varepsilon_{n+1}) < (N+1)^2\varepsilon_n.$$

In particular,

$$\rho(z_{n+1}(x), z_n(x)) < N\varepsilon_n + (N+1)^2\varepsilon_n + N\varepsilon_{n+1} < (N+1)^3\varepsilon_n, \quad x \in W.$$

Now, by choosing the sequence $\{\varepsilon_n\}$ so that $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon_0(N+1)^{-3}$ we obtain two uniformly Cauchy functional sequences $f_1, f_2, \dots, f_n, \dots$ and $z_1, z_2, \dots, z_n, \dots$ with the same limit

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} z_n(x) = z(x).$$

Thus such a limit gives a continuous mapping $f : W \rightarrow E$. To see that f is a selection of F it suffices to estimate the distance

$$\rho(f(x), z_1(x)) = \rho(z(x), z_1(x)) \leq \sum_{n=1}^{\infty} \rho(z_{n+1}(x), z_n(x)) < (N+1)^3 \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon_0.$$

Having by construction

$$z_1(x) \in \text{conv}_N(F(x) \cap D_0) \subset D(y_0, N\varepsilon_0),$$

we see that

$$f(x) \in \text{Cl}\{D(y_0, 2N\varepsilon_0)\} \subset D(y_0, 3N\varepsilon_0) \subset G.$$

So $f(x) \in F(x)$, by closedness of $F(x)$ in G . \square

The idea of control by using the constant N presented above is similar to van Mill [M, I.4.9].

As for Lemma 1.3 and Proposition 1.4 in nonlocally convex case, they are formally true, but sometimes look as meaningless because for example in the nonlocally convex space L_p , $0 < p < 1$, there is a unique nonempty open convex subset G , namely $G = L_p$. The question concerning Lemma 1.5 for nonlocally convex spaces is interesting and still open. In fact, both Lemma 1.5 and Theorem 1.7 imply an alternative proof of Theorem 1.1, but it seems, that they do not give equivalent approaches.

Note also that all statements above admit a unification with zero-dimensional selection theorem and selection theorems for countable domains in spirit of [M4]. For example, we have:

Theorem 1.8. *Let X be a paracompact space, $Z \subset X$ a subset with $\dim_X Z \leq 0$ and $C \subset X$ a countable subset. Then every LSC mapping $F : X \rightarrow Y$ into an open subset Y of a completely metrizable linear space E admits a continuous singlevalued selection provided that values $F(x)$ are closed in Y for every $x \in X \setminus C$ and that $\text{Cl}(F(x))$ are convex for every $x \in X \setminus Z$.*

2. The counterexample

Theorem 2.1. *In the Hilbert cube $Q = [0; 1]^{\mathbb{N}}$ there exist a convex subcompactum X and a convex G_{δ} -subset $Y \subset X$ such that some LSC mapping $F : X \rightarrow Y$ with nonempty, convex and closed in Y values admits no continuous singlevalued selections.*

Proof. First, we present the construction. It consists of the following three steps:

- (A) $X = \{x \in Q : x_1 = 1, x_n = x_{2n} + x_{2n+1}, n \in \mathbb{N}\}$;
- (B) $Y = \{x \in X : \sup\{x_n g_n^{-1} : n \in \mathbb{N}\} = \infty\}$, where $g \in X$ is arbitrarily chosen so that $\lim_{n \rightarrow \infty} g(n) = 0$ and $g(n) > 0$ for all $n \in \mathbb{N}$; and
- (C) $F(x) = \Phi(x) \cap Y$, where $\Phi : X \rightarrow X$ is defined by $\Phi(x) = \{y \in X : y_n = 0 \text{ whenever } x_n = 0\}$.

Second, we list some properties of the objects from (A)-(C). For those properties which admit a short proof, we make comments immediately after the statements. The others properties we only state, mark them with sign "-" and we present the proofs in the third part, after the key proof that the mapping F has no continuous selections. The idea of the explanation is based on [F].

