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SPLITTING ALONG A SUBMANIFOLD PAIR

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Abstract. The splitting obstruction groups arise naturally in closed manifold surgery problem and in computations of maps in the surgery exact sequence. In the present paper we introduce a group of obstructions for splittting a homotopy equivalence along a pair of submanifolds and study its properties. We describe relations between introduced groups and various surgery obstruction groups for the manifold triple. We study relations of these groups with the structure sets which arise for the triple of manifolds. The natural map from the surgery obstruction group of the ambient manifold to the introduced group gives a forbidden invariant for realization of elements of the Wall group by normal maps of closed manifolds.

1. Introduction.

Consider a simple homotopy equivalence $f : M \to X$ of closed n-dimensional topological manifolds. Such a map is called an s-triangulation of the manifold X. Two s-triangulations

$$f_i : M_i \to X,\ i = 1, 2$$

are said to be equivalent if there exists a preserving orientation homeomorphism $h : M_1 \to M_2$ fitting in the homotopy commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{h} & M_2 \\ & \searrow f_1 & \downarrow f_2 \\ & & X \end{array} \tag{1.1}$$

The set of equivalence classes of s-triangulations of the manifold X is denoted by $\mathcal{S}(X) = \mathcal{S}^s(X)$ (see [17] and [20]). The computation of the structure set $\mathcal{S}^s(X)$ for a manifold X is one the main problems of geometric topology.

Let $Y \subset X$ be a locally flat submanifold of codimension q in n-dimensional topological manifold X. A simple homotopy equivalence $f : M \to X$ splits along the submanifold

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Y (see [17] and [20]) if it is homotopy equivalent to a map g, transversal to Y such that $N = g^{-1}(Y)$ and satisfies the following properties:

$$\begin{array}{rr} i) & g|_N : N \to Y \text{ is a simple homotopy equivalence,} \\ ii) & g|_{(M\setminus N)} : M \setminus N \to X \setminus Y \text{ is a simple homotopy equivalence.} \end{array} \tag{1.2}$$

A simple homotopy equivalence $g : M \to X$ with the properties (1.2) is called an s-triangulation of the pair (X, Y). The set of concordance classes of such s-triangulations is denoted by $\mathcal{S}(X, Y, \xi)$ where ξ is the topological normal bundle of the submanifold Y in X (see [17, §7.2]).

Let U be a tubular neighborhood of the submanifold Y in X, and let ∂U denotes the boundary of U. Denote by

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(U) & \to & \pi_1(X) \end{pmatrix} \tag{1.3}$$

the push-out square of fundamental groups with orientations.

An obstruction to splitting the map f along the submanifold Y lies in the splitting obstruction group $LS_{n-q}(F)$ which depends only on $n - q \bmod 4$ and on the push-out square F.

In fact, the obstruction to splitting correctly defines the map [17] that fits in the following exact sequence

$$\cdots \to \mathcal{S}(X, Y, \xi) \to \mathcal{S}(X) \to LS_{n-q}(F). \tag{1.4}$$

The splitting obstruction groups are closely related to other obstruction groups which arise naturally for the manifold pair $Y \subset X$. The main relation is given by the following braid of exact sequences (see [17] and [20])

$$\begin{array}{ccccccccc} \to L_n(\pi_1(X\setminus Y) & & \longrightarrow & & L_n(\pi_1(X)) & & \longrightarrow & & LS_{n-q-1}(F) \to \\ \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\ & & LP_{n-q}(F) & & & & L_n(\pi_1(X\setminus Y) \to \pi_1(X)) & & \\ \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\ \to LS_{n-q}(F) & & \longrightarrow & & L_{n-q}(\pi_1(Y)) & & \to & & L_{n-1}(\pi_1(X\setminus Y) \to \end{array} \tag{1.5}$$

where $L_* = L_*^s$ denote the surgery obstruction groups and $LP_*(F) = LP_*^s(F)$ denote the surgery obstruction groups of the manifold pair (X, Y). The groups $LP_*(F)$ also depend only on $n - q \bmod 4$ and on the square F.

The main methods to compute the set $\mathcal{S}(X)$ (for $n \geq 4$) are based on the surgery exact sequence (see [16], [17], and [20])

$$\cdots \to L_{n+1}(\pi_1(X)) \to \mathcal{S}(X) \to [X, G/TOP] \xrightarrow{\sigma} L_n(\pi_1(X)) \tag{1.6}$$

where the set $[X, G/TOP]$ is isomorphic to the set of concordance classes of topological normal maps to the manifold X.

The set $\mathcal{S}(X, Y, \xi)$ fits in the surgery exact sequence for the manifold pair (X, Y)

$$\cdots \to LP_{n-q+1}(F) \to \mathcal{S}(X, Y, \xi) \to [X, G/TOP] \to LP_{n-q}(F). \tag{1.7}$$

The exact sequence (1.7) is the natural generalization of the exact sequence (1.6) to the case of a manifold pair.

The computation of the map σ in (1.6) is the basic problem to investigate the surgery exact sequence. For the manifolds with finite fundamental groups deep results in this direction were obtained in [4], [5], [8], [9], and [10]. The results of these papers are based on relations between the surgery exact sequence and the splitting problem for one-sided submanifold.

Let

$$Z^{n-q-q'} \subset Y^{n-q} \subset X^n \tag{1.8}$$

be a triple of closed topological manifolds. We shall consider only locally flat topological submanifolds equipped with the structure of normal topological bundle (see [17, pages 562–563]). Such triple of manifolds defines a stratified manifold $\mathcal{X}$ in the sense of Browder and Quinn (see [3], [13], [14],[15], and [20]).

