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AND PROPERLY 3-REALIZABLE  
GROUPS

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# AMALGAMATED PRODUCTS AND PROPERLY 3-REALIZABLE GROUPS

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ABSTRACT. In this paper, we show that the class of all properly 3-realizable groups is closed under amalgamated free products (and HNN-extensions) over finite groups. We recall that  $G$  is said to be properly 3-realizable if there exists a compact 2-polyhedron  $K$  with  $\pi_1(K) \cong G$  and whose universal cover  $\tilde{K}$  has the proper homotopy type of a 3-manifold (with boundary).

## 1. INTRODUCTION

We are concerned about the behavior of the property of being properly 3-realizable (for finitely presented groups) with respect to the basic constructions in Combinatorial Group Theory; namely, amalgamated free products and HNN-extensions. Recall that a finitely presented group  $G$  is said to be properly 3-realizable if there exists a compact 2-polyhedron  $K$  with  $\pi_1(K) \cong G$  and whose universal cover  $\tilde{K}$  has the proper homotopy type of a 3-manifold. In [1] it was shown that this property is preserved under amalgamated free products (HNN-extensions) over finite cyclic groups. See also [3, 4, 7] to learn more about properly 3-realizable groups and related topics. In this paper, we continue in the line of [1]. Our main result is the following generalization of ([1], Thm. 1.4):

**Theorem 1.1.** *The class of all properly 3-realizable groups is closed under amalgamated free products (and HNN-extensions) over finite groups.*

This generalizes to show that the fundamental group of a finite graph of groups with properly 3-realizable vertex groups and finite edge groups is properly 3-realizable, since such a group can be expressed as a combination of amalgamated free products and HNN-extensions of the vertex groups over the edge groups.

Recall that, given a finitely presented group  $G$  and a compact 2-polyhedron  $K$  with  $\pi_1(K) \cong G$  and  $\tilde{K}$  as universal cover, the number

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of ends of  $G$  is the number of ends of  $\tilde{K}$  which equals 0, 1, 2 or  $\infty$  [6] (see also [8, 12]). The 0-ended groups are the finite groups and the 2-ended groups are those having an infinite cyclic subgroup of finite index, and they are all known to be properly 3-realizable (see [1]). Note that Stallings' Structure Theorem [11] characterizes those groups  $G$  with more than one end as those which split as an amalgamated free product (or an HNN-extension) over a finite group (see also [12, 8]). In addition, Dunwoody [5] showed that this process of further splitting  $G$  must terminate after finitely many steps.

**Corollary 1.2.** *In order to show whether or not all finitely presented groups are properly 3-realizable it suffices to look among those groups which are 1-ended.*

## 2. MAIN RESULT

The purpose of this section is to prove Theorem 1.1. We will make use of the following result :

**Proposition 2.1** ([1], Prop. 3.1). *Let  $M$  be a manifold of the same proper homotopy type of a locally compact polyhedron  $K$  with  $\dim(K) < \dim(M)$ . Then, any Freudenthal end  $\epsilon \in \mathcal{F}(M)$  can be represented by a sequence of points in  $\partial M$ .*