(1) Linearity and continuity of the relations $x_n = x_{2n} + x_{2n+1}, n \in \mathbb{N}$, show that X is a convex subcompactum of Q .

(-2) For the sets $E(X)$ and $E(Q)$ of extreme points of the convex compacta X and Q one has that $E(X) = X \cap E(Q)$.

(3) Y is a G_{δ} -subset of X because

$$x \in Y \Leftrightarrow \forall k \in \mathbb{N} \exists n \in \mathbb{N} : \frac{x_n}{g_n} > k \Leftrightarrow x \in \bigcap_{k=1}^{\infty} \left(\bigcup_{n=1}^{\infty} \{x \in X : x_n > k g_n\} \right)$$

and for every natural k and n the sets $\{x \in X : x_n > k g_n\}$ are open (and convex) subsets of X .

(4) Y is convex because $2z = x + y, x \in Y, y \in Y$ imply that for each positive C there exists $n \in \mathbb{N}$ with $x_n > 2C g_n$ and hence $z_n = 0, 5(x_n + y_n) > C g_n$.

(-5) $E(X) \subset Y$ due to (-2).

(6) (4) and (5) together imply that Y is dense in X .

(7) Linearity and continuity of the relations $x_n = 0, n \in \mathbb{N}$ show that the values $\Phi(x)$ are convex compacta. Their nonemptiness is guaranteed by the example from $x \in \Phi(x)$.

(8) The mapping $\Phi : X \rightarrow X$ is LSC. In fact, for an arbitrary $x \in X$ and $y \in \Phi(x)$ choose a basic neighborhood

$$U = U(y; N, \varepsilon) = \{z \in X : |z_i - y_i| < \varepsilon, 1 \leq i \leq N\}.$$

Let $\delta = \min\{x_i : 1 \leq i \leq N, x_i > 0\}/2$ and $x' \in V = V\{x; N, \delta\}$. Then $(x'_i = 0) \Rightarrow (x_i = 0) \Rightarrow (y_i = 0)$ for all $1 \leq i \leq N$. So, we define $z \in Q$ by the equalities $z_n = 0$

whenever $x'_n = 0$, by the equalities $z_i = y_i$ for $1 \leq i \leq N$ and for other indices $n \in \mathbb{N}$ one can define z_n such that $z \in X$. Therefore $z \in \Phi(x') \cap U$, $x \in V$, i.e. the mapping Φ is LSC at x .

(9) $F : X \rightarrow Y$ is LSC because of (6), (8), Proposition 2.3 from [M1] and the equality

$$Cl(F(x)) = Cl(\Phi(x) \cap Y) = Cl(\Phi(x)) \cap Cl(Y) = \Phi(x) \cap X = \Phi(x).$$

In particular, $F(x) \neq \emptyset, x \in X$.

(-10) $\Phi\left(\sum_{i=1}^k \alpha_i e^i\right) = conv\{e^1, e^2, \dots, e^n\}$ for any $n \in \mathbb{N}$, arbitrary extreme points e^1, e^2, \dots, e^n and for any positive α_i with $\sum_{i=1}^k \alpha_i = 1$.

(11) Finally, $F : X \rightarrow Y$ is a LSC mapping with nonempty, convex and closed (in Y) values. Suppose to the contrary that $f : X \rightarrow Y$ is a continuous singlevalued selection of F . Then f is the identity map over the set $E(X)$, because $\Phi(e) = \{e\}, e \in E(X)$.

