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SOLVING FOUR-DIMENSIONAL SURGERY PROBLEMS USING CONTROLLED THEORY

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Abstract. In this paper the controlled surgery sequence of Ranicki, Pedersen and Quinn is applied to solve surgery problems in dimension four when the fundamental group is not known to be good. Our examples concern free nonabelian fundamental groups, surface fundamental groups, and special knot groups. Using results from our earlier paper (joint with Spaggiari) we state a general result from which our examples follow.

§ 1. Introduction

A surgery problem is written as a diagram

$$\begin{array}{ccc} \nu_M & \xrightarrow{\ b\ } & \xi \\ \big\downarrow & & \big\downarrow \\ M^n & \xrightarrow{\ f\ } & X^n \end{array}$$

where M^n is an n–manifold, ν_M the stable normal bundle of an embedding $M^n \subset \mathbb{R}^{n+l}$ (l large), an l–bundle ξ over an n–dimensional Poincaré complex X (in the sense of Wall [Wal]), a degree 1–map f and a bundle map $b : \nu_M \to \xi$ over f (being fiberwise an isomorphism). We assume that M^n is a closed topological manifold of dimension n.

There is an obvious notion of normal bordism between such surgery problems. Disjoint union defines a sum of equivalence classes. Thus defined bordism group is denoted $\Omega_n(X, \xi)$. Further, ν_M and ξ are topological $\mathbb{R}^l$–bundles. If we take the "union" of all $\Omega_n(X, \xi)$ with respect to all possible $\mathbb{R}^l$–bundles ξ over X we get a set which is in bijective correspondence with $[X, G/TOP]$. This bijective correspondence is not canonical. It depends on fixing a specific surgery problem, or equivalently a TOP reduction of the Spivak normal fibration over X (see [Wal], §10). So we have this correspondence if there is at least one surgery problem whose target is X.

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For any surgery problem (f, b) as described above there is a well-defined surgery obstruction in the Wall group $\Theta(f, b) \in L_n(\pi_1(X))$ such that: $\Theta(f, b) = 0$ if (f, b) is normally cobordant to

$$\begin{array}{ccc} \nu_M & \xrightarrow{c} & \xi \\ \downarrow & & \downarrow \\ N^n & \xrightarrow{g} & X \end{array}$$

where g is a simple homotopy equivalence. In the case $n \geq 5$ this is a necessary and sufficient condition. If $n \geq 5$ and $\Theta(f, b) = 0$ then a normal cobordism between (f, b) and (g, c) is constructed by a sequence of surgeries.

The surgery obstruction $\Theta(f, b) \in L_n(\pi_1(X))$ defines a map

$$\Theta : [X, G/TOP] \to L_n(\pi_1(X)).$$

The Wall group depends only on the fundamental group of X, and the orientation character $w : \pi_1(X) \to \{\pm 1\}$, which we shall ignore.

For $n \geq 5$ surgery theory is expressed as an exact sequence:

$$\mathcal{S}(X) \xrightarrow{\eta} [X, G/TOP] \xrightarrow{\Theta} L_n(\pi_1(X))$$

where $\mathcal{S}(X)$ is the structure set. An element of $\mathcal{S}(X)$ is represented by a simple homotopy equivalence $h : M \to X$, and $h : M \to X$ is equivalent to $h' : M' \to X$ if there is a homeomorphism $\phi : M' \to M$ such that $h \circ \phi$ is homotopic to h'. The map η associates to every (simple) homotopy equivalence its normal invariant: Let $h^{-1} : X \to M$ denote a homotopy inverse of $h : M \to X$, $\xi = (h^{-1})^*(\nu_M)$. Note that $h^*(\xi) = (h^{-1} \circ h)^*(\nu_M) \cong \nu_M$. Then $\eta[h : M \to X]$ is the surgery problem

$$\begin{array}{ccc} \nu_M \cong h^*(\xi) & \xrightarrow{\overline{h}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{h} & X \end{array}$$

where $\overline{h}$ is the canonical map. The surgery sequence can be extended to the left by a map $L_{n+1}(\pi_1(X)) \to \mathcal{S}(X)$. This will not be a subject of our paper.

