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# ON CONSTRUCTING BASIC EMBEDDINGS IN THE PLANE

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# On constructing basic embeddings in the plane

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**Abstract**

A compactum $K \subseteq \mathbb{R}^2$ is said to be basically embedded in $\mathbb{R}^2$ if for each continuous function $f: K \to \mathbb{R}$ there exist continuous functions $g, h: \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) + h(y)$ for each point $(x,y) \in K$. We give a new proof of the fact that if $K \subseteq \mathbb{R}^2$ does not contain any triple of points forming two orthogonal segments parallel to the coordinate axes, we say $E(K) = \emptyset$, then the embedding of $K$ in $\mathbb{R}^2$ is basic. The original proof uses a reduction to linear operators and non-trivial results of functional analysis. Our proof is elementary and constructive.

We approximate $f$ by $g+h$ on a neighbourhood of $K$, which imitates the property $E(K) = \emptyset$, using a certain labeled graph representing the properties of the neighbourhood.

**Keywords:** Basic embedding; Kolmogorov theorem on representation of functions.

**MSC 2000:** 54C30, 54C25.

## 1 Introduction

In solving Hilbert's 13th problem, Kolmogorov [Kol57, Kol56] and Arnold [Arn57, Arn59] proved that the $n$-dimensional unit cube $\mathbb{I}^n$ can be embedded into the $2n+1$-dimensional Euclidean space, $\psi(\mathbb{I}^n) \subseteq \mathbb{R}^{2n+1}$ in such a way that for every continuous real-valued function $f$ on $\psi(\mathbb{I}^n) \subseteq \mathbb{R}^{2n+1}$, $f \in C(\psi(\mathbb{I}^n))$, there exist functions $g_1, \dots, g_{2n+1} \in C(\mathbb{R})$ such that $f(x_1, \dots, x_{2n+1}) = g_1(x_1) + \ldots + g_{2n+1}(x_{2n+1})$, for all $(x_1, \dots, x_{2n+1}) \in \psi(\mathbb{I}^n) \subseteq \mathbb{R}^{2n+1}$. We say that the embedding $\mathbb{I}^n \subseteq \mathbb{R}^{2n+1}$ is *basic*.

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Ostrand [Ost65] showed that every $n$-dimensional compact metric space is basically embeddable in $\mathbb{R}^{2n+1}$, thus generalizing in a way the classical Nöbeling-Menger embedding theorem [HW41] which, when restricted to compact spaces, states that every $n$-dimensional compact space is embeddable into $\mathbb{R}^{2n+1}$. The parameter $2n+1$ in the embedding theorem can not be improved because there are $n$-dimensional compacta which can not be embedded into $\mathbb{R}^{2n}$ ([Flo35]). Sternfeld [Ste85] showed that the parameter $2n+1$ in the basic embedding theorem is the best possible in a much stronger sense: if a compactum is basically embeddable into $\mathbb{R}^{2n+1}$ then its dimension is at most $n$, for $n \geq 1$. Basic embeddability into the real line $\mathbb{R}^1$ is trivially equivalent to embeddability. The remaining problem of establishing when a compactum is basically embeddable into $\mathbb{R}^2$ was raised already by Arnold [Arn58], and a characterisation was given by Sternfeld [Ste89]:

**Theorem 1.1.** *For a compactum $K \subseteq \mathbb{R}^2$ the following conditions are equivalent:*

*(B) The embedding $K \subseteq \mathbb{R}^2$ is basic.*

*(E) Let $p$ and $q$ denote the two projections, $p(x,y) = x$ and $q(x,y) = y$, and let*

$$E(K) = \{(x,y) \in K \mid \ \operatorname{card}(p^{-1}(x) \cap K) \geq 2 \text{ and } \operatorname{card}(q^{-1}(y) \cap K) \geq 2\}.$$

*Then $E^n(K) = E(E(\ldots E(K)\ldots)) = \emptyset$ for some $n$.*

Using the geometric description $(E)$, Skopenkov [Sko95] gave a characterisation of Peano continua basically embeddable into the plane by means of forbidden subsets, which resembles Kuratowski's characterisation of planar graphs. When restricted to finite graphs it says that a graph is basically embeddable into the plane if and only if it does not contain a subset homeomorphic to any of the following: a circle a pentod and a cross with branched ends. In a similar way Kurlin [Kur00] characterized finite graphs basically embeddable into $\mathbb{R} \times T_n$, with $T_n$ a star with $n$-rays.