Moreover, for any extreme points e^1 and e^2 and for the segment $[e^1; e^2]$ joining them we get $f([e^1; e^2]) \subset \Phi([e^1; e^2]) = [e^1; e^2]$. Hence $f([e^1; e^2]) = [e^1; e^2]$ due to the continuity of f and equalities $f(e^1) = e^1$ and $f(e^2) = e^2$. Similarly, $f(conv\{e^1; e^2, e^3\}) = conv\{e^1; e^2, e^3\}$. We see by induction that $f(conv(E(X))) = conv(E(X))$. Therefore convexity and compactness of X imply that

$$X = \overline{conv}(E(X)) = Cl(f(conv(E(X)))) \subset Cl(f(X)) = f(X) \subset Y \subset X \setminus \{g\}.$$

Contradiction.

To complete the proofs of (2),(5) and (10) let us temporarily say that natural numbers $2n$ and $2n + 1$ are "sons" of the number n , which in turn we shall call the "father" of such "twins". So, each natural number has exactly 2 sons, 4 grandsons, etc, and the natural partial order, say \prec , on the set \mathbb{N} , immediately arises. With respect to \prec , the set \mathbb{N} can be represented as the binary tree T (see Fig. 2).

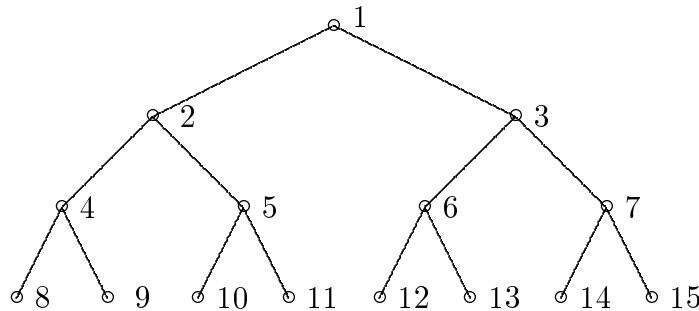


Figure 2

Every $x \in X$ is a mapping $x : T \rightarrow [0; 1]$ with $x_1 = 1, x_n = x_{2n} + x_{2n+1}, n \in \mathbb{N}$. Denote by \mathcal{M} the family of all maximal chains of T . Proofs of (2) and (5) are completely covered by the following assertion.

Assertion 2.2. For any $x \in X$ the following assertions are equivalent:

- (a) For every $n \in \mathbb{N}$ $x_n = 0$ or, $x_n = 1$;
- (b) There exists $\mu \in \mathcal{M}$ such that $x_n = 1, n \in \mu$ and $x_n = 0$ otherwise; and
- (c) $x \in E(X)$.

Proof. (a) \Rightarrow (b). By definition of X , the sum of all x_n on every horizontal level l of the tree T equals to 1. Hence, at each level l there exists a unique $x_{n_0} = 1$ and $x_n = 0$ for others n . So, passing from a previous level to the next one we define the desired maximal chain $\mu \in \mathcal{M}$.

(b) \Rightarrow (c). This is evident, because of $E(Q) \cap X \subset E(X)$ and $E(Q) = \{q \in Q; q_n = 0 \text{ or, } q_n = 1\}$;

(c) \Rightarrow (a). Suppose to the contrary, that the set $A(x) = \{n \in \mathbb{N} : 0 < x_n < 1\}$ is nonempty and $N = \min A(x)$. Then for the "twin" N' of N we clearly have that $0 < x_{N'} < 1$, too. Hence $N = 2n, N' = 2n + 1$ and $x_n = 1 = x_N + x_{N'}$. Pick $\varepsilon > 0$ such that $x_N \pm \varepsilon \in (0; 1)$ and therefore $x_{N'} \pm \varepsilon \in (0; 1)$.

