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THE MINOR CROSSING NUMBER

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Abstract

The minor crossing number of a graph G is defined as the minimum crossing number of all graphs that contain G as a minor. Basic properties of this new invariant are presented. Topological structure of graphs with bounded minor crossing number is determined and a new strong version of a lower bound based on the genus is derived. An inequality of Moreno and Salazar [15] between crossing numbers of a graph and its minors is generalized.

Keywords: crossing number, graph minor

AMS Subject Classification (2000): 05C10, 05C83

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1 Preliminaries

Crossing numbers of graphs have been thoroughly studied [18], yet only a few exact results are known and new ideas seem to be needed. Crossing numbers in general give a measure of nonplanarity of graphs. Unfortunately, they are not monotone with respect to graph minors. Seymour [1] asked "How to define a crossing number that would work well with minors?" In this paper we propose two possible answers to this question and study one of them in greater detail. Our approach is based on general principles of how a graph invariant can be transformed into a minor-monotone graph invariant [4].

Crossing numbers of graphs are believed to have applications in VLSI design where one wants a design of a (huge) electrical network such that the number of crossing edges (wires) is minimized [3, 9, 10]. However, today's chip manufacturers replace vertices of high degree by binary trees. The minor crossing number treated in this paper does precisely this – each vertex is expanded into a cubic tree in such a way that the resulting graph can be realized with as few crossings as possible. It turns out that this interpretation of crossing numbers has rich mathematical structure, whose basics are uncovered in this work.

Let $G = (V_G, E_G)$ be a graph and Σ a closed surface. If Σ has Euler characteristics χ, then the number $g = 2 - \chi$ is called the *Euler genus* of Σ. The nonorientable surface of Euler genus $g \geq 1$ is denoted by $\mathbb{N}_g$, and the orientable surface of Euler genus $2g$ ($g \geq 0$) is denoted by $\mathbb{S}_g$.

A *drawing* $D = (\varphi, \varepsilon)$ of G in (PL) surface Σ consists of a one-to-one mapping $\varphi : V_G \to \Sigma$ and a mapping $\varepsilon : E_G \to \Omega(\Sigma)$ that maps edges of G to simple (polygonal) curves in Σ, such that endpoints of $\varepsilon(uv)$ are $\varphi(u)$ and $\varphi(v)$, $\varphi(V_G)$ does not intersect interiors of images of edges, and the intersection of interiors of ε-images of any two distinct edges contains at most one point.

Let e and f be distinct edges of G, let r and s be their images in Σ, and suppose that $x \in r \cap s$. Let U be a neighborhood of x so that for each disk neighborhood $B \subseteq U$ of x both $B \cap r \cap s = \{x\}$ and $|\partial B \cap (r \cup s)| = 4$. We say that e and f or that r and s *cross* at x (and call x a *crossing*) if points of r and s interlace along ∂B for every such B, and say that r and s *touch* otherwise. In the latter case we call x a *touching* of r and s (or of e and f).

A drawing D is *normal* if it has no touchings and for each crossing x there are precisely two edges of G whose crossing is x.

Crossing number of a graph G in Σ, $\mathrm{cr}(G, \Sigma)$, is defined as the minimum number of crossings in any normal drawing of G in Σ, and with $\mathrm{cr}(G)$ we denote the crossing number of G in the sphere. For a drawing $D = (\varphi, \varepsilon)$

of G in Σ, connected regions of $\Sigma \setminus \varepsilon(E_G)$ are called *faces of D*. By our standards, a drawing of G in the plane $\mathbb{R}^2$ is a drawing of G in the sphere $\mathbb{S}_0$, equipped with an *infinite point* ∞ avoiding the image of G. The *infinite face* of a drawing of G in the plane is the face containing ∞. Further, an *embedding* is a drawing without crossings. Besides this terminology, the reader is referred to [14] for other notions related to graph embeddings.

For a given graph G, the *minor crossing number* is defined as the minimum crossing number of all graphs, which contain G as a minor:

$$\operatorname{mcr}(G, \Sigma) := \min\{\operatorname{cr}(H, \Sigma) \mid G \leq_m H\}\ . \tag{1.1}$$

By $\operatorname{mcr}(G)$ we denote $\operatorname{mcr}(G, \mathbb{S}_0)$.

Similarly, the *major crossing number* of G is the maximum crossing number taken over all minors of G:

$$\operatorname{Mcr}(G, \Sigma) := \max\{\operatorname{cr}(H, \Sigma) \mid H \leq_m G\}\ . \tag{1.2}$$

The following lemmas follow directly from the definitions:

Lemma 1.1 *For every graph G and every surface Σ,*

$$\operatorname{mcr}(G, \Sigma) \leq \operatorname{cr}(G, \Sigma) \leq \operatorname{Mcr}(G, \Sigma).$$

Lemma 1.2 *If G is a minor of H, then for every surface Σ,*

$$\operatorname{mcr}(G, \Sigma) \leq \operatorname{mcr}(H, \Sigma) \quad \textit{and} \quad \operatorname{Mcr}(G, \Sigma) \leq \operatorname{Mcr}(H, \Sigma).$$

Lemma 1.2 immediately yields:

Corollary 1.3 *Let $k \geq 0$ be an integer and Σ a surface. The families $\omega(k, \Sigma) := \{G \mid \operatorname{mcr}(G, \Sigma) \leq k\}$ and $\Omega(k, \Sigma) := \{G \mid \operatorname{Mcr}(G, \Sigma) \leq k\}$ are minor-closed.*

For each graph G there exists a graph $\bar{G}$, such that $G \leq_m \bar{G}$ and $\operatorname{mcr}(G, \Sigma) = \operatorname{cr}(\bar{G}, \Sigma)$. We call such a graph $\bar{G}$ a *realizing graph of G*, and an optimal drawing of $\bar{G}$ in Σ is called a *realizing drawing of G* (with respect to Σ). By no means a realizing graph or drawing are uniquely determined, but we shall always assume that G and $\bar{G}$ have the same number of connected components.