The proof of $(E) \Leftrightarrow (B)$ in [Ste89] is not direct, but uses a reduction to linear operators and advanced results of functional analysis. Therefore it would be desirable to find a straightforward, constructive proof, which would consequently provide an elementary proof of Skopenkov's characterisation. An elementary proof of $(B) \Rightarrow (E)$ was given in [MKT03]. Given a compactum $K \subseteq \mathbb{R}^2$ with $E^n(K) \neq \emptyset$ for every $n \in \mathbb{N}$, the authors gave a construction of a function $f \in C(K)$ not expressible in the form $f = g + h$ with $g, h \in C(\mathbb{R})$.

Let $K \subseteq \mathbb{R}^2$ be such that $E^n(K) = \emptyset$ for some $n$. If $K$ is a finite graph and $f \in C(K)$ then, using the $E(K)$ operation, it is possible to construct the functions $g$ an $h$ with $f = g + h$, see [Žs]. However, if $K$ is an arbitrary compactum then no construction of functions $g$ and $h$ is known, even if $E(K) = \emptyset$. In this paper we present such a construction provided that

$E(K) = \emptyset$. Despite the restriction $E(K) = \emptyset$, we hope that the the proof can be generalized for sets $K$ with $E^n(K) = \emptyset$ for any $n$.

**Theorem 1.2.** *Let $K \subseteq \mathbb{R}^2$ be compact and let $E(K) = \emptyset$. Then for every $f \in C(K)$ functions $g, h \in C(\mathbb{R})$ such that $f(x,y) = g(x) + h(y)$ for all $(x,y) \in K$ can be constructed.*

Functions $p, q : \mathbb{R}^2 \to \mathbb{R}$ will denote the two orthogonal projections: $p(x,y) = x$, $q(x,y) = y$. By saying that two subsets of $\mathbb{R}^2$ are *$p$-intersecting, $p$-disjoint*, *$p$-equal* or that one is a *$p$-subset* of the other, we shall mean that the relation holds for their $p$-projections. We shall also use these expressions when one of the sets is a subset of $\mathbb{R}$, then we shall consider it as a subset of the $x$-axis. Similarly for $q$.

In $\mathbb{R}^2$ we shall consider the maximum metric. The distance of two sets $X, X' \subseteq \mathbb{R}$ is defined by $\operatorname{dist}(X, X') = \inf\{|x - x'|; x \in X, x' \in X'\}$. The diameter of a bounded nonempty set $X$ in $\mathbb{R}$ or $\mathbb{R}^2$ is denoted by $\operatorname{diam} X$. The closure of a set $X$ is denoted by $\overline{X}$, the interior by $\operatorname{Int} X$. In the space $C(X)$ of continuous real valued functions on $X$ we shall consider the supremum norm denoted by $||f||$, for $f \in C(X)$.

We say that a uniformly continuous function $f \in C(\mathbb{R}^2)$ is *$(\delta, \epsilon)$-uniformly continuous* if $|(x,y) - (x',y')| \leq \delta$ implies $|f(x,y) - f(x',y')| \leq \epsilon$, for all $(x,y), (x',y') \in \mathbb{R}^2$.

By a maximal (minimal) subset with a certain property we shall mean maximal (minimal) with respect to inclusion.

*Convention.* Throughout the text we fix a compactum $K \subseteq \mathbb{R}^2$ with $E(K) = \emptyset$, $f \in C(K)$, $\epsilon > 0$ and $\delta$ such that the function $f$ is $(3\delta, \epsilon/2)$-uniformly continuous.

In this example we show a construction of functions $g, h$ with $f = g + h$ on a set $L$ with $E(L) = \emptyset$. Its analogy works for all graphs $G \subseteq \mathbb{R}^2$, with $E^n(G) = \emptyset$, for some $n \in \mathbb{N}$, see [Žs]. We also demonstrate why it can not be generalized for arbitrary compacta $K$, even if $E(K) = \emptyset$.

