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TOPOLOGY OF 2-DIMENSIONAL COMPLEXES

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TOPOLOGY OF 2-DIMENSIONAL COMPLEXES

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ABSTRACT. We present results concerning geometric topology of 2-dimensional polyhedra, with emphasis on those related to the problem of embeddability into 3-manifolds (and thickenings), the problem of resolving arbitrary 2-polyhedra by fake surfaces (including an application to reduce the classical Whitehead asphericity conjecture to special polyhedra) and existence of nonhomeomorphic 3-manifolds with equivalent spines. We also consider different regular neighborhoods of codimension 2 embeddings of polyhedra into manifolds of any dimension. A special section is devoted to algebraic topology of 2-polyhedra, cohomology of groups and universal covers.

1. ON EMBEDDABILITY OF 2-POLYHEDRA INTO 3-MANIFOLDS

In this section, we present the basic results on embeddability of 2-polyhedra into 3-manifolds. We omit $\mathbb{Z}_2$-coefficients from the notation of (co)homology groups. In our notation and terminology, we follow [43]. Throughout this paper we shall work in the PL category. By [4] the same results hold in the topological category. A vertex of a graph is *hanging* if its degree is one. An edge of a graph is *hanging* if one of its endpoints is hanging. A *link* of a point of X is its link in some sufficiently small triangulation of X.

1.1. **Fake surfaces and special 2-polyhedra.**

A finite 2-polyhedron Q is called a *fake surface* if each of its points has a neighborhood homeomorphic to one of the following: D^2, the book with three pages ($T \times I$), or the cone over the complete graph with four vertices (or over the 1-skeleton of the 3-simplex). See Figure 1.1 for an illustration of these three types of neighborhoods.

We will refer to points in fake surfaces as points of type 1, 2 and 3, respectively depending on which of the above three neighborhoods they have. Soap films in $\mathbb{R}^3$ exhibit singularities precisely of types 2 and 3. The notion of soap films from differential geometry has proved to be an important tool and object of investigation in algebraic and geometric topology.

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Key words and phrases. special polyhedron, spine, fake surface, thickening, regular neighborhood, embedding obstruction, 3-manifold, asphericity, cohomology of group, cell-like map.

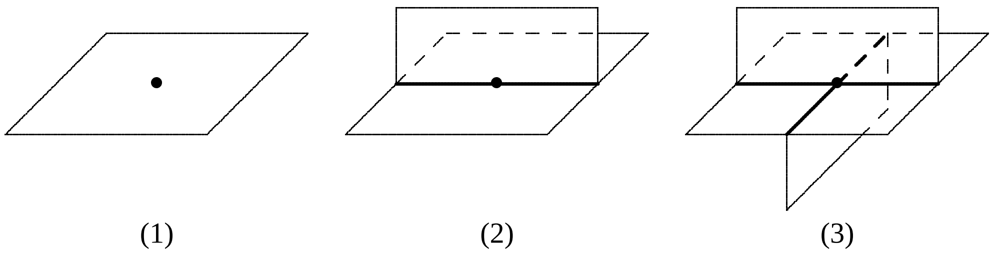

Figure 1.1

By Q' we shall denote the *intrinsic* 1-*skeleton* of a fake surface Q, i.e. the set of points of type 2 or 3. Obviously, Q' is a graph whose vertices have degrees 2 or 4. By Q'' we denote the finite set of points of Q which have type 3. A fake surface Q is called a *special* 2-*polyhedron,* if both $Q - Q'$ and $Q' - Q''$ are disjoint unions of open 2- and 1- disks, respectively (see [8]).

The following relationship exists among the classes of 2-polyhedra we shall consider:

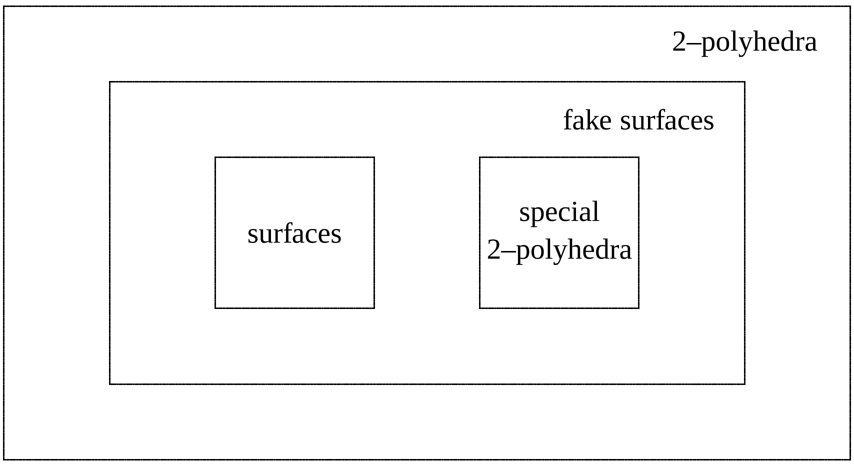

Figure 1.2

Examples of fake surfaces are the union of a torus with two disks, attached to the longitude and the meridian of the torus, and the *Bing house with two rooms* shown in Figure 1.3.

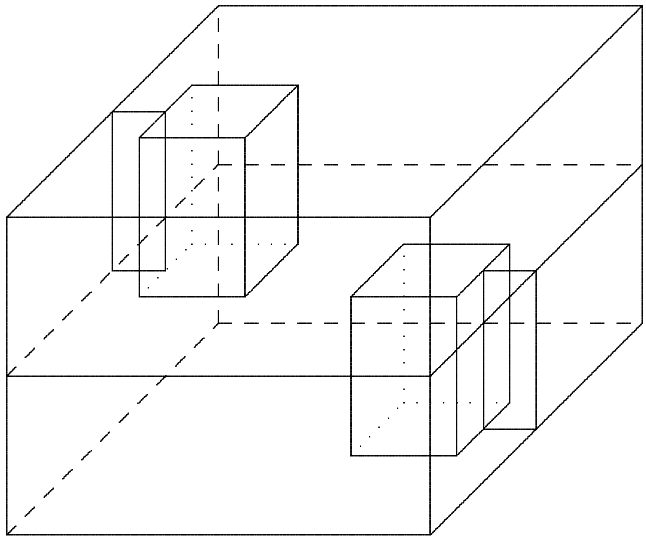

Figure 1.3

Note that the Dunce Hat (see [48, Fig. 1.3]) is not a fake surface. Fake surfaces were introduced by Ikeda [20].

1.2. **Thickenability of fake surfaces.**
A polyhedron P is said to be (orientably) *n-thickenable* if it embeds into some (orientable) n-manifold, which is not fixed in advance. An example is an embedding of the Klein bottle into some *orientable* 3-manifold. Indeed, let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then the 3-manifold

$$S^1 \times [-1,1] \times [0,1]/(z,t,0) \sim (\overline{z},-t,1)$$

is orientable and contains the Klein bottle

$$S^1 \times 0 \times [0,1]/(z,0,0) \sim (\overline{z},0,1)$$

Another example of an orientable 3-thickening of non-orientable 2-manifold is the regular neighborhood of $\mathbb{R}P^2$, standardly embedded into $\mathbb{R}P^3$. Note that any compact 2-manifold is orientably 3-thickenable. Recall that a 2-thickening μ of (or an I-bundle μ over) ∂N extends to a 3-thickening of (or extends over) N if and only if $\delta w_1(\mu) = 0 \in H^2(N, \partial N)$.

Theorem 1.1. *Let Q be a fake surface. Then*

(a$'$) Q is orientably 3*-thickenable if and only if it does not contain a union N of the Möbius band with an annulus where one of the boundary circles of the annulus attached to the middle circle of the Möbius band with a map of degree* 1 *(see* [2, 39]*).*

(a) Q is orientably 3*-thickenable if and only if the Matveev obstruction* $m(Q) \in H^1(Q')$ *is zero (see* [22, 39]*).*

(b) Q is 3*-thickenable if and only if the Matveev obstruction* $\delta m(Q) \in H^2(Q, Q')$ *is zero (see* [26, 39]*).*

Definition of $\mathbf{m(Q)}$ ***and proof of (a) and*** $(\mathbf{a}')$***.***
The Matveev obstruction $m(Q)$ is defined as follows. For every 3-thickening M of a fake surface Q, $m(Q) = w_1(M)|_{Q'}$. In order to prove that $m(Q)$ does not depend on M, observe that for a simple closed curve $J \subset Q'$ whether or not going around the curve J reverses orientation on M does not depend on M (*). Let $\gamma \in Z_1(Q')$ be a cocycle carried by J. Then the independence follows because (*) is equivalent to $m(Q) \cdot \gamma = 1$ mod 2 and to $w_1(M) \cdot \gamma = 1$.