As G is a minor of its realizing graph $\bar{G}$, G can be obtained as a contraction of a subgraph of $\bar{G}$. In other words, $G = (\bar{G} - R)/C$ for suitable edge sets $R, C \subseteq E_{\bar{G}}$. The edges of R are called *removed edges* and those in C

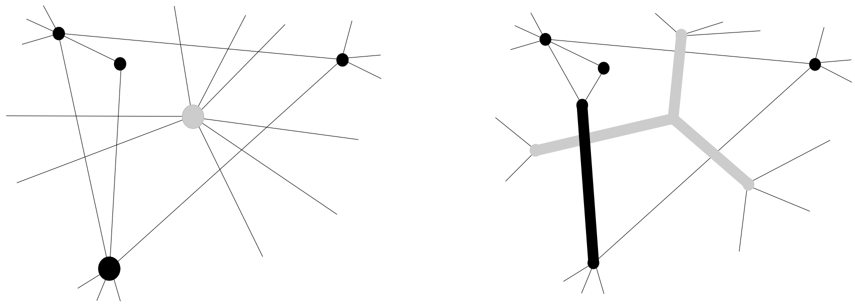

Figure 1: mcr as an extension of cr

are *contracted edges*. Note that $E_G = E_{\bar{G}} \setminus (R \cup C)$ are the original edges of G. It is clear that every graph G has a realizing graph $\bar{G}$ such that $R = \emptyset$.

To each vertex $v \in V_G$ corresponds a unique maximal tree $T_v \subseteq \bar{G}[C]$, such that T_v is contracted to v. In the figures, the original edges will be drawn as thin lines and the contracted edges as thick lines.

The minor crossing number can be considered as a natural extension of the usual crossing number. Clearly, if $e, f \in E_{\bar{G}}$ cross in a realizing drawing of G, then $e, f \in C \cup E_G$. If both belong to C, then their crossing is a *vertex-vertex* crossing, if both belong to E_G, then they cross in an *edge-edge* crossing, and otherwise they cross in an *edge-vertex* crossing. This point of view is illustrated in Figure 1. Note that by subdividing the original edges appropriately, all the crossings in the realizing drawing can be forced to be vertex-vertex crossings.

If G is a cubic graph, then clearly $\mathrm{mcr}(G, \Sigma) = \mathrm{cr}(G, \Sigma)$. Hliněný proved in [6] that computing planar crossing number of cubic graphs is NP-hard and has remarked that this implies that the same holds for computing $\mathrm{mcr}(G)$ for any graph G.

Proposition 1.4 *For every graph G and every surface Σ there exists a cubic realizing graph H. Moreover, if $\delta(G) \geq 3$, then G can be obtained from H by contracting edges only.*

Proof. Let H_0 be a realizing graph of G without removed edges, and let $D_0 = (\varphi, \varepsilon)$ be an optimal drawing of H_0. We shall describe H in terms of its drawing D obtained from D_0. For each vertex v of H_0 of degree $d := d_{H_0}(v) \neq 3$ let U_v be a closed disk containing $\varphi(v)$ in its interior, so that a small neighborhood of U_v contains no crossings, U_v is disjoint from U_u for $u \in V_{H_0} \setminus \{v\}$, and $U_v \cap \varphi(E_{H_0})$ is connected.

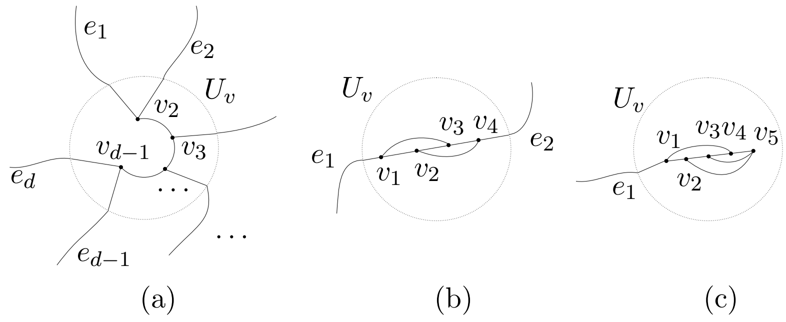

Figure 2: Drawing a cubic realizing graph, cf. Proposition 1.4

For each of the cases $d > 3$, $d = 2$ and $d = 1$, we modify D_0 in U_v as indicated in Figure 2. Let D be this new drawing and H the graph defined by D.

Clearly, $G \leq_m H$ and $\text{cr}(H, \Sigma) \geq \text{mcr}(G, \Sigma)$. As D contains no new crossings, we have $\text{mcr}(G, \Sigma) = \text{cr}(H_0, \Sigma) = \text{cr}(D, \Sigma) \geq \text{cr}(H, \Sigma)$. A combination of these two inequalities proves that $\text{cr}(H, \Sigma) = \text{mcr}(G, \Sigma)$.

If $\delta(G) \geq 3$ then we can assume $\delta(H_0) \geq 3$ which implies $|E_H| - |V_H| = |E_{H_0}| - |V_{H_0}|$. As $H_0 \leq_m H$, we can obtain G from H by contracting edges only. □

2 Minor crossing number and maximum degree

In this section we present a generalization and some corollaries of the following result (cf. also Section 6).

Theorem 2.1 (Moreno and Salazar [15]) *Let G be a minor of a graph H with $\Delta(G) \leq 4$. Then $\frac{1}{4}\,\text{cr}(G, \Sigma) \leq \text{cr}(H, \Sigma)$ for every surface Σ.*

Suppose that $G = H/e$ for $e = v_1v_2 \in E_H$. For $i = 1, 2$, let $d_i = \deg_H(v_i) - 1$ be the number of edges incident with v_i and distinct from e. We may assume that $d_1 \leq d_2$. As shown in Figure 3, any given drawing of H can be changed into a drawing of G such that every crossing involving e is replaced by d_1 new crossings.