A simple homotopy equivalence $f : M \to X$ is an s-triangulation of the triple if every pair of manifolds from this triple satisfies the properties that are similar to (1.2) for the pair (X, Y) (see [3], [15], and [20]). The set of concordance classes of such s-triangulations is denoted by $\mathcal{S}(\mathcal{X}) = \mathcal{S}(X, Y, Z)$.

The surgery theory is applicable for stratified spaces, and we have the following exact sequence

$$\cdots \to L^{BQ}_{n+1}(\mathcal{X}) \to \mathcal{S}(\mathcal{X}) \to [X, G/TOP] \to L^{BQ}_n(\mathcal{X}) \tag{1.9}$$

where $L^{BQ}_*(\mathcal{X})$ are the Browder-Quinn surgery obstruction groups of the stratified space $\mathcal{X}$. For these groups we have isomorphisms

$$L^{BQ}_n(\mathcal{X}) = LT_{n-q-q'}(X, Y, Z)$$

with surgery obstruction groups LT_* of the manifold triple (X, Y, Z) (see [13] and [15]).

In the present paper we develop the surgery theory for manifold triples to investigation of the splitting of a homotopy equivalence along a submanifold pair. By definition, a simple homotopy equivalence $f : M \to X$ splits along the submanifold pair $(Z \subset Y)$ if it is concordant to an s-triangulation g of the triple $Z \subset Y \subset X$. We introduce groups LSP_* of obstructions to splitting of a simple homotopy equivalence $f : M \to X$ along a pair of embedded submanifolds $(Z \subset Y) \subset X$ and describe their relations to classical obstruction groups in surgery theory. The group LSP_* is a natural straightforward generalization of the group LS_* if we consider the pair of submanifolds $(Z \subset Y)$ instead of the submanifold Y. The introduced groups give in a natural way a forbidden invariant in the problem of realization of elements of Wall groups by normal maps of closed manifolds. This invariant is equivalent to the pair of Hambleton's invariants (A and B) from paper [5].

2. Preliminaries.

In the papers [1],[7], [12], [15], and [20] the algebraic theory of surgery was developed. It is based on the application of spectra to surgery theory (see [16] and [17]). In this section we recall some necessary definitions and results from these papers.

Consider a triple of topological manifolds (1.8). Let ξ be the normal bundle of Y in X and F be the square of fundamental groups in the splitting problem for the pair $Y \subset X$. In the similar way we introduce the following objects:

the bundle η and the square Ψ for the pair $Z \subset Y$,

the bundle ν and the square Φ for the pair $Z \subset X$.

Let U_ξ be a space of the normal bundle ξ. We shall assume that the space U_ν of the normal bundle ν is identified with the space V_ξ of the restriction of the bundle ξ on the space U_η of the normal bundle η in such way that $\partial U_\nu = \partial U_\xi|_{U_\eta} \cup U_\xi|_{\partial U_\eta}$ (see [3], [14],[15], and [20]).

The conditions on the spaces of normal bundles for the manifold triple (1.8) yield a pair of manifolds with boundaries

$$(Y \setminus Z, \partial(Y \setminus Z)) \subset (X \setminus Z, \partial(X \setminus Z)) \tag{2.1}$$

where

$$\partial(Y \setminus Z) \subset \partial(X \setminus Z) \tag{2.2}$$

is the closed manifold pair. Denote by F_Z the square of fundamental groups in the splitting problem relative to boundary for the pair (2.1), and by F_U the square in the splitting problem for the pair (2.2).

For arbitrary group π with orientation we denote by $\mathbb{L}(\pi)$ [7] an Ω-spectrum with

$$\pi_n(\mathbb{L}(\pi)) = L_n(\pi).$$

Let $\mathbf{L}_\bullet$ be the 1-connected cover of the spectrum $\mathbb{L}(1)$ with $\mathbf{L}_{\bullet 0} = G/TOP$. For a topological space X we have the following cofibration (see [16] and [17])

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbb{L}(\pi_1(X)) \to \mathbb{S}(X). \tag{2.3}$$

The homotopy long exact sequence of the cofibration (2.3) gives the algebraic surgery exact sequence of Ranicki

$$\cdots \to L_{n+1}(\pi_1(X)) \to \mathcal{S}_{n+1}(X) \to H_n(X, \mathbf{L}_\bullet) \to L_n(\pi_1(X)) \to \cdots \tag{2.4}$$

with

$$\pi_{n+1}(\mathbb{S}(X)) = \mathcal{S}_{n+1}(X) \cong \mathcal{S}^{TOP}(X).$$

The left part of the exact sequence (2.4) is isomorphic to the exact sequence (1.6).

A similar result is valid for the exact sequences (1.4), (1.7), and (1.9). In particular, we have the cofibrations of spectra

$$\mathbb{S}(X, Y, \xi) \to \mathbb{S}(X) \to \Sigma^{q+1}\mathbb{L}S(F), \tag{2.5}$$

$$X_+ \wedge \mathbf{L}_\bullet \to \Sigma^q \mathbb{L}P(F) \to \mathbb{S}(X, Y, \xi), \tag{2.6}$$

and

$$X_+ \wedge \mathbf{L}_\bullet \to \Sigma^{q+q'} \mathbb{L}T(X, Y, Z) \to \mathbb{S}(X, Y, Z), \tag{2.7}$$

where Σ denotes the suspension functor on the category of Ω-spectra. These cofibrations generate exact sequences that contain the parts which are isomorphic to the exact sequences (1.4), (1.7), and (1.9), respectively.

Recall that for arbitrary pair (X, Y) of topological spaces equipped with orientation a spectrum $\mathbb{S}(X, Y)$ for the relative structure sets $\mathcal{S}_*(X, Y)$ is defined (see [13], [14], [16], and [17]).