Now let us prove that if Q does not contain N, then $m(Q) = 0$. Let T be a triod. Since lk$A$ is 3-connected for each $A \in Q'$, it follows by Menger's theorem that for each two vertices B, C of lk$A$, whose degrees are more than 2, there are three paths joining B to C and intersecting only at B, C (see [46]). Because of this, for every simple closed curve $J \subset Q'$ there is a T-fiber bundle over J, embedded in Q, whose zero-section is identified with J (see [47]).

There are three types of such bundles. They are obtained from $T \times I$ by identifying $T \times 0$ and $T \times 1$ by an self homeomorphism of T. The three types of self homeomorphisms are the identity, and a 3-cycle or 2-cycle permutation of the edges of T. If Q does not contain N, then for each J this bundle is of the first or the second type. It is easy to see that then $m(Q) = 0$. □

Note that the following plausible conjectures are false (see [39]).

(i) A fake surface is 3-thickenable if and only if it does not contain the union of the Möbius band and a 2-surface with one boundary circle (the boundary circle is attached to the middle circle of the Möbius band with a map of degree 1).

(ii) A special 2-polyhedron is 3-thickenable if and only if it does not contain the union of the Möbius band with a disk (the boundary circle of the disk attached to the middle circle of the Möbius band with a map of degree 1).

1.3. **Arbitrary polyhedra.**

Theorem 1.2. *For any* 2*-polyhedron there exists an algorithm of checking its (orientable) thickenability.*

This theorem is folklore – see the proof in [45], and also in [33]. □

Theorem 1.3. *A* 2*-polyhedron P is (orientably)* 3*-thickenable if and only if there exists a faithful embedding* $\varepsilon \in E(P)$ *such that (*$m(\varepsilon) = 0$*)* $\delta m(\varepsilon) = 0$*.*

For a proof see [39] and also [45], [35, Theorem 3.2]. □

Let us give the necessary definitions. By P' we denote the *intrinsic* 1*-skeleton* of a polyhedron P, i.e. the subpolyhedron of P formed by points having no neighborhood homeomorphic to the closed 2-disk. By P'' we denote the *intrinsic* 0*-skeleton* of P (or of P'), i.e. the finite subset of P' consisting of all points having no neighborhood in P' homeomorphic to a closed 1-disk. (or, equivalently, having no neighborhood in P homeomorphic to a book with n sheets for some $n \geq 1$). For any component of P' containing no points of P'', take a point in it. Denote by F the union of P'' and these points. Then P' is a graph whose vertices are either hanging or points of F.

Suppose that $\cup_{A\in F}\,\mathrm{lk}A$ embeds into S^2 and take a collection of embeddings $\{g_A : \mathrm{lk}A \to S^2\}_{A\in F}$. Take a non-hanging edge $d \subset P'$ and denote its vertices by A and B (possibly, $A = B$). The edge d meets $\mathrm{lk}A \cup \mathrm{lk}B$ at two points (distinct, even when $A = B$). Then, regular neighborhoods U and V of these points in $\mathrm{lk}A$ and in $\mathrm{lk}B$ are n-ods, which could be identified with the cone over $\mathrm{lk}\,d$. If for each such d the maps g_A and g_B give the same or opposite orders of rotation of the pages of the book at d then the collection $\{g_A\}$ is called *faithful*. This definition differs from the standard one - what is called *faithful* we call *orientably faithful*.

Faithful collections of embeddings $\{f_A : \mathrm{lk}A \to S^2\}_{A\in F}$ and $\{g_A : \mathrm{lk}A \to S^2\}_{A\in F}$ into (non-oriented) spheres are said to be *isopositioned*, if there is a family of homeomorphisms $\{h_A : S^2 \to S^2\}_{A\in F}$ such that $h_A \circ f_A = g_A$, for each $A \in F$. Evidently, isopositioned collections are faithful or not simultaneously. Denote by $E(P)$ the set of faithful collections up to isoposition.

The Matveev obstruction $m : E(P) \to H^1(P')$ is constructed as follows. For each $\varepsilon \in E(P)$ take its representative $\{g_A : \mathrm{lk}A \to S^2\}_{A\in F}$. For each non-hanging edge d of P', recall the rotations (the same or the opposite) from the definition of faithful collection of embeddings. Let $m(\varepsilon)$ be the class of the cocycle μ which assumes the value 0 or 1 on d if the rotations are the opposite or the same, respectively. The class $m(\varepsilon)$ is well-defined because for collections of embeddings, isopositioned via a family of homeomorphisms $\{h_A : S^2 \to S^2\}_{A\in \mathrm{lk}F}$, the cocycles μ differ by a coboundary of a cochain $\kappa \in C^0(P')$, which assumes value 1 or 0 on A depending on whether h_A reverses or preserves orientation of S^2, respectively.