More generally, let G be a minor of H. We assume that $G = (H - R)/C$. Then $E_G = E_H \setminus (R \cup C)$. Let $D_H = (\varphi_H, \varepsilon_H)$ be a normal drawing of H. Then D_H determines a normal drawing of $H - R$ in Σ in which no new crossings arise. On the other hand, by contracting the edges in C, the

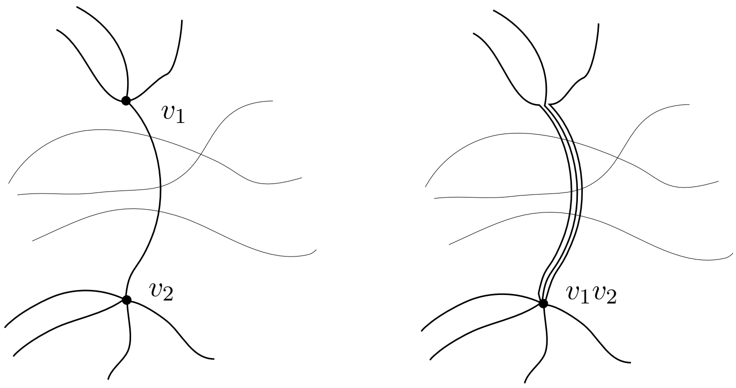

Figure 3: Contracting edges on a drawing

number of crossings can increase. If we perform edge-contractions one by one, and every time apply the redrawing procedure as described above, then we can control the number of new crossings. To do the counting properly, we need some additional notation.

Let us define $w(G,H) : E_H \to \mathbb{N}$ by setting $w(G,H,e) = 0$ if $e \in R$, and $w(G,H,e) = 1$ if $e \in E_G$. If $e \in C$, let T_v be the maximal tree induced by C containing e (which contracts to the vertex v in G). Let T_1, T_2 be the components of $T_v - e$, and let d_i $(i = 1,2)$ denote the number of edges in E_G that are incident with T_i. Then we set $w(G,H,e) = \min\{d_1,d_2\}$. For $e \in E_H$ we call $w(G,H,e)$ the *weight* of the edge e.

Let $G \leq_m H_1 \leq_m H$, so that $G = (H_1 - R_1)/C_1$, $H_1 = (H - R')/C'$, and $G = (H - R)/C$, where $R = R_1 \cup R'$ and $C = C_1 \cup C'$. Let D_H be a normal drawing of H. Further, let D_1 be a drawing of H_1, obtained from D_H by removing the edges of R' and applying the described contractions of the edges in C' one after another. When doing these contractions, we proceed similarly as shown in Figure 3 except that the criterion whether to contract towards v_1 or v_2 is not the degree of v_1 or v_2 but the quantities d_1 or d_2 introduced in the previous paragraph. Similarly, let D_G be obtained from D_1 by using R_1 and C_1. If D is a drawing, let $X(D)$ be the set of crossings of D, and for $x \in X(D)$, let e_x and f_x be the edges that cross at x.

Lemma 2.2 *Let G, H, H_1 and their drawings D_G, D_H, D_1 be as defined* *in the previous paragraph. Then*

$$\sum_{x\in X(D_1)} w(G,H_1,e_x)\,w(G,H_1,f_x) \le \sum_{x\in X(D_H)} w(G,H,e_x)\,w(G,H,f_x). \tag{2.1}$$

Proof. It is enough to prove this for the case when H_1 and H differ only in a single minor operation with respect to G, i. e. $R'\cup C' = \{e\}$. If $H_1 = H-e$, then $w(G,H,e) = 0$ and the sums are equal.

Suppose now that $H_1 = H/e$. As simplifying the image of e decreases the right-hand sum, we may assume that $\varepsilon_H(e)$ is a simple arc. We adopt the notation introduced above. The edge e is contracted, so $e \in C$. After the contraction of e, all weights remain the same, i.e. $w(G,H_1,f) = w(G,H,f)$ for every $f \in E_H - e$. Hence, the difference between the left and the right-hand side in (2.1) is that the crossings in D_H are replaced by newly introduced crossings in D_1 (as shown in Figure 3). Let $x \in X(D_H)$ with $e_x = e = v_1v_2$, and let E_1 be the set of edges incident with v_1. Since $\sum_{f\in E_1-e} w(G,H_1,f) = \sum_{f\in E_1-e} w(G,H,f) = w(G,H,e)$ and to each crossing x of e with some e' in D_1 correspond exactly the crossings of E_1-e with the edge e', the inequality (2.1) follows. □

Theorem 2.3 *Let G be a minor of a graph H, Σ be a surface, and $\tau := \left\lfloor \frac{1}{2}\Delta(G)\right\rfloor$. Then*

$$\mathrm{cr}(G,\Sigma) \le \tau^2\,\mathrm{cr}(H,\Sigma).$$

Proof. Let D_H be an optimal drawing of H and let D_G be the drawing of G, obtained from D_H as described before Lemma 2.2. We apply Lemma 2.2 with $H_1 = G$. Obviously, $\mathrm{cr}(G,\Sigma) \le \mathrm{cr}(D_G,\Sigma)$. As all edges in G have weight $w(G,G,e) = 1$, the left-hand side of inequality (2.1) equals the number of crossings in D_G. Since the weights $w(G,H,e)$ of edges in H are bounded from above by τ, the theorem follows. □

By using Theorem 2.3 together with definition (1.1) and Lemma 1.2, we obtain the following corollary.

Corollary 2.4 *Let G be a graph, Σ a surface, and $\tau := \left\lfloor \frac{1}{2}\Delta(G)\right\rfloor$. Then*

$$\mathrm{mcr}(G,\Sigma) \le \mathrm{cr}(G,\Sigma) \le \tau^2\mathrm{mcr}(G,\Sigma).$$

3 Minor crossing number and genus

In this section we derive some genus-related lower bounds for minor crossing number of graphs. For additional terminology, we refer the reader to [14].