It is not known if in dimension $n = 4$ the sequence exists in the general situation. By Freedman's result it holds for Poincaré 4–complexes with "good" fundamental groups (see [Fre], [FreQui], [FreTei1], [FreTei2], [KruQui]). However, there is a controlled surgery sequence in dimension 4 (see [PedQuiRan]). We will explain it in more detail in §3. The surgery obstruction group of this sequence not only depends on $\pi_1(X)$ but also on the topology of the space X, more precisely – on the control map $p : X \to B$ which must satisfy the UV^1–property. The UV^1–property is not an invariant of the homotopy type of X. Given X, there may exist a homotopy equivalent Poincaré complex Y, for which one can construct a good UV^1–map $Y \to B$. Therefore we have to transform our original surgery problem

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{f} & X^n \end{array}$$

into a surgery problem with target Y without changing the surgery obstruction $\Theta(f,b)$. This is well-known, in fact it follows from the formula for the surgery obstruction of a composition of normal maps (see [Wal], Chapter 17H, Lemma 2; [Ran1], Proposition 43; or [Jon], p. 322). Since this issue is basic in dimension 4 (on p. 264 of [Wal] it is stated that "This is clearly a basic result which should have been treated earlier in this book."), we will give a proof (different from the proofs cited above) in §2.

In §4 we will construct examples of UV^1–maps which in §5 will be used to state and prove the main results of this paper.

§ 2. Invariance of surgery obstructions under homotopy equivalence

We are only interested in 4–dimensional surgery, so we can restrict the discussion to $n = 2k$. Let the surgery problem be given

$$(*)\qquad \begin{array}{ccc} \nu_M & \xrightarrow{\ b\ } & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{\ f\ } & X^n \end{array}$$

Let Y^n be another n–Poincaré complex and $h : X \to Y$ be a (simple) homotopy equivalence. Our goal is to use h to transform it into a surgery problem

$$\begin{array}{ccc} \nu_M & \xrightarrow{\ c\ } & \eta \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{\ g\ } & Y, \end{array}$$

($g = h \circ f$ and the bundle map c has yet to be defined) so that $\Theta(g,c) = \Theta(f,b)$ under the identification $L_n(\pi_1(Y)) \leftrightarrow L_n(\pi_1(X))$, induced by $h_* : \pi_1(X) \underset{\cong}{\to} \pi_1(Y)$.

Let us assume that f is k-connected. Then $[\pi_{k+1}(f), \lambda, \nu] \in L_n(\pi_1(X))$ represents $\theta(f,b)$, where

$$\lambda : \pi_{k+1}(f) \times \pi_{k+1}(f) \to \Lambda = \mathbb{Z}[\pi_1(X)]$$

$$\mu : \pi_{k+1}(f) \to \Lambda/_{\{a-(-1)^k\overline{a}\ \mid\ a\in\Lambda\}} = Q_k$$

are the intersection and self-intersection forms. The operation $\overline{a}$ denotes the canonical anti-involution of Λ. To define $c : \nu_M \to \eta$ we choose a homotopy inverse $h^{-1} : Y \to X$ and homotopies

$$H : X \times I \to X \text{ of } h^{-1} \circ h \text{ and } \mathrm{Id}_X,$$

$$G : Y \times I \to Y \text{ of } h \circ h^{-1} \text{ and } \mathrm{Id}_Y.$$

Let $\eta = (h^{-1})^*(\xi)$ and $c : \nu_M \to \eta$ be the composition $\nu_M \xrightarrow{b} \xi \cong h^*(\eta) \xrightarrow{\overline{h}} \eta$, where $\overline{h}$ is the canonical bundle map and the isomorphism $\xi \cong h^*(\eta)$ is induced by the isomorphism $H^*(\xi) \cong \xi \times I$ being the identiy for $t = 0$ (i.e., $H_0^*(\xi) = \xi \times 0$, $H_1^*(\xi) = (h^{-1} \circ h)^*(\xi) = h^*(\eta)$). Then

$$\begin{array}{ccccc} \nu_M \times I & \xrightarrow{b\times 1} & \xi \times I \cong H^*(\xi) & \xrightarrow{\overline{H}} & \xi \\ \downarrow & & \downarrow & & \downarrow \\ M \times I & \xrightarrow{f\times 1} & X \times I & \xrightarrow{H} & X \end{array}$$

is a normal cobordism between

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \cong h^*(\eta) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X. \end{array}$$

hence the surgery obstructions coincide. Therefore we can assume that $\xi = h^*(\eta)$ and $c : \nu_M \to \eta$ is the composition $c = \overline{h} \circ b$. Since $h : X \to Y$ is a simple homotopy equivalence, the homomorphism

$$h_* : \pi_{k+1}(f) \to \pi_{k+1}(h \circ f)$$

is an isomorphism respecting the preferred bases and intersection forms. So the only difference can appear in the self-intersection forms $\mu : \pi_{k+1}(f) \to Q_k$ and $\mu' : \pi_{k+1}(f) = \pi_{k+1}(h \circ f) \to Q_k$.