*Proof of Theorem 1.1.* Let  $G_0, G_1$  be properly 3-realizable groups and  $F$  be a finite group with presentation  $\langle a_1, \dots, a_N; r_1, \dots, r_M \rangle$ . Consider monomorphisms  $\varphi_i : F \rightarrow G_i (i = 0, 1)$ , and denote by  $G_0 *_F G_1 = \langle G_0, G_1; \varphi_0(a_i) = \varphi_1(a_i), 1 \leq i \leq N \rangle$  the corresponding amalgamated free product. Let  $X_0, X_1$  be compact 2-polyhedra with  $\pi_1(X_i) \cong G_i$  and such that their universal covers have the proper homotopy type of 3-manifolds  $M_0, M_1$  respectively. Let  $L = \bigvee_{i=1}^N S^1$  and  $f_i : L \rightarrow X_i (i = 0, 1)$  be cellular maps such that  $Im f_{i*} \subseteq \pi_1(X_i)$  corresponds to the subgroup  $Im \varphi_i \subseteq G_i$ . We take the standard 2-dimensional CW-complex  $Y'$  associated to the above presentation of  $F$ , i.e.,  $Y'$  has one 1-cell  $e_i$  for each generator  $a_i (1 \leq i \leq N)$ , all of them sharing the only vertex in  $Y'$ , and one 2-cell  $d_j$  for each relation  $r_j (1 \leq j \leq M)$  attached via a map  $S^1 \rightarrow \bigvee_{i=1}^N e_i$  which "spells" the relation  $r_j$ . Consider the adjunction spaces  $Y = (\bigvee_{i=1}^N e_i) \times I \cup_{(\bigvee_{i=1}^N e_i) \times \{\frac{1}{2}\}} Y'$  (homotopy equivalent to  $Y'$ ) and  $Z = Y \cup_{f_0 \times \{0\} \cup f_1 \times \{1\}} (X_0 \sqcup X_1)$ . By van Kampen's Theorem,  $Z$  is a compact 2-polyhedron with  $\pi_1(Z) \cong G_0 *_F G_1$ . Let  $\tilde{Z}$  be the universal cover of  $Z$  with covering map  $p : \tilde{Z} \rightarrow Z$ . Then,  $p^{-1}(X_i)$  consists of a disjoint union of copies of the universal cover  $\tilde{X}_i$  of  $X_i$ , since the inclusion  $X_i \hookrightarrow Z$  induces a monomorphism  $G_i \hookrightarrow G_0 *_F G_1$  between the fundamental groups,  $i = 0, 1$  (see [9]). On the other hand,

let  $\Gamma$  be a connected component of  $p^{-1}(\bigvee_{i=1}^N e_i) \subset p^{-1}(Y')$  and  $\tilde{Y}'$  be the connected component of  $p^{-1}(Y')$  containing  $\Gamma$ . Observe that  $\tilde{Y}'$  is a copy of the universal cover of  $Y'$  (which is compact), as the inclusion  $Y' \hookrightarrow Z$  induces a monomorphism  $F \hookrightarrow G_0 *_F G_1$ . Then, it is easy to see that  $p^{-1}(Y)$  consists of a disjoint union of copies of the compact CW-complex  $K = (\Gamma \times I) \cup_{\Gamma \times \{\frac{1}{2}\}} \tilde{Y}'$ . Thus,  $\tilde{Z}$  comes together with the following data (see [12]) :

(a) The disjoint unions  $\bigsqcup_{p \in \mathbf{N}} \tilde{X}_{0,p}$  and  $\bigsqcup_{r \in \mathbf{N}} \tilde{X}_{1,r}$  of copies of  $\tilde{X}_0$  and  $\tilde{X}_1$

respectively;

(b) a disjoint union  $\bigsqcup_{p,q \in \mathbf{N}} K_{p,q}$  of copies of  $K$ ; and

(c) a bijective function  $\varphi : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N} \times \mathbf{N}$ ,  $(p, q) \mapsto (r, s)$  (given by the group action of  $G_0 *_F G_1$  on  $\tilde{Z}$ ), so that for each  $p, q \in \mathbf{N}$ ,  $\Gamma \times \{0\} \subset K_{p,q}$  is being glued to  $\tilde{X}_{0,p}$  via a lift  $\tilde{f}_{p,q}^0 : \Gamma \times \{0\} \longrightarrow \tilde{X}_{0,p}$  of the map  $f_0$ , and  $\Gamma \times \{1\} \subset K_{p,q}$  is being glued to  $\tilde{X}_{1,r}$  via a lift  $\tilde{f}_{r,s}^1 : \Gamma \times \{1\} \longrightarrow \tilde{X}_{1,r}$  of the map  $f_1$ .

Next, for each copy of  $\tilde{X}_i$ ,  $i = 0, 1$ , in  $\tilde{Z}$  (written as  $\tilde{X}_{0,p}$  or  $\tilde{X}_{1,r}$ ), we take one of the maps  $\tilde{f}_{\lambda,\mu}^i : \Gamma \times \{i\} \longrightarrow \tilde{X}_i$  and observe that this map is nullhomotopic so we can replace it (up to homotopy) with a constant map  $g_{\lambda,\mu}^i : \Gamma \times \{i\} \longrightarrow \tilde{X}_i$  with  $Im\ g_{\lambda,\mu}^i \subset Im\ \tilde{f}_{\lambda,\mu}^i$ , and we do this equivariantly using the group action of  $G_i$  on  $\tilde{X}_i$ . Since this action is properly discontinuous, the collection of all these homotopies gives rise to a proper homotopy equivalence between  $\tilde{Z}$  and a new 2-dimensional CW-complex  $W$  obtained from a collection of copies of  $K$  and a collection of copies of  $\tilde{X}_0$  and  $\tilde{X}_1$  by gluing each copy of  $\Gamma \times \{i\}$  to the corresponding copy of  $\tilde{X}_i$  via the bijection  $\varphi$  and the new maps  $g_{\lambda,\mu}^i$ ,  $i = 0, 1$ .