Theorem 3.1 *Let G be a graph with genus $g(G)$ and nonorientable genus $\tilde{g}(G)$. If Σ is an orientable surface of genus $g(\Sigma)$, then* $\mathrm{mcr}(G,\Sigma) \geq g(G) - g(\Sigma)$ *and* $\mathrm{mcr}(G,\Sigma) \geq \tilde{g}(G) - 2g(\Sigma)$.

If Σ is a nonorientable surface with nonorientable genus $g(\Sigma)$, then $\mathrm{mcr}(G,\Sigma) \geq \tilde{g}(G) - g(\Sigma)$.

Proof. Let D be an optimal drawing of a realizing graph $\bar{G}$ in an orientable surface Σ. For each crossing in D we add a handle to Σ and obtain an embedding of $\bar{G}$ in a surface Σ' of genus $g(\Sigma') = g(\Sigma) + \mathrm{mcr}(G,\Sigma)$. Using minor operations on D we can obtain an embedding of G in Σ', which yields $g(\Sigma') \geq g(G)$. Thus, we have $\mathrm{mcr}(G,\Sigma) \geq g(G) - g(\Sigma)$.

The other two claims can be proved in a similar way by adding crosscaps at crossings of D. Note also that adding a crosscap to an orientable surface of genus g results in a surface of nonorientable genus $2g+1$. □

When the genus of a graph is not known, one can derive the following lower bound using Euler formula and the same technique as in the preceding proof.

Proposition 3.2 *Let G be a graph with $n = |V_G|$, $m = |E_G|$ and girth r, and let Σ be a surface of Euler genus g. Then* $\mathrm{mcr}(G,\Sigma) \geq \frac{r-2}{r}m - n - g + 2$.

Proof. As in the proof of Theorem 3.1 we obtain an embedding D of G in $\mathbb{N}_{g+k}$, where $k = \mathrm{mcr}(G,\Sigma)$. Let f be the number of faces in D. All faces have length at least r, thus $f \leq \frac{2m}{r}$. Euler formula results in $2 - (g+k) = n - m + f \leq n - \frac{r-2}{r}m$, which yields the claimed bound. □

In Section 5 we derive a strong improvement over Proposition 3.2, see Theorem 5.6.

The following proposition relates minor crossing numbers in different surfaces with the one in the plane.

Proposition 3.3 *The inequality* $\mathrm{mcr}(G,\Sigma) \leq \max(0, \mathrm{mcr}(G) - g(\Sigma))$ *holds for every surface Σ and every graph G, where $g(\Sigma)$ denotes the (non)orientable genus of Σ.*

Proof. Let us start with a realizing drawing of G in the sphere. We can remove at least one existing crossing by adding either a crosscap (if the surface is nonorientable) or a handle. This increases the genus of the surface by 1, and the result follows. $\square$

4 Minor crossing number and connectivity

Let $G_1, \ldots, G_k$ be the components of a graph G. It is easy to see that $\mathrm{mcr}(G) = \sum_{i=1}^{k} \mathrm{mcr}(G_i)$. We shall extend this fact to the blocks (2-connected components) of G, even in the setting of the minor crossing number in a surface.

Let Σ be a surface and k a positive integer. We say that a collection $\Sigma_1, \ldots, \Sigma_k$ of surfaces is a *decomposition* of Σ and write $\Sigma = \Sigma_1 \# \cdots \# \Sigma_k$ if Σ is homeomorphic to the connected sum of $\Sigma_1, \ldots, \Sigma_k$.

Theorem 4.1 *Let Σ be a surface and let G be a graph with blocks $G_1, \ldots, G_k$. Then*

$$\sum_{i=1}^{k} \mathrm{mcr}(G_i, \Sigma) \le \mathrm{mcr}(G, \Sigma) \le \min \left\{ \sum_{i=1}^{k} \mathrm{mcr}(G_i, \Sigma_i) \;\middle|\; \Sigma = \Sigma_1 \# \cdots \# \Sigma_k \right\} .$$

Proof. Let D be an optimal drawing of a realizing graph $\bar{G}$ in Σ. For each G_i it contains an induced subdrawing D_i of some graph $\tilde{G}_i$ with G_i as a minor. G_i and G_j are either disjoint (implying $\tilde{G}_i$ and $\tilde{G}_j$ are disjoint), or they have a cutvertex v in common (implying that $\tilde{G}_i$ and $\tilde{G}_j$ intersect in a part of the tree T_v). As there are at least $\mathrm{mcr}(G_i, \Sigma)$ crossings in D_i and there are no crossings in the subdrawing induced by T_v for any $v \in V_G$, the lower bound follows.

Let us reorder the blocks of G in such way that for $i = 2, \ldots, k$ the block G_i shares at most one vertex with the graph $H_i := \bigcup_{j=1}^{i-1} G_j$. This can be done using the block-cutvertex forest of G.

Let $\Sigma_1, \ldots, \Sigma_k$ be a decomposition of Σ where the minimum is attained. For $i = 1, \ldots, k$ let the D_i be some optimal drawing of $\bar{G}_i$ in Σ_i. Set $\tilde{D}_1 = D_1$, $\tilde{H}_1 = \bar{G}_1$ and $\Pi_1 = \Sigma_1$. For $i = 2, \ldots, k$ we choose a face f_i of $\tilde{D}_{i-1}$ in Π_{i-1} and f_i' of D_i in Σ_i. If H_{i-1} and G_i share a vertex v, then we choose f_i incident with some vertex x_i of $T_v \subseteq \tilde{H}_{i-1}$ and f_i' incident with some vertex y_i of $T_v \subseteq \bar{G}_i$, otherwise the choice can be arbitrary. By constructing a connected sum of faces f_i, f_i' and, if necessary, connecting x_i with y_i in the new face $f_i \# f_i'$, we obtain a drawing $\tilde{D}_i$ of $\tilde{H}_i$ in $\Pi_i := \Pi_{i-1} \# \Sigma_i$.