However, in order to define the surgery obstructions we first have to make surgery below the middle dimension. In that range surgeries use framed embeddings $S^p \times D^{n-p} \to M$, and these are defined using $b : \nu_M \to \xi$. So different bundle maps can produce different Λ–modules $\pi_{k+1}(f)$. Moreover, the self intersection form μ is defined representing elements $x \in \pi_{k+1}(f)$ by framed immersions of type

$$\begin{array}{ccc} S^k \times D^k & \looparrowright & M \\ \downarrow & & \downarrow \\ D^{k+1} \times D^k & \longrightarrow & X \end{array}$$

These framed immersions are defined using $b : \nu_M \to \xi$. This goes as follows: Let $x \in \pi_{p+1}(f)$ be represented by

$$(**) \qquad \begin{array}{ccccc} S^p & \xrightarrow{\varphi} & M & \longleftarrow & \nu_M \\ \big\downarrow & & \big\downarrow f & & \big\downarrow \\ D^{p+1} & \xrightarrow{\psi} & X & \longleftarrow & \xi \end{array}$$

We can assume that $p \leq n - 2$. Embedd $M^n \subset \mathbb{R}^{n+l}$, l large, hence

$$\tau_M \oplus \nu_M = \varepsilon_M^{n+l}.$$

Then we have $\varphi^*(\tau_M) \oplus \varphi^*(\nu_M) = \varepsilon_{S^p}^{n+l} = \tau_{S^p} \oplus \varepsilon_{S^p}^{n+l-p}$ in a canonical way. But from diagram $(**)$ we get an isomorphism $\varphi^*(\nu_M) = \psi^*(\xi)|_{S^p} \cong \varepsilon_{S^p}^l$. Together this gives a fiberwise isomorphism

$$\begin{array}{ccc} \tau_{S^p} \oplus \varepsilon^{n+l-p} & \longrightarrow & \tau_M \oplus \varepsilon_M^l \\ \downarrow & & \downarrow \\ S^p & \xrightarrow{\varphi} & M \end{array}$$

which can be uniquely destabilized to

$$\begin{array}{ccc} \tau_{S^p} \oplus \varepsilon^{n-p} & \longrightarrow & \tau_M \\ \downarrow & & \downarrow \\ S^p & \xrightarrow{\varphi} & M \end{array}$$

because $p \leq n-2$ (so $\pi_p(BO(n)) \underset{\cong}{\to} \pi_p(BO)$). By the Häfliger–Hirsch theorem this defines an immersion $S^p \times D^{n-p} \looparrowright M$ unique up to regular homotopy.

It is now clear that we obtain the same regular immersion $S^p \times D^{n-p} \looparrowright M$ using diagram

$$(**')\qquad \begin{array}{ccccc} S^p & \xrightarrow{\varphi} & M & \longleftarrow & \nu_M \\ \downarrow & & \downarrow & & \downarrow b \\ D^{p+1} & \xrightarrow{\psi} & X & \longleftarrow & \xi \\ & \searrow & h\downarrow & & \downarrow \overline{h} \\ & & Y & \longleftarrow & \eta \end{array}$$

These completes the proof that $\Theta(f, b) = \Theta(h \circ f, c)$, i.e. composition with a homotopy equivalence $X \to Y$ defines a surgery problem with target Y and same surgery obstruction.

If $\Theta(f, b) = 0$ there exists a preferred Λ–base $\{e_1, \ldots, e_r, f_1, \ldots, f_r\}$ (after stabilization) such that $\lambda(e_i, f_j) = \delta_{ij}$, $\lambda(x, y) = 0$, for all other $x, y \in \{e_1, \ldots, e_r, f_1, \ldots, f_r\}$, and $\mu(e_i) = \mu(f_i) = 0$, $i = 1, \ldots, r$. Using the Whitney trick ($k \geq 3$), the regular homotopy class of immersion $S^k \times D^k \looparrowright M$ of any e_i contains an embedding on which surgeries are performed to obtain a homotopy equivalence.