A homomorphism of oriented groups $f : \pi \to \pi'$ induces a cofibration of Ω-spectra

$$\mathbb{L}(\pi) \longrightarrow \mathbb{L}(\pi') \longrightarrow \mathbb{L}(f) \tag{2.8}$$

where $\mathbb{L}(f)$ is the spectrum for relative L-groups of the map f.

For the manifold pair (X, Y) we have a homotopy commutative diagram of spectra (see [1], [7], and [14])

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \to & \Sigma^{-q}\mathbb{L}(\pi_1(\partial U) \to \pi_1(U)) & \to & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \\ & \searrow & \downarrow & & \downarrow \\ & & \Sigma^{1-q}\mathbb{L}(\pi_1(\partial U)) & \to & \Sigma^{1-q}\mathbb{L}(\pi_1(X \setminus Y)), \end{array} \tag{2.9}$$

where the left maps are the transfer maps on the spectra level, and the right horizontal maps are induced by inclusions.

The diagram (2.9) provides a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \to & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) & \to & \Sigma\mathbb{L}S(F) \\ \downarrow = & & \downarrow & & \downarrow \\ \mathbb{L}(\pi_1(Y)) & \to & \Sigma^{1-q}\mathbb{L}(\pi_1(X \setminus Y)) & \to & \Sigma\mathbb{L}P(F) \end{array} \tag{2.10}$$

in which the horizontal rows are cofibrations. The homotopy long exact sequences of the maps from the diagram (2.10) generates the diagram (1.5).

The triple (1.8) defines also on the spectra level the composition (see [15] and [21])

$$\mathbb{L}(\pi_1(Z)) \to \Sigma^{-q'+1}\mathbb{L}P(F_U) \to \Sigma^{-q'+1}\mathbb{L}P(F_Z) \tag{2.11}$$

where the first map is the transfer map, and the second map is induced by the inlusion in (2.1).

In accordance with [15] and [21] we have the cofibration

$$\mathbb{L}(\pi_1(Z)) \to \Sigma^{-q'+1}\mathbb{L}P(F_Z) \to \Sigma\mathbb{L}T(X, Y, Z) \tag{2.12}$$

where the first map is the composition (2.11).

Consider the composition

$$\mathbb{L}P(F) \to \mathbb{L}(\pi_1(Y)) \to \mathbb{S}(Y) \to \Sigma^{q'+1}\mathbb{L}S(\Psi). \tag{2.13}$$

The first map in (2.13) follows from (2.10), the second is the map from (1.3) for the manifold Y, and the third map is the map from (2.5) for the pair (Y, Z). In accordance with [13] and [15] we have the cofibration

$$\mathbb{L}P(F) \to \Sigma^{q'+1}\mathbb{L}S(\Psi) \to \Sigma^{q'+1}\mathbb{L}T(X, Y, Z). \tag{2.14}$$

From the cofibration (2.14) follows the homotopy pull-back square of spectra

$$\begin{array}{ccc} \mathbb{L}T(X,Y,Z) & \to & \Sigma^{-q'}\mathbb{L}P(F) \\ \downarrow & & \downarrow \\ \mathbb{L}P(\Psi) & \to & \Sigma^{-q'}\mathbb{L}(\pi_1(Y)) \end{array} \tag{2.15}$$

where cofibres of the vertical maps are naturally homotopy equivalent to the spectrum $\Sigma^{-q-q'+1}\mathbb{L}(\pi_1(X \setminus Y))$.

We have the commutative diagram of inclusions

$$\begin{array}{ccc} (Y \setminus Z) & \subset & (X \setminus Z) \\ \cap & & \cap \\ Y & \subset & X. \end{array} \tag{2.16}$$

The horizontal inclusions of submanifolds of codimensin q, provide as in (2.10), the transfer maps fitting in the homotopy commutative diagram

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y\setminus Z)) & \to & \Sigma^{-q}\mathbb{L}(\pi_1(X\setminus Y)\to\pi_1(X\setminus Z)) & \to & \Sigma\mathbb{L}S(F_Z) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}(\pi_1(Y)) & \to & \Sigma^{1-q}\mathbb{L}(\pi_1(X\setminus Y)) & \to & \Sigma\mathbb{L}P(F) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}(\pi_1(Y\setminus Z)\to(\pi_1(Y)) & \stackrel{tr^{rel}}{\to} & \Sigma^{1-q}\mathbb{L}(\pi_1(X\setminus Z)\to\pi_1(X\setminus Y)) & \to & \Sigma^{1+q'}\mathbb{L}NS \end{array} \tag{2.17}$$

in which the upper vertical maps are induced by the vertical maps from (2.16). The spectrum $\mathbb{L}NS = \mathbb{L}NS(X, Y, Z)$ is the spectrum for the relative L-groups of the map tr^{rel} (see [6] and [14]) with the homotopy groups

$$LNS_n = LNS_n(X, Y, Z) = \pi_n(\mathbb{L}NS).$$

Note that the diagram (2.17) generates the following commutative diagram [14]

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots \to & & LS_{n-q}(F_Z) & \to & L_{n-q}(\pi_1(Y \setminus Z)) & \to & L_n(\pi_1(X \setminus Y) \to \pi_1(W)) & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots \to & & LS_{n-q}(F) & \to & L_{n-q}(\pi_1(Y)) & \to & L_n(\pi_1(X \setminus Y) \to \pi_1(X)) & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots \to & & LNS_k & \to & L_{n-q}(\pi_1(Y \setminus Z) \to \pi_1(Y)) & \to & L_n(\pi_1(W) \to \pi_1(X)) & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array} \tag{2.18}$$

where $k = n - q - q'$ and $W = X \setminus Z$.