We will now manipulate the CW-complex  $K$  as follows. First, let  $K'$  be the CW-complex obtained from  $K$  by shrinking to a point  $v \times \{i\}$  each copy  $T \times \{i\}$  ( $i \in I$ ) of a maximal tree  $T \subset \tilde{Y}' \subset K$ . Next, we take  $K''$  to be the CW-complex obtained from  $K'$  by identifying the subcomplexes  $\Gamma \times \{i\}/T \times \{i\}$ ,  $i = 0, 1$ , to a (different) point which we will denote by  $[v \times \{0\}]$  and  $[v \times \{1\}]$ . Note that  $K''$  has a copy of  $\tilde{Y}'/T$  as a subcomplex. Since  $\tilde{Y}'/T$  is compact and simply connected, it follows from ([13], Prop. 3.3) that  $\tilde{Y}'/T$  is homotopy equivalent to a finite bouquet of 2-spheres  $\bigvee_{\alpha \in \mathcal{A}} S^2$  (which we may regard as a connected 2-dimensional CW-complex with no 1-cells). Moreover, we may assume that this homotopy equivalence is given by a cellular map

$\tilde{Y}'/T \longrightarrow \bigvee_{\alpha \in \mathcal{A}} S^2$  so that the 1-skeleton  $\Gamma/T$  of  $\tilde{Y}'/T$  is mapped to the wedge point. Finally, taking into account this homotopy equivalence, it is not difficult to see that  $K''$  is homotopy equivalent to the CW-complex  $\widehat{K}$  obtained from the disjoint union of a finite bouquet  $\bigvee_{\alpha \in \mathcal{A} \cup \mathcal{B}} S^2$  (where  $\text{Card}(\mathcal{B}) = 2 \text{rank}(\pi_1(\Gamma))$ ) and the unit interval  $I$  by identifying  $\frac{1}{2} \in I$  with the wedge point, so that  $I \subset \widehat{K}$  would correspond to the subcomplex  $v \times I \subset K'$  and  $0, 1 \in I$  would correspond to  $[v \times \{0\}], [v \times \{1\}] \in K''$ . Notice that  $\widehat{K}$  thickens to a 3-manifold  $P \searrow \widehat{K}$  containing 3-dimensional 1-handles  $H$  and  $H'$  (with a free end face each of them) corresponding to the edges  $[0, \frac{1}{2}], [\frac{1}{2}, 1] \subset I \subset \widehat{K}$  respectively.

According to the above, one can see that the CW-complex  $W$  (proper homotopy equivalent to  $\tilde{Z}$ ) is in turn proper homotopy equivalent to the quotient space obtained from the following data :

- (a) A disjoint union  $\bigsqcup_{p \in \mathbf{N}} \tilde{X}_{0,p}$  of copies of  $\tilde{X}_0$  together with a locally finite sequence of points  $\{x_q^p\}_{q \in \mathbf{N}} \subset \tilde{X}_{0,p}$ , for each  $p \in \mathbf{N}$ , corresponding to the images of the constant maps  $g_{p,q}^0 : \Gamma \times \{0\} \longrightarrow \tilde{X}_{0,p}$  considered above in the construction of  $W$ ;
- (b) a disjoint union  $\bigsqcup_{r \in \mathbf{N}} \tilde{X}_{1,r}$  of copies of  $\tilde{X}_1$  together with a locally finite sequence of points  $\{y_s^r\}_{s \in \mathbf{N}} \subset \tilde{X}_{1,r}$ , for each  $r \in \mathbf{N}$ , corresponding to the images of the constant maps  $g_{r,s}^1 : \Gamma \times \{1\} \longrightarrow \tilde{X}_{1,r}$  from the construction of  $W$ ;
- (c) a disjoint union  $\bigsqcup_{p,q \in \mathbf{N}} \widehat{K}_{p,q}$  of copies of  $\widehat{K}$ ; and
- (d) the bijective function  $\varphi : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N} \times \mathbf{N}$ ,  $(p, q) \mapsto (r, s)$ , so that  $0 \in I \subset \widehat{K}_{p,q}$  is being identified with  $x_q^p \in \tilde{X}_{0,p}$  and  $1 \in I \subset \widehat{K}_{p,q}$  is being identified with  $y_s^r \in \tilde{X}_{1,r}$  ( $(r, s) = \varphi(p, q)$ ), for each  $p, q \in \mathbf{N}$ .