§ 3. Controlled surgery theory

In this section we will describe the controlled surgery sequence of Pedersen-Quinn-Ranicki (see [PedQuiRan]). Its advantage is that it holds in dimension 4. Then we explain how it can be used to solve 4–dimensional surgery problems.

The Wall groups of the trivial group are

$$L_n(\{1\}) = \begin{cases} \mathbb{Z} & n \equiv 0(4) \\ \mathbb{Z}_2 & n \equiv 2(4) \\ 0 & n \text{ odd.} \end{cases}$$

There is a 4–periodic Ω–spectrum $\mathbb{L}$ such that $\pi_n(\mathbb{L}) = L_n$ (see [Qui1], [Nic] or [Ran2] for an algebraic approach). $\mathbb{L}$–homology of a space B can be computed using an Atiyah-Hirzebruch spectral sequence:

$$E^2_{pq} = H_p(B, \pi_q(\mathbb{L})) \Rightarrow H_{p+q}(B, \mathbb{L}).$$

The space B will be the control space, and it has to be a compact metric ANR space of finite dimension. Hence it is homotopy equivalent to a finite complex (following by the Borsuk conjecture proved by West [Wes]). To calculate $H_n(B, \mathbb{L})$ we can therefore assume that B is a finite complex. In fact, taking a regular neighborhood in some $\mathbb{R}^l$ we may assume that B is a triangulated manifold. An n–cycle x representing an element $[x] \in H_n(B, \mathbb{L})$ is a family of surgery problems

$$x(\sigma) : \quad \begin{array}{ccc} \nu_{M_\sigma} & \xrightarrow{b_\sigma} & \xi_\sigma \\ \downarrow & & \downarrow \\ M_\sigma & \xrightarrow{f_\sigma} & X_\sigma, \end{array}$$

σ a simplex in B, and $\dim M_\sigma = n - \dim \sigma$, together with a reference map $X_\sigma \to \sigma^*$ (σ^* the dual cell of σ). The cycle condition and the fact that B is a manifold implies that one can paste these together to a global surgery problem

$$\begin{array}{ccccc} \nu_M & \xrightarrow{b} & \xi & & \\ \downarrow & & \downarrow & & \\ M & \xrightarrow{f} & X & \longrightarrow & B \end{array}$$

over B. With this notations one can define the assembly map

$$A : H_n(B, \mathbb{L}) \to L_n(\pi_1(B))$$

by $A([x]) =$ image of $\Theta(f, b) \in L_n(\pi_1(X)) \to L_n(\pi_1(B))$ (see [Nic], Ch. 3 and [Ran2], Ch. 12).

Let us suppose that we have a map $p : X \to B$ from an n–Poincaré complex in B. By a theorem of Cohen (see [Coh]), p can be assumed to be transverse to any σ^*, $\sigma \in B$. Let $X_\sigma = p^{-1}(\sigma^*)$. Suppose that given any normal degree 1 map

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

such that $p \circ f$ is transverse regular for any $\sigma^* \subset B$. We get from it a family x of surgery problems

$$x(\sigma) : \quad \begin{array}{ccc} \nu_{M_\sigma} & \xrightarrow{b_\sigma} & \xi_\sigma \\ \downarrow & & \downarrow \\ M_\sigma & \xrightarrow{f_\sigma} & X_\sigma, \end{array}$$

where σ is a simplex in B, and a map $X_\sigma \to \sigma^*$. Note that $\dim M_\sigma = n - |\sigma| = \dim X_\sigma$. This defines a map

$$\Theta_B : [X, G/TOP] \to H_n(B, \mathbb{L}).$$

The construction shows that if $p_* : \pi_1(X) \to \pi_1(B)$ is an isomorphism then $A \circ \Theta_B = \Theta$ so for a given $[f, b] \in [X, G/TOP]$ the element $\Theta_B([f, b]) \in H_n(B, \mathbb{L})$ is a stronger obstruction than $\Theta([f, b])$.

If $\Theta([f, b]) = 0$ then surgery can be completed to get homotopy equivalences for each individual surgery problem $x(\sigma)$, if $n \geq 5$. This is because $\Theta_B([f, b]) = 0$ is equivalent to: $x(\sigma)$ normally bounds a surgery problem $y(\sigma)$ over σ^* for any simplex $\sigma \subset B$. This also works for $n = 4$ if the fundamental groups are good for all $x(\sigma)$. The family $\{y(\sigma) \mid \sigma \subset B\}$ can be assembled to give a normal surgery problem

$$\begin{array}{ccc} \nu_N & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ N^{n+1} & \longrightarrow & X \end{array}$$

which bounds (f, b).