3. Splitting a homotopy equivalence along a submanifold pair.

In this section for the triple of manifolds (1.8) we introduce the spectrum $\mathbb{L}SP_*(X, Y, Z)$ with homotopy groups

$$LSP_* = LSP_*(X, Y, Z) = \pi_n(\mathbb{L}SP_*(X, Y, Z)). \tag{3.1}$$

The groups $LSP_*(X, Y, Z)$ are the natural straightforward generalization of the splitting obstruction groups $LS_*(F)$ to the case when the manifold X contains a pair of embedded submanifolds $(Z \subset Y) \subset X$ instead of a submanifold Y. We describe relation of the groups $LSP_*(X, Y, Z)$ to classical obstruction groups and structure sets which arise naturally for a triple of manifolds. We also obtain relations of the introduced groups to surgery exact sequences of the manifolds from the given triple.

The bottom map in the diagram (2.15) and the commutative diagram (2.10) provide the homotopy commutative diagram of spectra

$$\begin{array}{ccc} \mathbb{L}P(\Psi) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \\ \downarrow= & & \downarrow \\ \mathbb{L}P(\Psi) & \to & \Sigma^{1-q-q'}\mathbb{L}(\pi_1(X \setminus Y)) \end{array} \tag{3.2}$$

in which the fiber of the bottom map is the spectrum $\mathbb{L}T(X, Y, Z)$ as follows from the pull-back square (2.15). Denote by $\mathbb{L}SP(X, Y, Z)$ the fiber of the upper horizontal map in (3.2). We obtain the following homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}P(\Psi) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) & \to & \Sigma\mathbb{L}SP(X, Y, Z) \\ \downarrow= & & \downarrow & & \downarrow \\ \mathbb{L}P(\Psi) & \to & \Sigma^{1-q-q'}\mathbb{L}(\pi_1(X \setminus Y)) & \to & \Sigma\mathbb{L}T(X, Y, Z) \end{array} \tag{3.3}$$

in which the right vertical map is induced by the two others vertical maps (see [19]). Remark, that the right square in (3.3) is a pull-back.

Proposition 3.1. *The groups $LSP_*(X, Y, Z)$ that are defined by (3.1) fit in the following braid of exact sequences*

$$\begin{array}{ccccccccccc} \to & L_n(C) & & \longrightarrow & & L_n(\pi_1(X)) & & \to & & LSP_{k-1} & \to \\ & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & \\ & & & LT_k(X, Y, Z) & & & & L_n(C \to D) & & & \\ & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & \\ \to & LSP_k & & \longrightarrow & & LP_k(\Psi) & & \longrightarrow & & L_{n-1}(C) & \to, \end{array} \tag{3.4}$$

where $C = \pi_1(X \setminus Y)$, $D = \pi_1(X)$, and $k = n - q - q'$. The diagram (3.4) is realized on the spectra level.

Proof. The left square in the diagram (3.3) is a pull-back. The homotopy long exact sequences of this square provide the commutative braid of exact sequences (3.4). □

Theorem 3.2. *There exists the commutative braid of exact sequences*

$$\begin{array}{ccccccccccc} \to & \mathcal{S}_{n+1}(X,Y,Z) & & \longrightarrow & & H_n(X,\mathbf{L}_\bullet) & & \to & & L_n(\pi_1(X)) & \to \\ & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & & \mathcal{S}_{n+1}(X) & & & & LT_{n-q-q'} & & & \\ & \searrow & \nearrow & & \searrow^{\alpha} & & \nearrow & & \searrow & & \nearrow \\ \to & L_{n+1}(\pi_1(X)) & & \longrightarrow & & LSP_{n-q-q'} & & \longrightarrow & & \mathcal{S}_n(X,Y,Z) & \to \end{array} \tag{3.5}$$

which is realized on the spectra level.

Proof. Consider the homotopy commutative square of spectra

$$\begin{array}{ccc} X\wedge \mathbf{L}_\bullet & \to & \mathbb{L}(\pi_1(X)) \\ \downarrow & & \downarrow = \\ \Sigma^{q+q'}\mathbb{L}T & \to & \mathbb{L}(\pi_1(X)) \end{array} \tag{3.6}$$

in which the upper horizontal map lies in (2.3), the left vertical map lies in (2.7), and the bottom horizontal map is the map from the diagram (3.4) on spectra level (see [13] and [14]). The diagram (3.6) induces the map of the fibres of the horizontal maps. We obtain the homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \Sigma^{-1}\mathbb{S}(X) & \to & X\wedge \mathbf{L}_\bullet & \to & \mathbb{L}(\pi_1(X)) \\ \downarrow & & \downarrow & & \downarrow = \\ \Sigma^{q+q'}\mathbb{L}SP & \to & \Sigma^{q+q'}\mathbb{L}T & \to & \mathbb{L}(\pi_1(X)) \end{array}$$

in which the left square is a push-out. The homotopy long exact sequences of this square give the diagram (3.5). □

The commutative diagram (3.5) is a natural generalization of the diagram from [17, Proposition 7.2.6 iv] to the case of a pair of submanifolds $Z\subset Y$ in the manifold X. The left vertical map in (3.5) induces a map

$$\alpha : \mathcal{S}_{n+1}(X) \to LSP_{n-q-q'}(X,Y,Z)$$

that on the algebraic level corresponds to taking of the obstruction to splittting along the submanifold pair $Z\subset Y$.

Now we describe relation of LSP_* to classical surgery obstruction groups for the triple (X,Y,Z) of manifolds (1.8).