For partial cases there are simpler criteria of 3-thickenability (see [35]).

2. 2-POLYHEDRA, COHOMOLOGY OF GROUPS AND UNIVERSAL COVERS

In this section, we discuss the relation between some open problems in (geometric) group theory and the proper homotopy invariants of the universal cover of a compact 2-polyhedron. We also pose new questions on 2-polyhedra derived from the above. For this, we need to recall some algebra from the topological viewpoint.

Given a group G and a $K(G,1)$-complex X, the augmented cellular chain complex of the universal cover $\tilde{X}$ of X

$$\mathcal{C} \equiv \{\cdots \to C_n(\tilde{X};\mathbf{Z}) \to \cdots \to C_0(\tilde{X};\mathbf{Z}) \to \mathbf{Z} \to 0\}$$

provides a free $\mathbf{Z}G$-resolution of $\mathbf{Z}$ (as trivial $\mathbf{Z}G$-module). This is because of the contractibility of $\tilde{X}$ and the natural free G-action we have on it.

The cohomology of $Hom_{\mathbf{Z}G}(\mathcal{C},\mathbf{Z}G)$ gives us the cohomology groups of G with $\mathbf{Z}G$-coefficients, denoted by $H^q(G;\mathbf{Z}G), q \geq 0$.

More generally, let G be a group and X be a 2-polyhedron with $\pi_1(X) \cong G$ and $\tilde{X}$ as universal cover. Under certain hypothesis, some of the proper invariants of

$\tilde{X}$ are, in fact, invariants of G which appear to be related to the low dimensional cohomology groups $H^q(G;\mathbf{Z}G)$. For instance, if G is finitely generated and the 1-skeleton of X is compact, then we can define the *number of ends of* G as the number of ends of the universal cover $\tilde{X}$ of any such 2-polyhedron X, and this number equals $1+rank(H^1(G;\mathbf{Z}G)) = 0,1,2$ *or* ∞, with $H^1(G;\mathbf{Z}G)$ being in fact free abelian (see [14]).

Recall that two proper maps $w,w' : [0,\infty) \longrightarrow \tilde{X}$ *define the same end* if their restrictions to the natural numbers are properly homotopic. The 0-ended groups are the finite groups and the 2-ended groups are those having an infinite cyclic subgroup of finite index. On the other hand, Stallings' splitting theorem characterizes those groups with more than one end (see [44] for a general reference).

One dimension up, the cohomology group $H^2(G;\mathbf{Z}G)$ of a finitely presented group G is related to the semistability at infinity of the universal cover $\tilde{X}$ of a compact 2-polyhedron X with $\pi_1(X) \cong G$, i.e., whether any two proper rays in $\tilde{X}$ defining the same end are in fact properly homotopic. If so, we say the group G *is semistable at infinity*, as this property does not depend on the choice of such a 2-polyhedron X (see [27]). Geoghegan and Mihalik [17] established the relation mentioned above by showing that if G is semistable at infinity then $H^2(G;\mathbf{Z}G)$ is free abelian.

At this point, there are two questions that still remain open:

Question 2.1. Is any finitely presented group G semistable at infinity?

Question 2.2. Is the cohomology group $H^2(G;\mathbf{Z}G)$ free abelian for any finitely presented group G?

If a finitely presented group G is semistable at infinity, then new invariants, such as proper analogues of the fundamental group, may be defined (see [16]). Mihal̃ik [28] showed that Question 2.1 may be reduced to the case G is 1-ended, and there are results in the literature (see [27]) showing that many 1-ended groups are semistable at infinity. On the other hand, Mihalik and Tschantz [29] showed that this property is preserved under amalgamated products (HNN-extensions) over finitely generated groups. All one-relator groups are also known to be semistable at infinity [30].

As indicated above, an affirmative answer to Question 2.1 would also give us an affirmative answer to Question 2.2. It is known that $H^2(G;\mathbf{Z}G)$ is torsion free [17], and Farrell [15] showed that if G contains an element of infinite order then $H^2(G;\mathbf{Z}G)$ is either $0,\mathbf{Z}$ or is not finitely generated. Of course, only those finitely presented groups which are infinite are of interest, since otherwise the above cohomology group is trivial.