*Example* 1.3. Let a set $L \subseteq \mathbb{R}^2$ be formed by the four points $(1,-1)$ $(1,1)$, $(-1,-3)$, $(-1,3)$ and the following segments, which converge to them, with $i = 0, 1, 2, \ldots$ (see Figure 1):

$$[(1 - \frac{1}{4i+1}, 1 + \frac{1}{i+1}), (1 - \frac{1}{4i+2}, 1 + \frac{1}{i+2})]$$

$$[(1 - \frac{1}{4i+3}, -1 - \frac{1}{i+1}), (1 - \frac{1}{4i+4}, -1 - \frac{1}{i+2})]$$

$$[(-1 + \frac{1}{4i+2}, -3 + \frac{1}{i+1}), (-1 + \frac{1}{4i+3}, -3 + \frac{1}{i+2})]$$

$$[(-1 + \frac{1}{4i+4}, 3 - \frac{1}{i+1}), (-1 + \frac{1}{4i+5}, 3 - \frac{1}{i+2})]$$

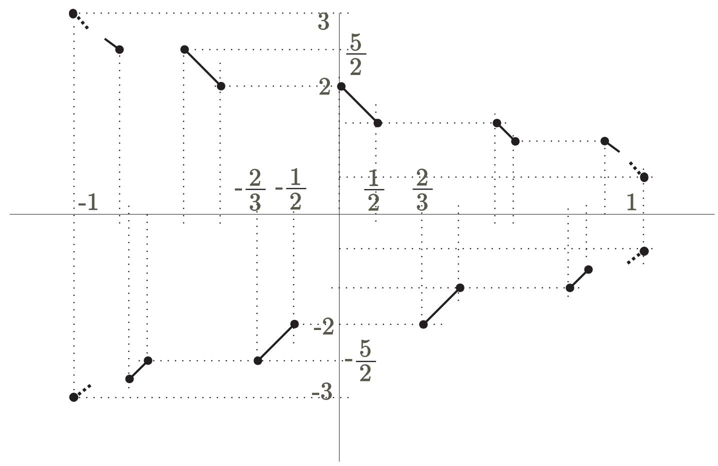


Figure 1: Set $L$

Thus $E(L) = \emptyset$. For any set $K \subseteq \mathbb{R}^2$ define

$$D_B(K) := \{(x,y) \in K \mid \operatorname{card}(p^{-1}(x) \cap K) \geq 2\} \tag{1.1}$$

$$D_A(K) := K - D_B(K) \tag{1.2}$$

thus $D_B(L) := \{(1,1),(1,-1),(-1,-3),(-1,3)\}$. The property $E(K) = \emptyset$ is equivalent to saying that the restrictions $p|D_A(K)$ and $q|D_B(K)$ are 1-to-1, in other words:

$$\begin{array}{rcll} \operatorname{diam}(p^{-1}(x) \cap K) & = & 0, & \forall (x,y) \in D_A(K) \\ \operatorname{diam}(q^{-1}(y) \cap K) & = & 0, & \forall (x,y) \in D_B(K). \end{array} \tag{1.3}$$

To express $f$ as $g + h$ on $L$ let

1. $g := 0$ on $p(D_B(L))$ and $h := f$ on $q(D_B(L))$, i.e. $g(1) = g(-1) = 0$, $h(1) = f(1,1)$, $h(-1) = f(1,-1)$, $h(3) = f(-1,3)$, $h(-3) = f(-1,-3)$,
2. extend $h$ continuously from $q(D_B(L))$ to $q(L) = [-3,-1] \cup [1,3]$ and let $g := f - h$ on $p(D_A(L))$.

Finally extend $g$, $h$ from $p(L)$ and $q(L)$ to $\mathbb{R}$.

However if we started by letting $g := f, h := 0$ on $D_A(L)$ which, unlike $D_B(L)$, is not compact then continuity would imply $g = f, h = 0$ on $D_B(L)$, which is impossible.

In general both the sets $D_A(K)$, $D_B(K)$ may be non-compact. Our approach for finding $g, h$ is described bellow. $\square$

The idea is to find $g$ and $h$ which are arbitrarily near to a given $f$, that is to find $g$ and $h$ with $||f - g - h|| \leq \epsilon$ on $K$ for any $\epsilon > 0$ (see Proposition 1.4).

A compactum $K \subseteq \mathbb{R}^2$ with $E(K) = \emptyset$, $f \in C(K)$ and $\epsilon > 0$ is fixed. The set $K$ is covered by a certain finite family of sets called 'squares' (Section 2) with properties that mimic $E(K) = \emptyset$. Their size is such that the change of $f$ on each one of them is bounded by $\epsilon/4$. Then, if $g, h \in C(\mathbb{R})$ are such that each of these sets contains a point $(x, y)$ with $|f(x, y) - g(x) - h(y)| \leq \epsilon/4$, while the change of $g$ and $h$ on every set is bounded by $\epsilon/4$, then the functions $g$ and $h$ are as desired, $||f - g - h|| \leq \epsilon$.