Theorem 3.3. *There exist braids of exact sequences*

$$\begin{array}{ccccccccccc} \to & LS_{n-q}(F_Z) & & \longrightarrow & & LT_k(X,Y,Z) & & \to & & L_n(\pi_1(X)) & \to \\ & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & & LSP_k & & & & LP_k(\Phi) & & & \\ & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\ \to & L_{n+1}(\pi_1(X)) & & \longrightarrow & & LS_k(\Phi) & & \longrightarrow & & LS_{n-q-1}(F_Z) & \to, \end{array} \tag{3.7}$$

$$
(3.8)\qquad
\begin{array}{ccccccccccccc}
\to & & LS_{n-q}(F_Z) & & \longrightarrow & & LS_{n-q}(F) & & \to & & LS_{k-1}(\Psi) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & LSP_k & & & & LNS_k & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & LS_k(\Psi) & & \longrightarrow & & LS_k(\Phi) & & \longrightarrow & & LS_{n-q-1}(F_Z) & & \to,
\end{array}
$$

and

$$
(3.9)\qquad
\begin{array}{ccccccccccccc}
\to & & LS_{n-q+1}(F) & & \longrightarrow & & LS_k(\Psi) & & \to & & LT_k & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & LP_{n-q+1}(F) & & & & LSP_k & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & LT_{k+1} & & \longrightarrow & & L_{n+1}(\pi_1(X)) & & \longrightarrow & & LS_{n-q}(F) & & \to,
\end{array}
$$

where $k = n - q - q'$. The braids (3.7), (3.8), and (3.9) are realized on the spectra level.

Proof. The natural forgetful maps (see [14])

$$\mathbb{L}T(X, Y, Z) \to \mathbb{L}P(\Phi) \to \Sigma^{-q-q'}\mathbb{L}(\pi_1(X))$$

provide the homotopy commutative square

$$
(3.10)\qquad
\begin{array}{ccc}
\mathbb{L}T(X, Y, Z) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)) \\
\downarrow & & \downarrow= \\
\mathbb{L}P(\Phi) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X))
\end{array}
$$

which induces the map of the fibres of the horizontal maps (see [19]). Thus we obtain the homotopy commutative diagram

$$
\begin{array}{ccccc}
\mathbb{L}SP(X, Y, Z) & \to & \mathbb{L}T(X, Y, Z) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)) \\
\downarrow & & \downarrow & & \downarrow= \\
\mathbb{L}S(\Phi) & \to & \mathbb{L}P(\Phi) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X))
\end{array}
$$

in which the left square is a push-out (and hence a pull-back). Now, similarly to the Proposition 3.1, we obtain the diagram (3.7).

The diagram (2.10) for the pair (Y, Z) provides the homotopy commutative pull-back square of spectra

$$
(3.11)\qquad
\begin{array}{ccc}
\mathbb{L}P(\Psi) & \to & \Sigma^{-q'}\mathbb{L}(\pi_1(Y)) \\
\downarrow & & \downarrow \\
\mathbb{L}(\pi_1(Z)) & \to & \Sigma^{-q'}\mathbb{L}(\pi_1(Y \setminus Z) \to \pi_1(Y)).
\end{array}
$$

We can write down the homotopy commutative pull-back square of spectra

$$
(3.12)\qquad
\begin{array}{ccc}
\Sigma^{-q'-q}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) & \overset{=}{\to} & \Sigma^{-q'-q}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \\
\downarrow & & \downarrow \\
\Sigma^{-q'-q}\mathbb{L}(\pi_1(X \setminus Z) \to \pi_1(X)) & \overset{=}{\to} & \Sigma^{-q'-q}\mathbb{L}(\pi_1(X \setminus Z) \to \pi_1(X))
\end{array}
$$

in which the vertical maps are induced by the natural inclusion. The transfer maps and diagrams (2.17) and (3.3) give the map of the diagram (3.11) to the diagram (3.12). The cofibers of this map of the diagrams provide a homotopy commutative pull-back square of spectra (see [19])

$$\begin{array}{ccc} \Sigma\mathbb{L}SP & \rightarrow & \Sigma^{-q'+1}\mathbb{L}S(F) \\ \downarrow & & \downarrow \\ \Sigma\mathbb{L}S(\Phi) & \rightarrow & \Sigma\mathbb{L}NS, \end{array} \tag{3.13}$$

as follows from (2.9), (2.18), and (3.3). The diagram (3.8) follows from the square (3.13) similarly to the previous case.

The natural forgetful maps from the diagram (2.15)

$$\mathbb{L}T(X,Y,Z) \rightarrow \Sigma^{-q'}\mathbb{L}P(F) \rightarrow \Sigma^{-q-q'}\mathbb{L}(\pi_1(X))$$

provide the homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}T(X,Y,Z) & \rightarrow & \Sigma^{-q'}\mathbb{L}P(F) & \rightarrow & \Sigma\mathbb{L}S(\Psi) \\ \downarrow = & & \downarrow & & \downarrow \\ \mathbb{L}T(X,Y,Z) & \rightarrow & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)) & \rightarrow & \Sigma\mathbb{L}SP(X,Y,Z), \end{array} \tag{3.14}$$

in which the rows are cofibrations, and the right vertical map is defined by [19]. Hence the right square in (3.14) is a pull-back. From this the diagram (3.9) follows. □

Corollary 3.4. *There exist exact sequences*

$$\cdots \rightarrow LSP_k \rightarrow LS_{n-q}(F) \rightarrow LS_{k-1}(\Psi) \rightarrow \cdots,$$

$$\cdots \rightarrow LSP_k \rightarrow LS_k(\Phi) \rightarrow LS_{n-q-1}(F_Z) \rightarrow \cdots,$$

and

$$\cdots \rightarrow LSP_k \rightarrow LP_k(\Psi) \rightarrow L_{n-1}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) \rightarrow \cdots,$$

in which the left maps are natural forgetful maps.