It is clear that $G \leq_m \tilde{H}_k$ and that $\tilde{D}_k$ is a drawing of $\tilde{H}_k$ in Σ with at most $\sum_{i=1}^{k} \mathrm{mcr}_{\Sigma_i}(G_i)$ crossings. The upper bound follows. □

There exist graphs for which both lower-bound and upper-bound inequalities are strict in some surface $\Sigma \neq \mathbb{S}_0$. However, such a situation is not possible in case of graphs drawn in the sphere:

Corollary 4.2 *Let G be a graph with blocks $G_1, \ldots, G_k$. Then*

$$\mathrm{mcr}(G) = \sum_{i=1}^{k} \mathrm{mcr}(G_i).$$

Proof. To prove this, one just has to observe that for $\Sigma = \mathbb{S}_0$, the left-hand side and the right-hand side in the inequalities in Theorem 4.1 are equal. □

5 Structure of graphs with bounded $\mathrm{mcr}(G, \Sigma)$

As mentioned in Section 1, the family $\omega(k, \Sigma)$ of all graphs, whose $\mathrm{mcr}(G, \Sigma)$ is at most k, is minor closed. Let us denote by $F(k, \Sigma)$ the set of minimal forbidden minors for $\omega(k, \Sigma)$. $F(k)$ and $\omega(k)$ stand for $F(k, \mathbb{S}_0)$ and $\omega(k, \mathbb{S}_0)$, respectively.

Graphs in $\omega(0, \Sigma)$ have a simple topological characterization — they are precisely the graphs that can be embedded in Σ. A similar topological characterization holds for graphs in $\omega(1)$. They are precisely the graphs that can be embedded in the projective plane with face-width at most 2. This was observed by Robertson and Seymour [16], where they determined the set $F(1)$ of minimal forbidden minors for $\omega(1)$:

Theorem 5.1 (Robertson and Seymour [16]) *The set $F(1)$ contains precisely the 41 graphs $G_1, \ldots, G_{35}$ and $Q_1, \ldots, Q_6$, where $G_1, \ldots, G_{35}$ are the minimal forbidden minors for embeddability in the projective plane and $Q_1, \ldots, Q_6$ are projective planar graphs that can be obtained from the Petersen graph by successively applying the $Y\Delta$ and ΔY operations.*

This theorem establishes the following linear time algorithm for testing if $\mathrm{mcr}(G) \leq 1$: first embed G in the projective plane [12] and then check whether the face-width of the embedding is less or equal 2 [13].

Let us remark that the forbidden minors for the projective plane have been determined by Glover et al. [7] and Archdeacon [2]. There are 7 graphs that can be obtained from the Petersen graph by $Y\Delta$ and ΔY operations (known as the Petersen family), but one of them cannot be embedded in the projective plane, and is one of the forbidden minors for the projective plane.

We will prove that every family $\omega(k,\Sigma)$ has a similar topological representation, for which we need some further definitions.

Let γ be a onesided simple closed curve in a nonorientable surface Π of Euler genus g. Cutting Π along γ and pasting a disk to the resulting boundary yields a surface denoted by Π/γ of Euler genus $g-1$. We say that Π/γ is obtained from Π by *annihilating* a crosscap at γ.

Let us call a set of pairwise noncrossing, onesided, simple closed curves $\Gamma = \{\gamma_1,\ldots,\gamma_k\}$ in a nonorientable surface Π a *k-system* in Π. It is easy to see that for distinct $\gamma_i,\gamma_j \in \Gamma$ the surface $(\Pi/\gamma_i)/\gamma_j$ is homeomorphic to $(\Pi/\gamma_j)/\gamma_i$. Therefore the order in which we annihilate the crosscaps at prescribed curves is irrelevant and we define $\Pi/\Gamma := \Pi/\gamma_1/\ldots/\gamma_k$. We say that the k-system Γ in Π is an *orienting k-system*, if the surface Π/Γ is orientable.

Suppose that D is a drawing of G in a nonorientable surface Π with at most c crossings. If there exists an (orienting) k-system Γ in Π with each $\gamma \in \Gamma$ intersecting D in at most two points, then we say that D is (*orientably*) *(c,k)-degenerate*, and we call Γ an (*orienting*) *k-system* of D. If $c = 0$ then D is an embedding and we also say that it is *k-degenerate*. Let us observe that an embedding of a graph in the projective plane is 1-degenerate precisely when the face-width of the embedding is at most 2.

Lemma 5.2 *Let Σ be an* (*orientable*) *surface of Euler genus g and let $k \geq 1$ be an integer. Then, for any $l \in \{1,\ldots,k\}$, the family $\omega(k,\Sigma)$ consists precisely of all those graphs $G \in \omega(k-l,\mathbb{N}_{g+l})$, for which there exists a graph $\tilde{G}$ that contains G as a minor and that can be drawn in the nonorientable surface $\mathbb{N}_{g+l}$ of Euler genus $g+l$ with* (*orienting*) *degeneracy $(k-l,l)$.*

Proof. Let $G \in \omega(k,\Sigma)$ and let $\bar{G}$ be its realizing graph, drawn in Σ with at most k crossings. Choose a subset of l crossings of $\bar{G}$. By replacing a small disk around each of the chosen crossings with a Möbius band, we obtain a drawing of $\bar{G}$ in $\mathbb{N}_{g+l}$ with (orienting) degeneracy $(k-l,l)$.

For the converse we first prove the induction basis $l = 1$.