Suppose $\Theta_B([f, b]) = 0 \in H_n(B, \mathbb{L})$. We assume that $p : X \to B$ satisfies $\pi_1(X) \underset{\cong}{\xrightarrow{p_*}} \pi_1(B)$. Instead of trying to glue together the traces of the individual surgery problems on $x(\sigma)$, one does small controlled surgeries on the global problem (f, b), so that they perhaps can be considered as a surgery over some σ. Completing controlled surgeries leads to a controlled surgery obstruction group $L_n(B, p, \varepsilon, \delta)$. Of course, if this obstruction vanishes we get also a controlled (simple) homotopy equivalence (using the controlled Hurewicz-Whitehead theorem).

Then there exists the assembly map $A_{Y_a} : H_n(B, \mathbb{L}) \to L_n(B, p, \varepsilon, \delta)$ defined by Yamasaki [Yam1], and it can be proved that it is an isomorphism for suitable ε, δ. This was done recently by Pedersen-Quinn-Ranicki (see [PedQuiRan]). An alternative approach was given by Ferry [Fer]. The proofs are given under the hypothesis that $p : X \to B$ is UV^1 (or $UV^1(\delta)$) for a sufficiently small δ, i.e. the local surgery problems $x(\sigma)$ all have trivial fundamental group. This also makes it possible in dimension 4 by work of Quinn. The general case, where the local fundamental group is nontrivial, or even nonconstant remains unsettled (cf. [Yam2] for difficulties given in the general case).

Before we state the result of [PedQuiRan] we recall the definition of a UV^1–map.

Definition. A continuous map $p : X \to B$, B a metric space with metric d, is said to be a $UV^1(\delta)$–map, $\delta > 0$, if every commutative diagram of continuous maps

$$\begin{array}{ccc} K_0 & \xhookrightarrow{x_0} & X \\ \downarrow & & \downarrow \\ K & \xrightarrow{\alpha} & B \end{array}$$

where K is a 2–complex and $K_0 \subset K$ a subcomplex, can be completed by $\overline{\alpha} : K \to X$ such that $\overline{\alpha}|_{K_0} = \alpha_0$ and $d(p\,\overline{\alpha}(u), \alpha(u)) < \delta$, for any $u \in K$. A UV^1–map is a $UV^1(\delta)$–map for any $\delta > 0$.

This is the definition used in controlled topology and algebra. There is another aspect of UV^1–maps useful for the application which will be explained in §4.

Remark. The $UV^1(\delta)$–property of p depends strongly on the topology and metric of the space B. Changes of B we made before, in order to define geometrically the assembly map, disturb it. Doing controlled surgery, one has to stick to the $UV^1(\delta)$–map. There is also a spectral definition of the assembly map given, for instance, in [Ran2], §12.

Theorem 3.1. (ε–δ–surgery sequence [PedQuiRan]) *Let B be a finite dimensional compact metric ANR and $n \geq 4$. Then there exist $\varepsilon_0 > 0$, depending on B and n, such that for all $\varepsilon > 0$ with $0 < \varepsilon < \varepsilon_0$, there exists $\delta > 0$ with the following property:*

If X^n is a (closed) n–manifold, $p : X \to B$ is a $UV^1(\delta)$–map, then the following controlled surgery sequence is exact:

$$H_{n+1}(B, \mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(X, p) \xrightarrow{\eta} [X, G/TOP] \xrightarrow{\Theta_B} H_n(B, \mathbb{L}).$$

Here $\mathcal{S}_{\varepsilon,\delta}(X, p)$ are equivalence classes of (M, g), M a closed n–manifold, $g : M \to X$ a δ–homotopy equivalence over $p : X \to B$ (i.e., there exists a homotopy inverse $g^{-1} : X \to M$ and homotopies $H : M \times I \to M$ and $G : X \times I \to X$ between $g^{-1} \circ g$ and Id_M, $g \circ g^{-1}$ and Id_X such that $\{d(p\ circf \circ H(x,t)) \mid t \in I\} \subset B$ and $\{d(p \circ G(y,t)) \mid t \in I\} \subset B$ have diameter $< \delta$ for any $x \in M$, $y \in X$ respectively). Two elements (M, g), (M', g') are equivalent if there is a homeomorphism $h : M \to M'$ such that

$$\begin{array}{ccc} M & \xrightarrow{h} & M' \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle g'} & \\ & X & \end{array}$$

is ε–commutative over $p : X \to B$ (i.e., for any $x \in M$, $d(pg(x), pg'h(x)) < \varepsilon$).