Now we describe some relations between the introduced groups LSP_* and various structure sets which arise for the triple of manifolds (X, Y, Z).

Theorem 3.5. *There exist braids of exact sequences*

$$\begin{array}{ccccccccccccc} \rightarrow & \mathcal{S}_n(X) & & \longrightarrow & & LSP_{k-1} & & \rightarrow & & \mathcal{S}_{l-1}(Y,Z,\eta) & & \rightarrow \\ & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ & & & \mathcal{S}_n(X, X\setminus Y) & & & & \mathcal{S}_{n-1}(X,Y,Z) & & & & \\ & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ \rightarrow & \mathcal{S}_l(Y,Z,\eta) & & \longrightarrow & & \mathcal{S}_{n-1}(X\setminus Y) & & \longrightarrow & & \mathcal{S}_{n-1}(X) & & \rightarrow, \end{array} \tag{3.15}$$

(3.16)
$$\begin{array}{ccccccccccccc}
\to & H_l(Y,\mathbf{L}_\bullet) & & \longrightarrow & & L_n(\pi_1(X\setminus Y)\to\pi_1(X)) & & \to & & LSP_{k-1} & & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & LP_k(\Psi) & & & & \mathcal{S}_n(X,X\setminus Y) & & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & LSP_k & & \longrightarrow & & \mathcal{S}_l(Y,Z,\eta) & & \longrightarrow & & H_{l-1}(Y,\mathbf{L}_\bullet) & & \to,
\end{array}$$

$$\begin{array}{ccccccccccccc}
\to & LS_l(F_Z) & & \longrightarrow & & \mathcal{S}_n(X,Y,Z) & & \to & & \mathcal{S}_n(X) & & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & LSP_k & & & & \mathcal{S}_n(X,Z,\nu) & & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & \mathcal{S}_{n+1}(X) & & \longrightarrow & & LS_k(\Phi) & & \longrightarrow & & LS_{l-1}(F_Z) & & \to,
\end{array} \tag{3.17}$$

and

$$\begin{array}{ccccccccccccc}
\to & LS_{l+1}(F) & & \longrightarrow & & LS_k(\Psi) & & \to & & \mathcal{S}_n(X,Y,Z) & & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & \mathcal{S}_{n+1}(X,Y,\xi) & & & & LSP_k & & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & \mathcal{S}_{n+1}(X,Y,Z) & & \longrightarrow & & \mathcal{S}_{n+1}(X) & & \longrightarrow & & LS_l(F) & & \to,
\end{array} \tag{3.18}$$

where $l = n-q, k = n-q-q'$. *The diagrams (3.15)–(3.18) are realized on the spectra level.*

Proof. The transfer map gives the commutative diagram (see [17])

$$\begin{array}{ccc}
H_{n-q}(Y,\mathbf{L}_\bullet) & \xrightarrow{\cong} & H_n(X,X\setminus Y;\mathbf{L}_\bullet) \\
& \searrow & \downarrow \\
& & H_{n-1}(X\setminus Y;\mathbf{L}_\bullet).
\end{array} \tag{3.19}$$

Consider the commutative triangle

$$\begin{array}{ccc}
LP_{n-q-q'}(\Psi) & \to & L_n(\pi_1(X\setminus Y)\to\pi_1(X)) \\
& \searrow & \downarrow \\
& & L_{n-1}(\pi_1(X\setminus Y))
\end{array} \tag{3.20}$$

which lies in the commutative diagram (3.4).

The results of [17, Proposition 7.2.6] provide the maps of the groups from the diagrm (3.19) to the corresponding groups of the diagram (3.20). On the spectra level the cofibres of this map give a homotopy commutative triangle of spectra

$$\begin{array}{ccc}
\mathbb{S}(Y,Z,\eta) & \to & \Sigma^{-q}\mathbb{S}(X,X\setminus Y) \\
& \searrow & \downarrow \\
& & \Sigma^{-q+1}\mathbb{S}(X\setminus Y).
\end{array} \tag{3.21}$$

By [19] the diagram (3.21) induces the homotopy commutative diagram

$$\begin{array}{ccccc} \mathbb{S}(Y,Z,\eta) & \to & \Sigma^{-q}\mathbb{S}(X, X\setminus Y) & \to & \Sigma^{q'+1}\mathbb{L}SP \\ \downarrow = & & \downarrow & & \downarrow \\ \mathbb{S}(Y,Z,\eta) & \to & \Sigma^{-q+1}\mathbb{S}(X\setminus Y) & \to & \Sigma^{-q+1}\mathbb{S}(X,Y,Z) \end{array}$$

in which the rows are cofibrations, and the right square is a pull-back. The homotopy long exact sequences of the maps of this square give the braid (3.15). In a similar way, the maps from (3.19) to (3.20) provide the pull-back square

$$\begin{array}{ccc} \Sigma^{q'}\mathbb{L}P(\Psi) & \to & \Sigma^{-q}\mathbb{L}(\pi_1(X\setminus Y)\to \pi_1(X)) \\ \downarrow & & \downarrow \\ \mathbb{S}(Y,Z,\eta) & \to & \Sigma^{-q}\mathbb{S}(X, X\setminus Y) \end{array}$$

in which the cofibers of the vertical maps are homotopy equivalent to the spectrum $Y_+\wedge\mathbf{L}_\bullet$. From this square we obtain the braid of exact sequences (3.16). The diagram (3.17) is obtained in a similar way if we consider on the spectra level the homotopy commutative triangle of the cofibers of the map from $H_n(X,\mathbf{L}_\bullet)$ to the triangle of natural forgetful maps