We now follow an argument similar to the proof of ([1], Lemma 3.2). Fix proper homotopy equivalences  $h : \tilde{X}_0 \longrightarrow M$  and  $h' : \tilde{X}_1 \longrightarrow N$ . Given the above data, we set  $A = \mathbf{N} \times \mathbf{N}$  and consider maps  $i : A \longrightarrow \bigsqcup_{p \in \mathbf{N}} \tilde{X}_{0,p}$ ,  $i' : A \longrightarrow \bigsqcup_{r \in \mathbf{N}} \tilde{X}_{1,r}$  given by  $i(p, q) = x_q^p$  and  $i'(p, q) = y_s^r$ , where  $(r, s) = \varphi(p, q)$ . It is easy to check that  $i$  and  $i'$  are proper cofibrations, as the corresponding sequences of points are locally finite. Next, we take exhaustive sequences  $\{A_m^p\}_{m \in \mathbf{N}}$  and  $\{B_n^r\}_{n \in \mathbf{N}}$  of copies  $M_p$  and  $N_r$  of the 3-manifolds  $M$  and  $N$  respectively by compact submanifolds, and define proper cofibrations  $j : A \longrightarrow$

$\bigsqcup_{p \in \mathbf{N}} M_p$ ,  $j' : A \longrightarrow \bigsqcup_{r \in \mathbf{N}} N_r$  as follows. Given  $(p, q) \in A$  and the proper

homotopy equivalences  $h_p = h : \tilde{X}_{0,p} \longrightarrow M_p$ ,  $h'_r = h' : \tilde{X}_{1,r} \longrightarrow N_r$ 
 (with  $(r, s) = \varphi(p, q)$ ), we take  $m(q), n(s) \in \mathbf{N}$  to be the least natural
 numbers such that  $h_p \circ i(p, q) \notin A_{m(q)}^p \subset M_p$  and  $h'_r \circ i'(p, q) \notin B_{n(s)}^r \subset N_r$ .
 Then, using Proposition 2.1, we define  $j(p, q)$  and  $j'(p, q)$  to be
 points  $j(p, q) = a_{p,q} \in \partial M_p - A_{m(q)}^p$  and  $j'(p, q) = b_{r,s} \in \partial N_r - B_{n(s)}^r$  so
 that (i)  $j, j'$  are one-to-one maps (note that  $h, h'$  need not be one-to-
 one); and (ii)  $a_{p,q}$  and  $h_p \circ i(p, q)$  (resp.  $b_{r,s}$  and  $h'_r \circ i'(p, q)$ ) are in the
 same path component of  $M_p - A_{m(q)}^p$  (resp.  $N_r - B_{n(s)}^r$ ). Notice that  $j$ 
 and  $j'$  are proper maps by construction. Consider now maps

$$G : \left( \bigsqcup_{p \in \mathbf{N}} \tilde{X}_{0,p} \right) \times \{0\} \cup (i(A) \times I) \longrightarrow \bigsqcup_{p \in \mathbf{N}} M_p$$

$$H : \left( \bigsqcup_{r \in \mathbf{N}} \tilde{X}_{1,r} \right) \times \{0\} \cup (i'(A) \times I) \longrightarrow \bigsqcup_{r \in \mathbf{N}} N_r$$