It is worth mentioning that in higher dimensions ($q \geq 3$) there are finitely presented groups G for which $H^q(G;\mathbf{Z}G)$ is a finite non-trivial group (see [16]). We can translate Question 2.2 into a more geometric question Question 2.3 as follows. Let G be a finitely presented group, and X be a compact 2-polyhedron having G as fundamental group. Then, we have an isomorphism $H^2_c(\tilde{X};\mathbf{Z}) \cong H^2(G;\mathbf{Z}G) \oplus (free\ abelian)$ (see [17]), where H^*_c stands for cohomology with compact support. Thus, we have:

Question 2.3. Is the cohomology group with compact support $H^2_c(\tilde{X};\mathbf{Z})$ free abelian for any compact 2-polyhedron X?

Note that this would help us to find a condition for recognizing whether a given non-compact simply connected 2-polyhedron covers a compact polyhedron.

In trying to answer Question 2.2 (equivalently 2.3), one could ask the following stronger question for a finitely presented group G (see [23]):

Question 2.4. Does there exist a compact 2-polyhedron X having G as fundamental group and whose universal cover $\tilde{X}$ has the proper homotopy type of a 3-manifold M (with boundary)?

Indeed, an affirmative answer to Question 2.4 would also give us an affirmative answer to Question 2.3, since the cohomology group with compact support $H^2_c(\tilde{X};\mathbf{Z})$ is isomorphic to $H^2_c(M;\mathbf{Z})$. Lefschetz duality assures that $H^2_c(\tilde{X};\mathbf{Z}) \cong H_1(M,\partial M;\mathbf{Z})$ is free abelian, since simply connected manifolds are orientable. Note that we can not ask for such a proper homotopy equivalence to be equivariant under the G-action on $\tilde{X}$, for there are groups which are not 3-manifold groups.

In case there is a positive answer to Question 2.4 for G, we say that the group G is *properly* 3-*realizable.* Of course, the most intuitive way of $\tilde{X}$ having the proper homotopy type of a 3-manifold is by means of 3-dimensional thickenings, for which there is a well defined obstruction for the class of those (possibly non-compact) 2-polyhedra which are *fake surfaces* (see [23, 39]). Using this obstruction, it is shown that if G belongs to the class $\mathcal{C}$ of those groups which are the fundamental group of a compact fake surface with no points of type 3, then G is properly 3-realizable (see [23]). See also [24] for an extension of this result in case all points of type 3 are *sufficiently separated*.

On the other hand, it is shown in [6] that there are properly 3-realizable (non 3-manifold groups) which are not in the class $\mathcal{C}$. More recently, it has been proved in [1] that all 0-ended and 2-ended groups are properly 3-realizable. This gives a step towards proving that all ∞-ended groups are also properly 3-realizable, assuming one could show all 1-ended groups were properly 3-realizable. See also [1, 7] for recent results regarding the behavior of this property with respect to direct products and free products with amalgamation (and HNN-extensions). Finally, the question of whether or not every finitely presented group is properly 3-realizable remains open.

3. CONTRACTIBLE RESOLUTIONS OF 2-POLYHEDRA BY SPECIAL 2-POLYHEDRA

3.1. **Contractible resolutions.**

A *resolution* of a space P is a pair (Q, f), where Q is a space and $f : Q \to P$ is an onto map. We usually think of Q as better in some sense than P and we think of f as being a *good* map.

We construct resolutions of polyhedra by maps with simple point-inverses (i.e. contractible maps) up to polyhedra with simple singularities and structure (i.e. fake surfaces and special polyhedra). We begin with a straightforward one-dimensional analogue of our result. A graph is *cubic* if its vertices have degree 1, 2 or 3.

Proposition 3.1.
(a) Every graph is homotopy equivalent to a cubic graph.
(b) Every graph P has a resolution $f : Q \to P$ such that Q is cubic and point-inverses of f are points or arcs. Moreover, if P has no isolated points, isolated circles or hanging edges, then we can obtain Q without vertices of degree 1.

Proposition 3.1 is proved essentially by blowing up the vertices into arcs:

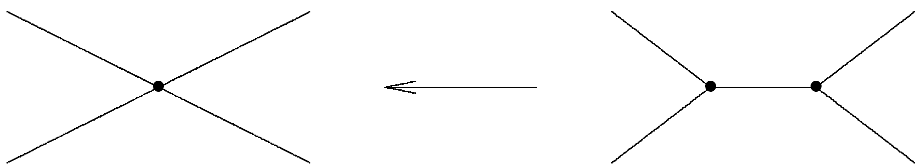

Figure 3.1

The concept of fake surfaces and special polyhedra as "general position" 2-polyhedra is formalized by the following result.