This relation is reflexive and symmetric. It is part of the theorem that it is also transitive.

It should be remarked that the proof given in [PedQuiRan] holds also when X^n is a δ–Poincaré complex with a sufficiently small δ. In particular, it holds when X^n is a generalized n–manifold. The map $H_{n+1}(B, \mathbb{L}) \to [X, G/TOP]$ is a controlled version of Wall's realization of surgery obstruction. Under the assumptions of the theorem it is clear that we have the following commutative diagram (we now consider only $n = 4$)

$$\begin{array}{ccccc} \mathcal{S}_{\varepsilon,\delta}(X) & \xrightarrow{\eta} & [X, G/TOP] & \xrightarrow{\Theta_B} & L_4(B, \mathbb{L}) \\ \downarrow & & \| & & \downarrow{\scriptstyle A_B} \\ \mathcal{S}(X) & \xrightarrow{\eta} & [X, G/TOP] & \xrightarrow{\Theta} & L_4(\pi_1(B)) \end{array}$$

The left vertical map is just forgetting ε, δ.

This leads to the following program: Given a 4–dimensional surgery problem

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{f} & X^n \end{array}$$

with surgery obstruction $\mathcal{O}$, construct B (as in the theorem) and a map $p : X \to B$ such that

(1) p is UV^1; and
(2) A_B is injective.

Then surgery on (f, b) can be completed to get a δ–homotopy equivalence over B. Note the two extreme cases:

(a) $p : X \to \{*\}$,
(b) $p = \mathrm{Id} : X \to X$,

where in (a) p (in general) is not UV^1 and in (b) A_B is (in general) not injective. So B has to sit "well–balanced" between $\{*\}$ and X.

We will give examples in §4. For a general manifold X^4 no construction for such $p : X \to B$ is in sight. At this point we have to mention that Frank Quinn has an approach to solve a generic 4–dimension surgery problem over a Poincaré 4–complex X with vanishing total surgery obstruction [Qui2].

§ 4. Examples of UV^1–maps

The examples given here are not particulary difficult, but using results of [CavHeg], [CavHegRep] and [HegRepSpa], they lead to interesting results. First we will describe an alternative aspect of UV^1–maps. A subset $A \subset X$ has the UV^1–property if for each neighborhood U of A there is a neighborhood V of A such that $V \subset U$, $\pi_1(V) \to \pi_1(U)$ is zero for any point in V, and any two points $x, y \in V$ can be joint by an arc in U. The following theorem is a special case of the approximate lifting theorem (see [Dav], p. 126).

Theorem 4.1. *Suppose X is a metric space and G is an upper semicontinuous UV^1–decomposition of X (i.e. each member $A \in G$ is a UV^1–subset). Let $B = X/G$, and $p : X \to B$ the projection. Then p is a UV^1–map.*

As corollaries we immediately obtain the following:

Lemma 4.2. *Let M^n be an n–manifold which is homeomorphic to a connected sum $M_1^n \# M_2^n$. If $h : M \to M_1 \# M_2$ is a homeomorphism, then the composition*

$$p : M \xrightarrow{h} M_1 \# M_2 \xrightarrow{c} M_1 \vee M_2$$

is UV^1, where c is the collapsing map and $n \geq 3$.

Proof. The inverse images of c are single points or S^{n-1}, so gives a UV^1 decomposition. □

Lemma 4.3. *Suppose M is homeomorphic to conected sum $M_1 \# M_2$ with $\pi_1(M_2) = \{1\}$. If $h : M \to M_1 \# M_2$ is a homeomorphism then $p : M \xrightarrow{h} M_1 \# M_2 \xrightarrow{c} M_1$ is UV^1, where c is the collapsing map.*

Proof. As above. □

Lemma 4.4. *Let $p : X \to B$ be a fibration between manifolds with simply connected fibers. Then p is UV^1.*

Proof. We may choose tubular neighborhoods of the fibers to see that $\{p^{-1}(b) \mid b \in B\}$ is a UV^1–decomposition. □

Since a compositions of UV^1–maps is again a UV^1 map we get in particular:

Lemma 4.5. *Let M^4 be homeomorphic to $(\mathop{\#}\limits_1^r S^1 \times S^3)\#M'$, $\pi_1(M') = \{1\}$. Then the composition p*

$$M \xrightarrow{h} (\mathop{\#}\limits_1^r S^1 \times S^3)\#M' \xrightarrow{c_1} \mathop{\#}\limits_1^r S^1 \times S^3 \xrightarrow{c_2} \bigvee_1^r S^1 \times S^3 \xrightarrow{c_3} \bigvee_1^r S^1$$

is UV^1. Here c_1, c_2, c_3 are the obvious collapsing and projecting maps.