To encode the $\epsilon$-change requirement and to find $g$ and $h$, a labeled graph (Section 3) is constructed. The functions $g$ and $h$ will be defined in the graph vertices which are points of $K$, labeled by the values of $f$. The edges and their labels contain information on how much the functions $g$ and $h$ can change from one vertex to the next.

Then (Theorem 3.4) it is proved that if the functions $g$ and $h$ are defined on the graph vertices so that $g + h$ approximates $f$ and the conditions imposed by the labels on the edges are satisfied, then they can be extended to $\mathbb{R}$ so that the sum $g + h$ approximates $f$ on $K$.

In Sections 4 and 5 it is shown that if certain local transformations of the graph are applied then, provided that the size of the squares is small enough, the resulting graph allows to define functions $g$ and $h$. Finally (Section 6) functions $g$ and $h$ for such a graph are constructed.

The approximation approach is justified by the following statement:

**Proposition 1.4.** *[Rud91] Let $K \subseteq \mathbb{R}^2$ be compact, and let $f \in C(K)$ be any function. If for every $\epsilon > 0$ there exist functions $g_\epsilon, h_\epsilon \in C(\mathbb{R})$ such that*

$$||f - (g_\epsilon + h_\epsilon)|| < \epsilon, \text{ on } K$$
$$||g_\epsilon|| \leq k||f|| \quad \text{and} \quad ||h_\epsilon|| \leq k||f||, \text{ on } K$$

*then there exist functions $g, h \in C(\mathbb{R})$ such that*

$$f(x, y) = g(x) + h(y), \quad (x, y) \in K$$
$$||g|| \leq k||f|| \quad \text{and} \quad ||h|| \leq k||f||, \text{ on } K$$

*In particular it follows that $K$ is basically embedded in $\mathbb{R}^2$.*

The constructive elementary proof can be found in [Rud91], Theorem 4.13, $(b) \Rightarrow (c)$, with $T(f) = g + h$.

## 2 Squares

We shall introduce the special closed sets, called squares, that imitate the property $E(K) = \emptyset$.

**Definition 2.1.** A product $[a,a'] \times [b,b'] \subseteq \mathbb{R}^2$ with nonempty interior $(a,a') \times (b,b') \subseteq \mathbb{R}^2$ is said to be an *A-square* if

$$\overline{(a,a') \times \mathbb{R} - (a,a') \times (b,b')} \cap K = \emptyset.$$

It is said to be a *B-square* if

$$\overline{\mathbb{R} \times (b,b') - (a,a') \times (b,b')} \cap K = \emptyset.$$

**Lemma 2.2.** *Let $(x,y) \in D_A(K)$ (i.e. $p^{-1}(x) \cap K = \{x\}$, see (1.2)). Then for every interval $y \in (b,b') \subseteq \mathbb{R}$ there exists an interval $x \in (a,a') \subseteq \mathbb{R}$ such that $[a,a'] \times [b,b']$ is an A-square. Moreover there exists an interval $(u,u')$, $(a,a') \subseteq (u,u')$ such that $(u,u') \times \mathbb{R} - (u,u') \times (b,b') \cap K = \emptyset$.*

*Proof.* Assume that for some point $(x,y) \in K$ and for some interval $y \in (b,b')$ there does not exist any interval $x \in (a,a')$ with $[a,a'] \times [b,b']$ an $A$-square. Then for every $n \in \mathbb{N}$ there exists a point $(x^n, y^n)$ in

$$\overline{(1/n, 1/n) \times \mathbb{R} - (1/n, 1/n) \times (b,b')} \cap K.$$

Since $K$ is compact, there is a convergent subsequence with the limit $(x^{\lim}, y^{\lim}) \in K$. We have $x^{\lim} = x$ and $y^{\lim} \in \overline{\{x\} \times \mathbb{R} - \{x\} \times (b,b')}$ so $y \neq y^{\lim}$. The second part of the statement follows from the first one by a similar argument. □

Since $E(K) = \emptyset$, for points $(x,y) \in D_B(K) = K - D_A(K)$ we have $q^{-1}(y) \cap K = \{y\}$, see (1.3). Thus a statement analogous to Lemma 2.2 holds, yielding $B$-squares.