with  $G|_{\tilde{X}_{0,p} \times \{0\}} = h_p = h$  and  $H|_{\tilde{X}_{1,r} \times \{0\}} = h'_r = h'$  ( $p, r \in \mathbf{N}$ ), and so
 that  $\alpha_{p,q} = G|_{i(p,q) \times I}$  (resp.  $\beta_{r,s} = H|_{i'(p,q) \times I}$ ) is a path in  $M_p - A_{m(q)}^p$ 
 from  $h_p \circ i(p, q)$  to  $a_{p,q}$  (resp. a path in  $N_r - B_{n(s)}^r$  from  $h'_r \circ i'(p, q)$ 
 to  $b_{r,s}$ ). Observe that  $G$  and  $H$  are proper maps, since  $h, h', j$  and  $j'$  are
 proper. By the Homotopy Extension Property, the maps  $G, H$  extend
 to proper maps

$$\hat{G} : \left( \bigsqcup_{p \in \mathbf{N}} \tilde{X}_{0,p} \right) \times I \longrightarrow \bigsqcup_{p \in \mathbf{N}} M_p, \quad \hat{H} : \left( \bigsqcup_{r \in \mathbf{N}} \tilde{X}_{1,r} \right) \times I \longrightarrow \bigsqcup_{r \in \mathbf{N}} N_r$$

which yield commutative diagrams

$$\begin{array}{ccc}
 & A & \\
 i \swarrow & & \searrow j \\
 \bigsqcup_{p \in \mathbf{N}} \tilde{X}_{0,p} & \xrightarrow{\hat{h}} & \bigsqcup_{p \in \mathbf{N}} M_p
 \end{array}
 \quad
 \begin{array}{ccc}
 & A & \\
 i' \swarrow & & \searrow j' \\
 \bigsqcup_{r \in \mathbf{N}} \tilde{X}_{1,r} & \xrightarrow{\hat{h}'} & \bigsqcup_{r \in \mathbf{N}} N_r
 \end{array}$$

where  $\hat{h} = \hat{G}|_{(\bigsqcup_{p \in \mathbf{N}} \tilde{X}_{0,p}) \times \{1\}}$  and  $\hat{h}' = \hat{H}|_{(\bigsqcup_{r \in \mathbf{N}} \tilde{X}_{1,r}) \times \{1\}}$  are proper ho-
 motopy equivalences. Moreover,  $\hat{h}$  and  $\hat{h}'$  are proper homotopy equiva-
 lences under  $A$ , by ([2], Prop. 4.16) (compare with [10], Chap. 6, § 5).
 Hence, they induce a proper homotopy equivalence between the quo-
 tient space described above (proper homotopy equivalent to  $W$ ) and
 the following 3-manifold obtained as the quotient space given by the
 data :

- (a) The disjoint unions  $\bigsqcup_{p \in \mathbf{N}} M_p$  and  $\bigsqcup_{r \in \mathbf{N}} N_r$  of copies of the 3-manifolds  $M$  and  $N$  respectively;
- (b) a disjoint union  $\bigsqcup_{p, q \in \mathbf{N}} P_{p, q}$  of copies of the compact 3-manifold  $P \searrow \widehat{K}$ ; and
- (c) the bijective function  $\varphi : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N} \times \mathbf{N}$ ,  $(p, q) \mapsto (r, s)$ , so that for each  $p, q \in \mathbf{N}$ , the free ends of the corresponding 3-dimensional 1-handles  $H_{p, q}, H'_{p, q} \subset P_{p, q}$  considered above are being identified homeomorphically with small disks  $D_{p, q} \subset \partial M_p$  and  $D'_{r, s} \subset \partial N_r$  about the points  $a_{p, q}$  and  $b_{r, s}$  respectively.

In the case of an HNN-extension  $G *_F = \langle G, t; t^{-1} \psi_0(a_i) t = \psi_1(a_i), 1 \leq i \leq N \rangle$  (with monomorphisms  $\psi_i : F \longrightarrow G, i = 0, 1$ ), let  $X$  be a compact 2-polyhedron with  $\pi_1(X) \cong G$  and whose universal cover has the proper homotopy type of a 3-manifold, and let  $f_i : \bigvee_{i=1}^N S^1 \longrightarrow X$  ( $i = 0, 1$ ) be cellular maps so that  $Im f_{i*} \subseteq \pi_1(X)$  corresponds to the subgroup  $Im \psi_i \subseteq G$ . Let  $Y$  be the 2-dimensional CW-complex constructed as above and consider the adjunction space  $Z = Y \cup_{f_0 \times \{0\} \cup f_1 \times \{1\}} X$ , with  $\pi_1(Z) \cong G *_F$ . Then, the proof goes just as the one given above for the amalgamated free product. □

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