$$\begin{array}{ccc} LT_{n-q-q'} & \to & LP_{n-q-q'}(\Phi) \\ & \searrow & \downarrow \\ & & L_n(\pi_1(X)) \end{array} \tag{3.22}$$

which follows from the square (3.10). We obtain the diagram (3.18) in a similar way to the diagram (3.17). To do this we have to consider the commutative triangle

$$\begin{array}{ccc} LT_{n-q-q'} & \to & LP_{n-q}(F) \\ & \searrow & \downarrow \\ & & L_n(\pi_1(X)) \end{array}$$

instead of the triangle (3.22). □

Let $Y^{n-q}\subset X^n$ be a manifold pair with $n-q\geq 5$ and $q\geq 3$. Then by [17] we have isomorphisms

$$LS_n(F)\cong L_n(\pi_1(Y)),\ LP_n(F)\cong L_{n+q}(\pi_1(X))\oplus L_n(\pi_1(Y)). \tag{3.23}$$

Consider the triple of manifolds (1.8) with the conditions

$$n-q-q'\geq 5,\ q\geq 3,\ q'\geq 3. \tag{3.24}$$

Then by [13, Theorem 3] we have isomorphisms

$$LT_{n-q-q'}\cong L_n(\pi_1(X))\oplus L_{n-q}(\pi_1(Y))\oplus L_{n-q-q'}(\pi_1(Z)). \tag{3.25}$$

Now we obtain similar results for the LSP_*-groups.

Theorem 3.6. *Suppose the triple of manifolds (1.8) satisfy the conditions (3.24). Then*

$$LSP_{n-q-q'}(X, Y, Z) \cong L_{n-q}(\pi_1(Y)) \oplus L_{n-q-q'}(\pi_1(Z)).$$

Proof. The result follows considering the diagram (3.9) and using the isomorphisms (3.25) and (3.23). □

Theorem 3.7. *Suppose the triple of manifolds (1.8) satisfy the conditions $n - q - q' \geq 5$ and $q \geq 3$. Then*

$$LSP_{n-q-q'}(X, Y, Z) \cong LP_{n-q-q'}(\Psi).$$

Proof. We have isomorphisms

$$LS_n(F_Z) \cong L_n(\pi_1(Y \setminus Z)),\ LS_n(F) \cong L_n(\pi_1(Y)),\ LS_n(\Phi) \cong L_n(\pi_1(Z)),$$

since $q \geq 3$. From the diagram (2.18) follows the isomorphism

$$LNS_n \cong L_{n+q'}(\pi_1(Y \setminus Z) \to \pi_1(Y))$$

since $q \geq 3$. Now the result follows considering the diagram (3.8). □

The Theorems 3.6 and 3.7 explain the geometrical meaning of the obstruction groups LSP_*. These groups give obstructions to surgery on the submanifold pair (Y, Z) inside the ambient manifold X.

4. Examples and applications.

A pair of manifolds $Y \subset X$ is called the Browder-Livesay pair if Y is an one-sided submanifold of codimension 1 and the horizontal maps in the square (1.3) are isomorphisms (see [2], [4], [5], [10], and [11]). In this case the splitting obstruction groups are denoted by

$$LN_n(\pi_1(X \setminus Y) \to \pi_1(X)) = LS_n(F).$$

Let the pairs of manifolds (X, Y) and (Y, Z) in the triple (1.8) be Browder-Livesay pairs. In this case $q = q' = 1$. Denote by r_p the map

$$L_n(\pi_1(X)) \to LSP_{n-3}(X, Y, Z)$$

from the braid (3.4). Let

$$r : L_n(\pi_1(X)) \to LS_{n-2}(F) = LN_{n-2}(\pi_1(X \setminus Y) \to \pi_1(X))$$

be the map from the braid (1.5). The map r gives the Browder-Livesay invariant of an element $x \in L_n(\pi_1(X))$. If $r(x) \neq 0$ then the element x is not realized by a normal map of closed manifolds [4].

In the paper [5] the invariants A and B were defined. The invariant A coincides with r, and the invariant B is defined on the kernel of the invariant A. The invariant B is called second Browder-Livesay invariant [10]. It is proved in [5] that if $B(x) \neq 0$ then the element x is not realized by a normal map of closed manifolds.

Proposition 4.1. *Let the pairs of manifolds* (X, Y) *and* (Y, Z) *be Browder-Livesay pairs. Then* $r_p(x) \neq 0$ *if and only if* $A(x) \neq 0$ *or* $B(x) \neq 0$*.*

Proof. Consider the exact sequence fitting in the diagram (3.7)

$$\cdots \to LT_{n-2}(X, Y, Z) \to L_n(\pi_1(X)) \xrightarrow{r_p} LSP_{n-3}(X, Y, Z) \to \cdots$$

Now the result follows from this exact sequence and [14, Theorem 3]. □

Corollary 4.2. *If* $r_p(x) \neq 0$ *then the element* $x \in L_n(\pi_1(X))$ *is not realized by a normal map of closed manifolds.*

Now we give several examples of computation of LSP-groups. Consider the triple

$$(Z \subset Y \subset X) = (\mathbb{RP}^n \subset \mathbb{RP}^{n+1} \subset \mathbb{RP}^{n+2}) \tag{4.1}$$

of real projective spaces with $n \geq 5$. The orientation homomorphism

$$w : \pi_1(\mathbb{RP}^k) = \mathbb{Z}/2 \to \{\pm 1\}$$

is trivial for k odd and nontrivial for k even. We have the following table for surgery obstruction groups (see [11] and [20])