Let $\tilde{G}$ be the graph that contains G as a minor and is drawn in $\mathbb{N}_{g+1}$ with at most $k-1$ crossings, and let us assume that a onesided curve γ intersects the drawing of $\tilde{G}$ in at most two points, x and y. After cutting the surface along γ and pasting a disc Δ on the resulting boundary, we get a surface

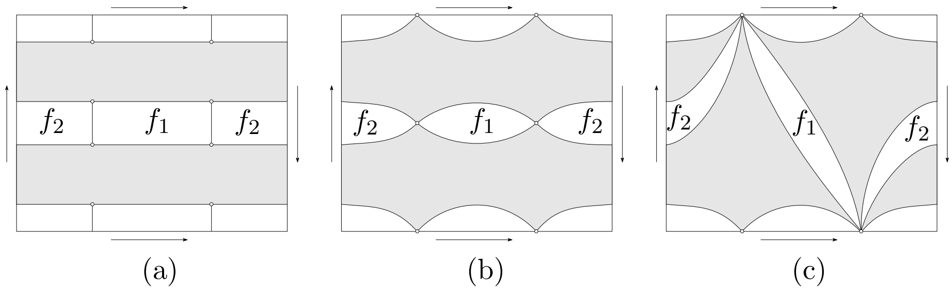

Figure 4: Embeddings in the Klein bottle with orienting degeneracy 2.

of Euler genus g. On the boundary of Δ, two copies of x and y interlace. By adding paths P_x and P_y joining the copies of x and y (respectively), we obtain a drawing D' of a graph G', which contains $\tilde{G}$ (and hence also G) as a minor. Clearly, D' has one crossing more (the one between P_x and P_y) than the drawing of $\tilde{G}$. So, D' is $(k-1,1)$-degenerate.

If $l \geq 2$, we may annihilate the crosscaps consecutively, as the curves in the corresponding l-system are noncrossing. Note that if the l-system is orienting, we obtain an orientable surface Σ. □

Lemma 5.3 *Let $\tilde{G}$ be a graph with an* (*orientably*) *k-degenerate embedding in a surface Σ. If G is a surface minor of $\tilde{G}$, then G is also* (*orientably*) *k-degenerate.*

Proof. It suffices to verify the claim for edge-deletions and edge-contractions. For edge-deletions, there is noting to be proved, and for edge contractions, one only has to show that a k-system for $\tilde{G}$ can be transformed into a k-system for $\tilde{G}/e$. We leave the details to the reader. □

Lemma 5.3 can be extended to drawings with crossings, if we restrict edge-contraction to edges that are not involved in crossings.

As a direct consequence of Lemmas 5.2 and 5.3 we have:

Theorem 5.4 *Let Σ be an* (*orientable*) *surface of Euler genus g and let $k \geq 1$ be an integer. Then $\omega(k, \Sigma)$ consists of precisely all the graphs that can be embedded in the nonorientable surface $\mathbb{N}_{g+k}$ of Euler genus $g+k$ with* (*orienting*) *degeneracy k.*

Figure 4(a) exhibits the geometric structure of a realizing drawing in the Klein bottle; (b) shows the general structure of its minors G with $\mathrm{mcr}(G) \leq$

2, and (c) is a degenerate example of this structure in which the curves of the corresponding 2-system $\{\gamma_1, \gamma_2\}$ touch twice.

Theorem 5.4 can be used to express a more intimate relationship between the graphs in $\omega(k, \Sigma)$ and $\omega(0, \Sigma)$:

Corollary 5.5 *Let Σ be a surface of Euler genus g, $k \geq 0$ an integer, and let $G \in \omega(k, \Sigma)$. Then there exists a graph H, which embeds in Σ, such that G can be obtained from H by identifying at most k pairs of vertices.*

Theorem 5.4 can be used to improve the lower bound of Proposition 3.2.

Theorem 5.6 *Let G be a simple graph with $n = |V_G|$, $m = |E_G|$ and let Σ be a surface of Euler genus g. Then*

$$\mathrm{mcr}(G, \Sigma) \geq \tfrac{1}{2}(m - 3(n + g) + 6).$$

Two technical lemmas are needed for the proof of this result. Let Σ be a closed surface and $x, y \in \Sigma$. Let $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ be a k-system of onesided noncrossing simple closed curves in Σ such that $\gamma_i \cap \gamma_j = \{x, y\}$ for all $1 \leq i < j \leq k$. Let $\gamma_i = \gamma_i^1 \cup \gamma_i^2$ where γ_i^l is an arc from x to y. If a curve $\gamma_i^l \cup \gamma_j^m$ $(i \neq j)$ bounds a disk in Σ whose interior contains no segment of curves in Γ, then we say that $\gamma_i^l \cup \gamma_j^m$ is a Γ-*digon*.

Lemma 5.7 *Every k-system Γ has at most $k - 1$ Γ-digons.*

Proof. We assume the notation introduced above. Let us contract one of the segments, say γ_1^1. Then each other γ_i^l becomes a loop in Σ. Since Γ is a k-system of onesided noncrossing loops, the loops in Γ generate a k-dimensional subspace of the first homology group $H_1(\Sigma; \mathbb{Z}_2)$. Therefore the $2k - 1$ loops $L = \{\gamma_i^l \mid 1 \leq i \leq k, l = 1, 2\} \setminus \{\gamma_1^1\}$ also generate at least k-dimensional subspace. If there are k Γ-digons, then k of the loops could be removed from L and the remaining $k - 1$ loops would still generate the same k-dimensional subspace. This contradiction completes the proof. □

Let G be a graph and D its k-degenerate embedding in a surface Σ. Let $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ be the corresponding k-system of D. The curves γ_i are pairwise noncrossing, so we may assume that γ_i and γ_j $(i \neq j)$ intersect (touch) only in points where they intersect the graph. We subdivide edges of D in such a way that every γ_i intersects D only at vertices. If γ_i intersects D at vertices u_i and v_i, we add to D two new edges e_i, f_i with ends u_i, v_i whose embedding in Σ coincides with γ_i. (If $u_i = v_i$, we add one loop e_i

at v_i.) We call the resulting embedding D' a *k-augmented embedding* of D and the corresponding graph G' a *k-augmented graph* of G (with respect to Γ). Let us observe that we may assume that curves in Γ intersect D only at vertices. In that case, subdivision of edges is not necessary and then G is a subgraph of G'.