Put $y_N = x_N - \varepsilon, z_N = x_N + \varepsilon, y_{N'} = x_{N'} + \varepsilon, z_{N'} = x_{N'} - \varepsilon$ and $y_k = x_k = z_k$ for all k with $n \neq k$. For "sons" of $N = 2n$ and $N' = 2n + 1$ define y_k and z_k by the following proportions

$$\frac{y_{4n}}{y_{2n}} = \frac{x_{4n}}{x_{2n}} = \frac{z_{4n}}{z_{2n}}, \quad \frac{y_{4n+1}}{y_{2n}} = \frac{x_{4n+1}}{x_{2n}} = \frac{z_{4n+1}}{z_{2n}},$$

$$\frac{y_{4n+2}}{y_{2n+1}} = \frac{x_{4n+2}}{x_{2n+1}} = \frac{z_{4n+2}}{z_{2n+1}}, \quad \frac{y_{4n+3}}{y_{2n+1}} = \frac{x_{4n+3}}{x_{2n+1}} = \frac{z_{4n+3}}{z_{2n+1}}.$$

and by similar proportions define y_k and z_k for all $k = 2^m n + r, 0 \leq r < 2^m$.

Then y and z are different points of X and $2x = y + z$ which contradicts the fact that $x \in E(X)$. \square

It now remains to check (10). To this end let e^1, e^2, \dots, e^k be extreme points of X , i.e. for some $\mu_1 \in \mathcal{M}, \dots, \mu_k \in \mathcal{M}$ we have $e_n^i = 1, n \in \mu_i$ and $e_n^i = 0$ otherwise, $i = 1, 2, \dots, k$. If $x = \sum_{i=1}^k \lambda_i e^i$ for positive λ_i with $\sum_{i=1}^k \alpha_i = 1$ then $x_n = 0 \iff n \notin \mu_1 \cup \mu_2 \cup \dots \cup \mu_k$. Therefore for $y \in \Phi(x)$ we definitely have that $y_n = 0$ provided that $n \notin \mu_1 \cup \mu_2 \cup \dots \cup \mu_k$.

For every maximal chain $\mu_i \in \mathcal{M}$ the sequence $y_n, n \in \mu_i$ is stable for all sufficiently large n , i.e. $y_n \equiv \beta_i \geq 0$. By definition of X we see that $\sum_{i=1}^k \beta_i = 1$ and $y = \sum_{i=1}^k \beta_i e^i$. Hence $\Phi(x) \subset \text{conv}\{e^1, \dots, e^k\}$. The inverse inclusion holds by the definition of the mapping Φ (see (C)).

Theorem is thus proved. \square

3. Open questions

We conclude by stating some open questions which are related to the omission of closedness in selection theorems. The first one is the hypothesis that that the affirmative answer to the G_δ -problem for an arbitrary Banach space B characterizes the C -property of the domain [RS3,p.452]:

Problem 3.1. *Let X be a paracompact space with the property that for any Banach space B and any of its G_δ -subsets Y , every LSC mapping $F : X \rightarrow Y \subset B$ with nonempty convex, closed in Y values, admits a singlevalued continuous selection. Is it true that X has the C -property? Even the cases of metric spaces, or metric compacta domains are interesting.*

The second questions concerns the C -property not for domains, but for the ranges of multivalued mappings.

Problem 3.2. *Let X be a paracompact space, B a Banach space and Y a G_δ -subset of B . Is it true that every LSC mapping $F : X \rightarrow Y \subset B$ with nonempty convex, closed in Y values admits a singlevalued continuous selection provided that all values $F(x)$ have the C -property? Again, even the cases of metric spaces, or metric compacta domains are interesting.*

The other two questions look more special.

Question 3.3. *Is it true that for any subcompactum C of a convex closed subset F of an open convex subset G of a completely metrizable linear space E the closed convex hull $\overline{\text{conv}}C$ lies in F ?*

Question 3.4. *Is it true that every LSC convex-valued mapping from the Sorgenfrey line into a nonseparable Banach space admits a continuous selection whenever all values are finite-dimensional?*

Note, that the answer to Question 3.4. is affirmative for perfectly normal domains and separable ranges [M1, Theorem 3.1"']. Naturally, a similar question concerning paracompact, but not perfectly normal domains and separable ranges is also interesting.

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