For the next example, consider a given finitely presented group π realized as fundamental group of a 2–complex K. One can embed K into $\mathbb{R}^5$ (in general into $\mathbb{R}^n$, $n \geq 5$) and take the boundary of a regular neighborhood N of K. Then N is a 4–manifold (in general a $(n-1)$–manifold) and the neighborhood retraction defines a map $p : N \to K = B$.

Lemma 4.6. *Suppose that π is the fundamental group of the complement of a torus knot. Then the construction described above gives a UV^1–map $p : N \to K = B$.*

Proof. We apply Theorem 3.1 and show that the inverse images of points are well embedded 2–spheres. So let k be a torus knot of type (k, l). The torus divides S^3 into solid tori T and T^*, and $k \subset T \cap T^* = S^1 \times S^1$. Let $M = S^3 \setminus \overset{o}{N}(k)$, where $N(k)$ is a small tubular neighborhood around $k \subset S^3$. The spine of M consists of subcomplexes $S \subset T$ and $S^* \subset T^*$ which intersect in a circle in $T \cap T^* = S^1 \times S^1$ parallel to the knot k.

If we look at slices of the solid tori T and T^*, the pictures of S and S^*, respectively are as in the Figure 1 below (where we take $(k, l) = (3, 2)$, i.e. the trefoil knot)

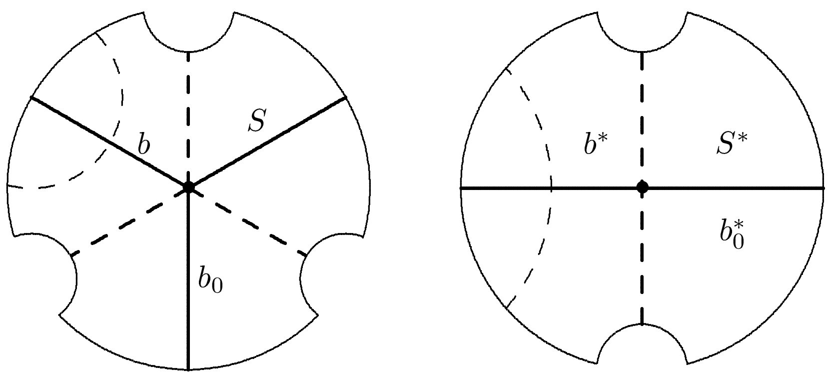

Figure 1

If $c : M \to S \cup S^* = B$ is the collapsing map, then for $b \in S$, $c^{-1}(b)$ is a wedge of a finite number of segments intersecting in k (the dotted segments). Similarly for $b^* \in S^*$.

We now consider the embedding $B \subset \mathbb{R}^5$ given by $B \subset M \subset S^3 \subset S^3 \times D^1 \subset \mathbb{R}^4 \subset \mathbb{R}^5$. So the boundary of a regular neighborhood of $B \subset \mathbb{R}^5$ is $\partial(M \times D^2) =$

$\partial M \times D^2 \cup M \times \partial D^2 = X$ and the neighborhood collapsing map is the composition of the projection $M \times D^2 \xrightarrow{\pi} M$ followed by c, that is

$$p = c \circ \pi|_X : X \to B.$$

Then it can be fairly easily seen that $p^{-1}(b)$ is a PL 2–sphere, hence p is a UV^1–map. □

§ 5. Results

Krushkal and Lee proved the following

Theorem 5.1. see [KruLee] *Let X be a 4–dimensional Poincaré complex with free nonabelian fundamental group, and assume that the intersection form on X is extended from the integers. Let $f : M \to X$ be a degree one normal map, where M is a closed 4–manifold. Then vanishing of the Wall obstruction implies that f is normally bordant to a (simple) homotopy equivalence.*

This is remarkable because the Disk theorem is supposed to be false for such fundamental groups (see [FreTei1]).