	$n=0$	$n=1$	$n=2$	$n=3$
$L_n(1)$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0
$L_n(\mathbb{Z}/2^+)$	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$L_n(\mathbb{Z}/2^-)$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

The superscript "+" denotes the trivial orientation of the corresponding group and superscript "−" denotes the nontrivial orientation. For the Browder-Livesay pairs from (4.1) we have the following squares of fundamental groups

$$F^{\pm} = \begin{pmatrix} 1 & \to & 1 \\ \downarrow & & \downarrow \\ \mathbb{Z}/2^{\mp} & \to & \mathbb{Z}/2^{\pm} \end{pmatrix}.$$

We have the following isomorphisms (see [11, page 15] and [20])

$$LS_n(F^+) = LN_n(1 \to \mathbb{Z}/2^+) = BL_{n+1}(+) = L_{n+2}(1)$$

and

$$LS_n(F^-) = LN_n(1 \to \mathbb{Z}/2^-) = BL_{n+1}(-) = L_n(1).$$

At first we recall intermediate computations of the groups $LP_*(F^{\pm})$ and $LT^*(X, Y, Z)$ from [13]. The computation of LP_*-groups for a pair $Y \subset X$ use the braid of exact sequences (1.5) (see, also [18]). The natural map that forget the manifold X

$$LS_n(F^{\pm}) \to L_n(\mathbb{Z}/2^{\mp})$$

coincides with the map

$$l_n : BL_n(\pm) \to L_{n-1}(\mathbb{Z}/2^{\mp})$$

from [11, page 35]. Using this result and a diagram chasing in the diagram (1.5) we obtain surgery obstruction groups (see also [13])

$$LP_n(F^+) = LP_{n-1}(F^-) = \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}$$

for $n = 0,\ 1,\ 2,\ 3 \pmod 4$, respectively.

Using the connections between these groups and the $LT_*(X, Y, Z)$-groups in [13] the following result was obtained.

Proposition 4.3. *Let M^{n-k} be a closed simply connected topological manifold. For the triple of manifolds*

$$(Z^n \subset Y^{n+1} \subset X^{n+2}) = (M^{n-k} \times \mathbb{RP}^k \subset M^{n-k} \times \mathbb{RP}^{k+1} \subset M^{n-k} \times \mathbb{RP}^{k+2})$$

with $n \geq 5$ we have the following results.

For k odd, the groups LT_n are isomorphic to

$$\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively.

For k even, $LT_0 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $LT_1 \cong \mathbb{Z}/2$. The groups LT_3 and LT_2 fit in the exact sequence

$$0 \to LT_3 \to \mathbb{Z} \to \mathbb{Z} \to LT_2 \to \mathbb{Z}/2 \to 0.$$

□

We applay these results to compute the LSP_*-groups in the considered cases.

Theorem 4.4. *Under assumptions of the Proposition 4.3 we have the following results.*

For k odd, the groups LSP_n are isomorphic to

$$\mathbb{Z}, \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively.

For k even, we have isomorphisms $LSP_0 \cong LSP_1 \cong \mathbb{Z}/2$. The groups LSP_3 and LSP_2 fit in the exact sequence

$$0 \to LSP_3 \to \mathbb{Z} \to \mathbb{Z} \to LSP_2 \to 0.$$

Proof. Consider the case when k is odd. From [13] we conclude that all the maps $LT_n \to LP_{n+1}(F^+)$ are epimorphisms. Now it is easy to describe the maps $LP_n(F^+) \to L_{n+1}(\mathbb{Z}/2^+)$ from the diagram (1.5). For $n = 1 \bmod 4$ and $n = 2 \bmod 4$ these maps are isomorphisms $\mathbb{Z}/2 \to \mathbb{Z}/2$. For $n = 0 \bmod 4$ the map is trivial since the group $L_1(\mathbb{Z}/2^+)$ is trivial. The map

$$\mathbb{Z} = LP_3(F^+) \to L_0(\mathbb{Z}/2^+) = \mathbb{Z} \oplus \mathbb{Z}$$

is an inclusion on a direct summand. The image of this map coincides with the image of the map $L_0(1) \to L_0(\mathbb{Z}/2^+)$ that is induced by the inclusion $1 \to \mathbb{Z}/2^+$. This follows from the commutative triangle

$$\begin{array}{ccc} & \mathbb{Z} & \\ & \| & \\ & LP_3(F^+) & \\ & \cong\nearrow \quad \searrow & \\ L_0(1) & \xrightarrow{mono} & L_0(\mathbb{Z}/2^+) \\ \| & & \| \\ \mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} \end{array}$$

which lies in the diagram (1.5). From the diagram (3.4) we obtain an exact sequence

$$\cdots \to LT_n \xrightarrow{\tau} L_{n+2}(\mathbb{Z}/2^+) \to LSP_{n-1} \to LT_{n-1} \to \cdots$$

The map τ is the composition

$$LT_n \to LP_{n+1}(F^+) \to L_{n+2}(\mathbb{Z}/2^+)$$

of the maps that we already know. Now we can compute the map τ. It is trivial for $n = 3$, an isomorphism $\mathbb{Z}/2 \to \mathbb{Z}/2$ for $n = 1$, an epimorphism $\mathbb{Z}\oplus\mathbb{Z}/2 \to \mathbb{Z}/2$ with a kernel $\mathbb{Z}$ for $n = 0$, and a homomorphism $\mathbb{Z}\oplus\mathbb{Z}/2 \to \mathbb{Z}\oplus\mathbb{Z}$ with kernel $\mathbb{Z}/2$ and cokernel $\mathbb{Z}$ for $n = 2$. Now considering the exact sequence (4.24) we get the result for k odd. The case of k even is obtained in a similar way. □

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