Lemma 5.8 *Let D be a k-degenerate embedding of a simple graph G in a nonorientable surface Σ and let D' be a k-augmented embedding of D. Then D' has at most k faces of length two and has no faces of length one.*

Proof. We shall use the notation introduced before the lemma. Since G is a simple graph, any face of length 1 or 2 involves some edge e_i, f_i ($i \in \{1, \dots, k\}$). If e_i is a loop, it cannot bound a face since γ_i is a onesided curve in Σ. Two loops cannot form a facial boundary since then they would be homotopic, and homotopic onesided curves always cross each other. So, an edge e_i or f_i can be part of a face of length two only when $u_i \neq v_i$.

For simplicity of notation, suppose that $\gamma_1, \dots, \gamma_t$ all contain the same pair of vertices u_1 and v_1. It suffices to see that the edges e_i, f_i ($i = 1, \dots, t$) and possible edge $e_0 = u_1v_1$ of G together form at most t faces of length 2. By Lemma 5.7, $\{e_i, f_i \mid 1 \leq i \leq t\}$ form at most $t - 1$ faces of length 2, and e_0 can give rise to one additional such face. This proves the claim, and the application of this claim to all pairs u_i, v_i completes the proof of the lemma. □

Proof of Theorem 5.6. Let $\mathrm{mcr}(G, \Sigma) = k$. By Theorem 5.4, there exists an embedding D of G in $\mathbb{N}_{g+k}$ with crossing degeneracy k. Let D' be a k-augmented embedding of D, and let G' be its graph. By Lemma 5.8, removing at most k edges from G' yields an embedding D'' without faces of length two, implying $|F_{D''}| \leq \frac{2}{3}|E_{D''}|$. Euler formula implies $n - |E_{D''}| + |F_{D''}| = 2 - (g + k)$. The stated inequality follows. □

If one would like to extend the bound of Proposition 3.2 for graphs of girth $r \geq 4$, additional arguments would be needed.

6 Examples

So far, we have developed some tools to find lower bounds of the minor crossing number. In this section, they are applied to several families of graphs. In general, Theorem 2.3 yields better bounds for graphs of small maximum degree (cubes, $C_n \Box C_m$) while Theorem 3.1 suits graphs with

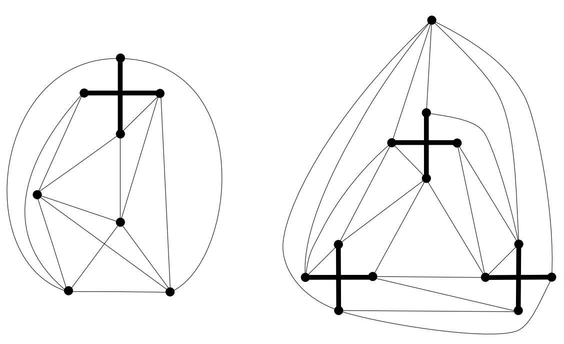

Figure 5: Realizing drawings of K_6, K_7, and K_8, respectively

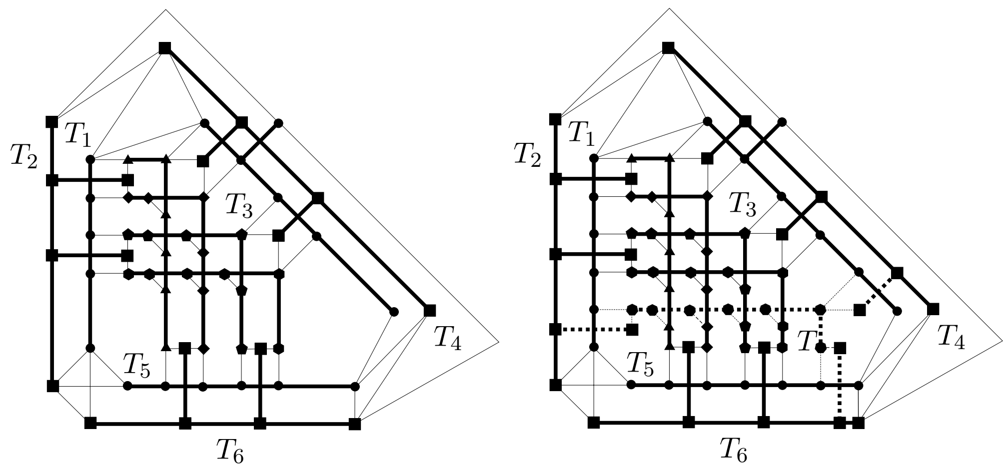

Figure 6: Drawings of graphs $\tilde{K}_{10}$ and $\tilde{K}_{11}$

large maximum degree better, e.g. complete bipartite graphs. Theorem 5.6 performs best on dense graphs of girth three, for instance complete graphs.

6.1 Complete graphs

Theorem 5.6 implies the following inequality, which is sharp for $n \in \{3, \dots, 8\}$, as demonstrated in Figure 5:

Proposition 6.1 *Let* $n \geq 3$*. Then* $\operatorname{mcr}(K_n) \geq \left\lceil \frac{1}{4}(n-3)(n-4) \right\rceil$.

The following proposition establishes an upper bound:

Proposition 6.2 *For* $n \geq 9$*,* $\operatorname{mcr}(K_n) \leq \left\lfloor \frac{1}{2}(n-5)^2 \right\rfloor + 3$.

Sketch of a proof. We shall exhibit graphs $\tilde{K}_n$ ($n \geq 9$) together with their drawings D_n so that $\tilde{K}_n$ contains K_n as a minor and that $\mathrm{cr}(D_n) = \left\lfloor \frac{1}{2}(n-5)^2 \right\rfloor + 3$. Figure 6 presents drawings of $\tilde{K}_{10}$ and $\tilde{K}_{11}$. Different vertex symbols (diamond, circle, triangle, ...) represent vertices in the same tree T_v, $v \in V_{K_n}$, which contracts to the vertex v in the K_n minor. By contracting the thick edges of the graphs in Figure 6, we obtain K_{10} and K_{11}, respectively.