Theorem 3.2.
(a) Every finite 2*-polyhedron is homotopy equivalent to a special polyhedron. (See* [47, Proposition 1]*).*
(b) For every 2*-polyhedron P there exists a fake surface Q and an onto map $f : Q \to P$ with contractible preimages (they are actually either points or arcs or* 2*-disks). Moreover, if P is dimensionally homogeneous and its manifold set is disjoint union of open disks, then we can obtain Q to be special (see* [42, Theorem 1]*).*

The *manifold set* of a 2-polyhedron is the set of points having a neighborhood homeomorphic to D^2. A 2-polyhedron P is *dimensionally homogeneous* if every point of P has an arbitrarily small 2-dimensional neighborhood. Theorem 3.2(a) can be obtained from ([47], Proposition 1) or from Theorem 3.2(b) by applying the construction from ([19], p. 37).

We conclude this subsection with a conjecture (due to A. Onischenko) on a higher-dimensional generalization of Theorem 3.2(b). Let Θ^k be the union of S^k with $k+1$ disks D^k attached to S^k along the main equators $S^{k-1} \subset S^k$.

Conjecture 3.1. *For every n-polyhedron P there is an onto map $f : Q \to P$ with contractible preimages and such that every point $x \in Q$ has a regular neighborhood homeomorphic to the product $I^{n-k-1} \times Cone(\Theta^k)$.*

Note that the class of "resolving" polyhedra from this conjecture does not coincide with the class of higher-dimensional special polyhedra (see [26]).

3.2. **Non-existence of contractible resolutions.**
Note that in Theorem 3.2(b) the class of fake surfaces cannot be replaced by certain smaller classes of 2-polyhedra. Clearly, there exists a 2-polyhedron P which is not homotopy equivalent to a surface. Indeed, we can take any polyhedron P with $\pi_1(P) \cong \mathbb{Z}_3$, because $\mathbb{Z}_3$ is not the fundamental group of any surface.

Also, there exists a 2-polyhedron P which is not homotopy equivalent to a fake surface without points of type 3. Indeed, we can take as P a 2-spine of a homology 3-sphere, because [23, Proposition 1.1] asserts that the fundamental group of a fake surface without points of type 3 cannot be a non-trivial perfect group.

Note that a point has a *contractible* resolution by a special 2-polyhedron (e.g. by the Bing house with two rooms) but has no *collapsible* resolution by a special 2-polyhedron. This is because every collapsible polyhedron has a “free” face and hence cannot be special.

Also, the 2-sphere S^2 has a contractible resolution by a special 2-polyhedron. The union of a torus with two disks, attached to the longitude and the meridian of the torus, is mapped to S^2 by shrinking each disk to a point (see Figure 3.2).

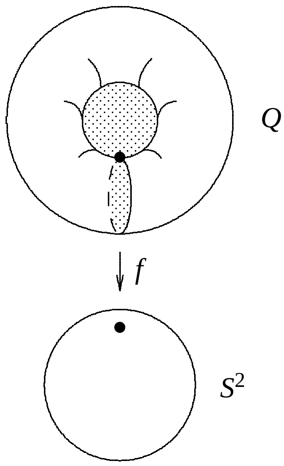

Figure 3.2

We conjecture that for a connected 2-polyhedron, distinct from the point and S^2, the conditions of Theorem 3.2(b) are not only sufficient but also necessary for the existence of a contractible resolution by a special 2-polyhedron.

3.3. **Applications of contractible resolutions.**
The *Whitehead Asphericity Conjecture* states that any subcomplex of an aspherical 2-complex is itself aspherical. Whitehead stated this conjecture in 1930’s motivated by the problem of description of relative π_2 in terms of generators and relators. From a more general perspective, computation of π_2 of 2-complexes, including the asphericity of 2-complexes, is a difficult problem that lies at the heart of the general problem of computing the homotopy groups of 2-complexes.

Examples of aspherical 2-complexes are all surfaces of genus > 0 and also spines of knot spaces. It is interesting to note that Whitehead’s question was also motivated by unsuccessful attempts at that time to prove the asphericity of knot spaces. This was verified much later in the 1950's by Papakyriakopoulous using the Sphere theorem.