Proof. This result can be very well understood using controlled surgery together with results from [CavHeg]: Since the intersection form $\lambda : H_2(X, \Lambda) \times H_2(X, \Lambda) \to \Lambda$, $\Lambda = \mathbb{Z}[\pi_1(X)]$, is extended from the intersection form $H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}$, Theorem 1 of [CavHeg] implies that X is (simple) homotopy equivalent to $Y = (\#_1^r S^1 \times S^3) \# M'$, with $\pi_1(M') = \{1\}$. □

Transform the surgery problem to one with target Y and surgery obstruction zero (as explained in §2). Then there is a UV^1 map $p : Y \to \bigvee_1^r S^1 = B$, by Lemma 4,5,

The proof now follows by the well-known fact that $A : H_4(B, \mathbb{L}) \underset{\cong}{\to} L_4(\pi_1(B))$ (see [Cap]).

Theorem 5.2. *Let X be spin Poincaré 4–complex and suppose it has the fundamental group of a closed oriented aspherical surface. Let us assume that the Λ–intersection form is extended from the $\mathbb{Z}$–intersection form (as in Theorem 5.1). Then any degree 1 normal map $f : M \to X$ with vanishing Wall obstruction is normally bordant to a (simple) homotopy equivalence.*

Proof. It follows from Theorem 4.6 of [CavHegRep] that X is (simple) homotopy equivalent to $F \times S^2 \# M' = Y$, F being the aspherical surface with $\pi_1(X) = \pi_1(F)$, $\pi_1(M') = \{1\}$. As before we transform the surgery problem to one over Y and observe that $p : Y \xrightarrow{c_1} F \times S^2 \xrightarrow{\pi_F} F = B$ is UV^1.

It follows from [Cap], Theorem 18 (using the spectral sequence $H_p(B, \mathbb{L}_q) \Rightarrow H_{p+q}(B, \mathbb{L})$), that $A : H_4(B, \mathbb{L}) \xrightarrow{\cong} L_4(\pi_1(B))$ □

From Lemma 4.6 we obtain

Theorem 5.3. *Let X be the manifold constructed as boundary of a regular neighborhood in $\mathbb{R}^5$ of the 2–complex defined by the fundamental group of the complement of a torus knot. Then any surgery problem with target X and vanishing Wall obstruction is normally bordant to a (simple) homotopy equivalence.*

Proof. Let B be the spine and $p : X \to B$ the UV^1–map of Lemma 4.6. It is well known that B is an aspherical space with $H_p(B) = \mathbb{Z}$ for $p = 0, 1$ and trivial otherwise. The Atiyah-Hirzebruch spectral sequence implies $A : H_4(B, \mathbb{L}) \overset{\cong}{\to} L_4(\pi_1(B))$. □

A more general result can be obtained using the following fact proved in [HegRepSpa] (Theorem A): Let X be a connected closed oriented 4–manifold with fundamental group to be infinite. Suppose $G \subset H_2(X, \Lambda)$ is a Λ–submodule such that:

(i) G is Λ–free and the adjoint of the Λ–intersection pairing induces $G \underset{\cong}{\to} G^* \equiv \mathrm{Hom}_\Lambda(G, \Lambda)$;
(ii) either (a) $H^2(B\pi_1, \Lambda) = 0$ or (b) $H_2(X, \Lambda)/_G$ is Λ–trivial;
(iii) the restriction of the Λ–intersection form to G is extended from the $\mathbb{Z}$–intersection form.

Then X is (simple) homotopy equivalent to $B\#M'$, where B is a Poincaré 4–complex, $\pi_1 M' = \{1\}$, M' a 4–manifold with $H_2(M', \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \Lambda = G$. Applying the controlled surgery sequence one gets:

Theorem 5.4. *Suppose that* X, $G \subset H_2(X, \Lambda)$ *satisfy the hypotheses above. Then there is* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *there is* $\delta > 0$ *with*

$$H_5(B, \mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(X) \to [X, G/TOP] \to H_4(B, \mathbb{L})$$

is exact, where $p : X \to B$ *is given by Lemma 4.3.*

Remark. Theorems 5.1 and 5.2 above are special cases of Theorem 5.4. In Theorem 5.1 we are in case (ii)(a) and the case (ii)(b) applies to Theorem 5.2. In both cases we have $A_B : H_4(B, \mathbb{L}) \underset{\cong}{\to} L_4(\pi_1)$.

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