The reader should have no difficulty placing the tree T_{n+1} into D_n in order to obtain D_{n+1}. The tree T_{n+1} crosses precisely each T_v with $7 \leq v \leq n$. To connect T_{n+1} with the trees $T_1, \ldots, T_6$, we need three new crossings if n is even (T_1 with T_2, T_3 with T_4 and T_5 with T_6) and no new crossing if n is odd.

Let c_n denote the number of crossings in the drawing of $\tilde{K}_n$ described above, and let $a_k = c_{2k}$. We have $a_4 = 6$, $a_5 = 14$, $a_6 = 26$ and a recurrence equation

$$a_{k+1} = a_k + 4k - 8,$$

whose solution is $a_k = 2k^2 - 10k + 14$. For even n this yields

$$c_n = \tfrac{1}{2}((n-5)^2 + 3)$$

and for odd n

$$c_n = \tfrac{1}{2}(n-5)^2 + 3.$$

□

Corollary 6.3 *Let Σ be a fixed surface. For $n \in \mathbb{N}$, let $c_n = \frac{\mathrm{mcr}(K_n,\Sigma)}{n(n-1)}$. The sequence $\{c_n\}_{n=1}^{\infty}$ is nondecreasing and*

$$c_\infty := \lim_{n\to\infty} c_n \in \left[\tfrac{1}{4}, \tfrac{1}{2}\right].$$

Proof. First we prove the following claim: Let $\mathrm{mcr}(K_n, \Sigma) \geq c\,n(n-1)$. Then $\mathrm{mcr}(K_m, \Sigma) \geq c\,m(m-1)$, for every $m \geq n$.

Clearly it suffices to prove this for $m = n+1$. Let $\bar{D}$ be a realizing drawing of K_{n+1} in Σ. Let T_i be the tree in $\bar{D}$ which contracts to the vertex i of K_{n+1}. If we remove T_i and all incident edges from $\bar{D}$, we obtain a drawing of a graph with K_n minor. This can be done in $n+1$ different ways. These $n+1$ drawings contain at least $(n+1)\,\mathrm{mcr}(K_n, \Sigma)$ crossings altogether. We may assume there are no removed edges in $\bar{D}$, as their number can only increase the number of crossings. Then each crossing from $\bar{D}$ appears in

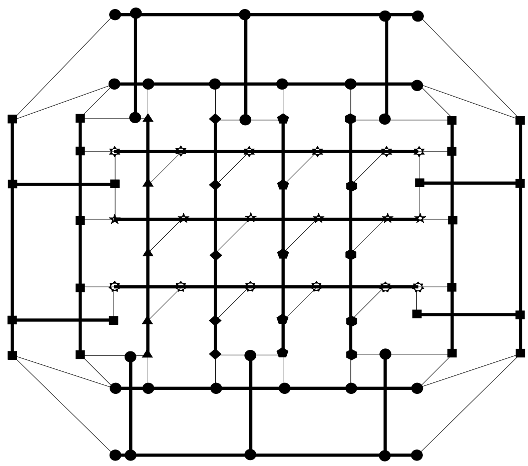

Figure 7: A drawing of the graph $\tilde{K}_{8,7}$ with 22 crossings

at most $n-1$ of these drawings. Therefore, $(n-1)\,\mathrm{mcr}(K_{n+1},\Sigma) \geq (n+1)\,\mathrm{mcr}(K_n,\Sigma) \geq c\,(n+1)n(n-1)$.

The stated bounds on c_∞ follow easily from Proposition 6.1 and Proposition 6.2. □

6.2 Complete bipartite graphs

The genus of complete bipartite graphs [14, Theorem 4.4.7] in combination with Theorem 3.1 establishes the following proposition:

Proposition 6.4 *Let* $3 \leq m \leq n$. *Then*

$$\mathrm{mcr}(K_{m,n}) \geq \left\lfloor \tfrac{1}{2}(m-2)(n-2) \right\rfloor .$$

For the upper bound, consider a set of graphs $\tilde{K}_{m,n}$. They are constructed in a similar way as their complete analogues $\tilde{K}_n$, and an example is presented in Figure 7.

Proposition 6.5 *Let* $3 \leq m \leq n$. *Then*

$$\mathrm{mcr}(K_{m,n}) \leq (m-3)(n-3) + 5.$$

6.3 Hypercubes

Applying Proposition 3.2 to hypercubes yields

Proposition 6.6 *Let* $n \geq 4$. *Then* $\mathrm{mcr}(Q_n) \geq (n-4)2^{n-2} + 2$.

Using the best known lower bound for crossing number of hypercubes: $\mathrm{cr}(Q_n) > 4^n/20 - (n^2+1)2^{n-1}$ by Sýkora and Vrťo [17] in combination with Theorem 2.3, we can deduce an alternative lower bound, which is stronger for large values of n:

Proposition 6.7 *Let* $n \geq 4$. *Then* $\mathrm{mcr}(Q_n) > \frac{1}{n^2}\left(\frac{1}{5}\,4^n - 2^{n+1}\right) - 2^{n+1}$.

Following the same idea as in [11, Figures 2, 3], one can obtain a family of graphs $\tilde{Q}_n$ and their drawings D_n with $\Delta(\tilde{Q}_n) = 4$ and $\tilde{Q}_n$ having Q_n as a minor. They establish the following upper bound:

Proposition 6.8 *Let* $n \geq 2$. *Then* $\mathrm{mcr}(Q_n) \leq 2^{n-2}(7 \cdot 2^{n-5} + 2n - 4)$.

6.4 Cartesian products of cycles $C_m \Box C_n$

Combining the results presented in [5] and Theorem 2.3 implies the following fact.

Suppose that $n \geq m$ and either $m \leq 7$, or $m \geq 3$ and $n \geq m(m+1)$. Then $\frac{1}{4}(m-2)n \leq \mathrm{mcr}(C_m \Box C_n) \leq (m-2)n$.

We believe that the value of $\mathrm{mcr}(C_m \Box C_n)$ is equal (or close) the upper bound, and that this example shows the need for stronger results than Theorem 2.3.

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