It follows from Theorem 3.2(b) that in order to prove the Whitehead asphericity conjecture, it would suffice to verify it only for the case of fake surfaces (see [42]). Indeed, let $f : Q \to P$ be a resolution provided by Theorem 3.2(b). Since the restriction of f to the preimage $f^{-1}(P_1)$ of every subpolyhedron $P_1 \subset P$ is a homotopy equivalence (see [21]), the reduction would then follow immediately.

Another application of Theorem 3.2(b) is motivated by the well-known fact that every 2-manifold embeds into $\mathbb{R}^4$:

Theorem 3.3. *There exists a fake surface (even a special* 2*-polyhedron)* Q *which does not embed into* $\mathbb{R}^4$.

Proof. Indeed, let Q be a resolution, given by Theorem 3.2(b), of the 2-skeleton P of the standard 6-simplex. Suppose that Q embeds into $\mathbb{R}^4$. It follows from the proof of Theorem 3.2(b) that the non-trivial preimages of the resolution are those of the points of the 1-skeleton of P. Hence by contracting in $\mathbb{R}^4$ the preimages of the resolution we obtain $\mathbb{R}^4$ in which P is embedded. This follows from the 1-LCC shrinking theorems for ANR's and the remark at the bottom of p.2 in [3]. The latter is well-known to be impossible. □

It is worth mentioning that the resolutions obtained from Theorem 3.2(b) are special cases of *cell-like resolutions*, which play an important role in geometric topology (see [32]). A polyhedron is said to be *cell-like* if and only if it is contractible. Note that this definition agrees with the standard one, since polyhedra are ANR's. An onto map is said to be *cell-like*, if it is a proper surjective map with cell-like point-inverses.

3.4. **Proof of Theorem 3.2(b).**

Our proof is an application of the so-called *Banana and pineapple trick* (see Figure 3.3). Observe that the (harder) construction of a special spine from the fake surface spine can be mimicked to obtain *some* resolution of P. However, certain fibers of this resolution are circles, and hence this resolution is by no means *cell-like*.

Step 1. Blow up every vertex to a disk. Then enlarge every 1-simplex to a ribbon, joining the disks which lie over the respective vertices, so that you get a disk with handles.

Step 2. Next, choose circles, which lie on this disk with handles, so that they intersect transversely, one circle for each 2-simplex below.

Step 3. Finally, attach 2-disks along these circles, so they intersect transversely, and define the resolution map to bijectively map each 2-disk to the corresponding 2-simplex below.

Step 4. Verify the properties of the resolution.

For the proof of the "moreover" part, see [42].

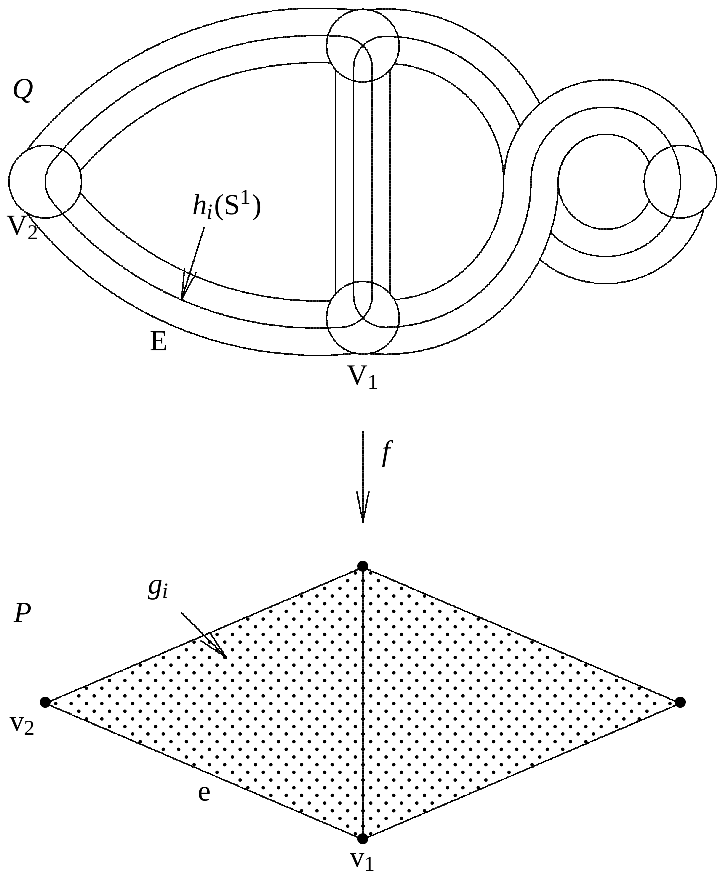

Figure 3.3

4. Regular neighborhoods of codimension 2 polyhedra

A polyhedron K in a closed 3-manifold M is called a *spine* of M, if there is a 3-ball $D \subset M - K$ such that $(M - D) - K \cong \partial D \times [0, 1)$. This property is equivalent to the property that $M - D$ collapses to K. The Bing house with 2 rooms is an example of a special spine of the 3-sphere.

A folklore result asserts that each 3-manifold has a spine which is a special polyhedron [19]. Casler [8] proved that if two 3-manifolds have the same special spines, then they are homeomorphic. Casler's uniqueness theorem was generalized by Brodsky-Repovš-Skopenkov [39] to a classification of 3-manifolds with the same (not necessarily special) spine, up to a homeomorphism, relative to the spine.

However, a 3-manifold can have several spines. Matveev [26] proved that if special 2-polyhedra P_1 and P_2 are spines of the same 3-manifold, then P_1 can be obtained from P_2 by *Tietze moves*. Conversely, non–homeomorphic 3-manifolds can have the same (non-special) spine: for 3-manifolds with boundary of genus one this was first proved by Mitchell, Przytycki, and Repovš [31], and for 3-manifolds with boundary of higher genera by Cavicchioli, Lickorish, and Repovš [11].

Recently, Hog-Angeloni and Glock [18] have shown that there are four obstruction to uniqueness of regular neighborhoods for compact connected 2-complexes in compact, connected orientable 3-manifolds. If the 3-manifold is prime and not a Poincaré counterexample, and if the regular neighborhood does not contain essential annuli and has connected boundary, then the regular neighborhood is determined by the 2-complex (see also earlier related results [5], [37] and [38]).

More generally, let us consider inequivalent embeddings of codimension 2 polyhedra into arbitrary PL manifolds. For a polyhedron $X \subset M$ we denote by $R_M(X)$ the regular neighborhood of X in M. Consider the following problem (for dimension 3 see [38]): *Find all pairs* (m,k) *such that if* K *is a compact* k*-polyhedron and* M *a PL* m*-manifold, then*

(*) $R_M(fK) \cong R_M(gK)$*, for each two homotopic PL embeddings* $f,g : K \to M$.

The property (*) holds for $m \geq 2k+2$ by general position. In general, many invariants of $R_M(fK)$ and $R_M(gK)$ coincide: the homotopy classes, the homology rings, the higher Massey products, the (classifying maps of) tangent bundles (and hence also all the invariants deduced from the tangent bundle, e.g. characteristic classes and numbers). This implies that (*) also holds for $m = 2k+1$ (since in this case an m-thickening is completely determined by its tangent bundle [25] and for $m = 2$ (since a 2-manifold N with boundary is completely determined by $H_1(N)$ and the intersection form). Also, for $l \geq 2k+1-m$,

$$R_M(fK) \times I^l \cong R_{M\times I^l}(fK) \cong R_{M\times I^l}(gK) \cong R_M(gK) \times I^l.$$

The homology ring of $\partial R_M(fK)$ does not depend on a homotopy of f (this follows from the exact sequence of the pair $(R_M(fK), \partial R_M(fK))$ and the Poinaré duality). Also, $\partial R_M(fK)$ is l-connected for $m \geq k+l+2$ and l-connected K. This shows that distinguishing $R_M(fK)$ and $R_M(gK)$ is a non-trivial problem for $m \geq k+3$. The property (*) holds for $m = k+1 \geq 3$ and a *fake surface* K (see [8], [26], [38], [39]).

The property (*) fails for:

a) $m = k+1 \geq 3$, $M = S^m$ and $K = S^k \vee S^1 \vee S^1$ (this was proved using non connectedness of $\partial R_M(fK)$) (see [38]),

b) $M = S^3$, some K and $R_M(fK), R_M(gK)$ complements of knots (see [31], [11]), and

c) $M = S^4$ and a certain *2-polyhedron* (the Dunce Hat) K (this was proved by means of $\pi_1(\partial R_M(fK))$) (see [48]).

The same problems for $M = S^m$ (then f and g are always homotopic) and also for K a PL manifold are also interesting. A version of this problem, when the homeomorphism $R_M(fK) \cong R_M(gK)$ is required to be an extension of the homeomorphism $g \circ f^{-1} : fK \to gK$, is better known [25]. For further results on these problems see [9], [10], [12], [13], [34], [36], [40], and [41].

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