Let $K$ be covered by $A$ and $B$-squares. Since their interiors are nonempty and $K$ is compact, there exists a finite minimal subcover

$$K \subseteq \bigcup_{i=1}^{m_A} A_i \cup \bigcup_{i=1}^{m_B} B_i = \mathbf{A} \cup \mathbf{B}.$$

Throughout the text we shall frequently define and prove some properties for one type of squares ($A$ or $B$) and one of the projections ($p$ or $q$). If not stated otherwise we shall use such properties also with $A$, $B$ and $p$, $q$ replaced.

The squares mimic the property $E(K) = \emptyset$ in the following way (compare with (1.3)):

$$\begin{aligned}&\operatorname{diam}\{p^{-1}(x) \cap K\} \leq \mu, \text{ for every } (x,y) \in \mathbf{A}\\&\operatorname{diam}\{q^{-1}(y) \cap K\} \leq \mu, \text{ for every } (x,y) \in \mathbf{B}\end{aligned}$$

where $\mu$ is a bound on the diameters of the squares.

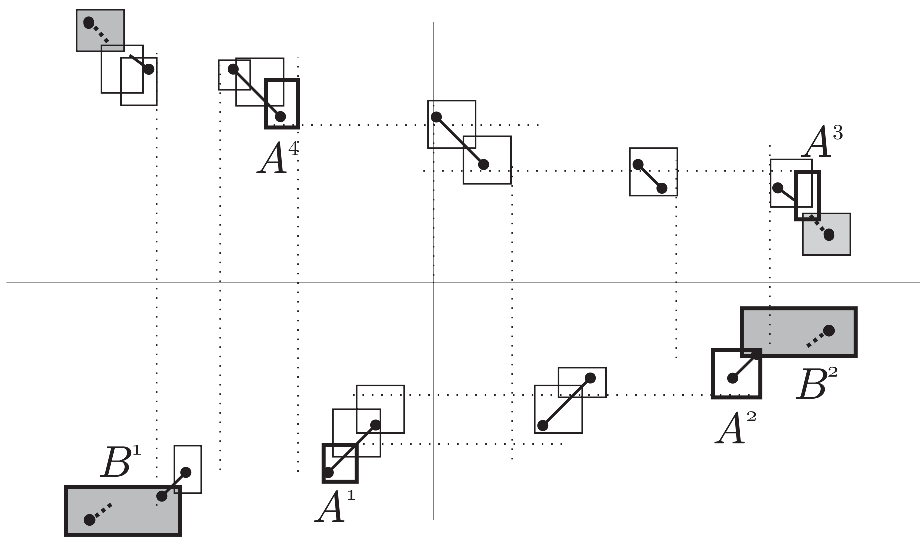


Figure 2: Cover for $L$

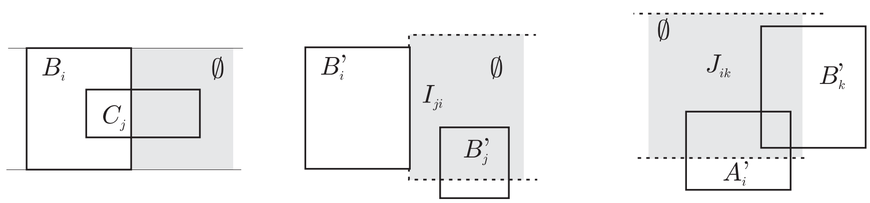


Figure 3: Adjusting the cover

*Example* 2.3. A cover by $A$ and $B$-squares for the set $L$ is drawn on Figure 2 ($A^i$, $B^i$ do not have any meaning in this stage). The $B$-squares are the shaded, the $A$-squares are not shaded. Any cover of the set $L$ contains exactly four $B$-squares, and a number of $A$-squares. Each of the four $B$-squares is intersected by an $A$-square. □

The minimality of the cover implies that no square is $p$-contained in an $A$-square or $q$-contained in a $B$-square

$$p(C_j) \not\subseteq p(A_i) \quad \text{and} \quad q(C_j) \not\subseteq q(B_i) \tag{2.4}$$

for any $C \in \{A, B\}$ and any $i, j$. Namely $p(C_j) \subseteq p(A_i)$ is impossible because by Definition 2.1 of $A$-squares the part of $C_j$ not contained in $A_i$ is empty in $K$, i.e. $(C_j - A_i) \cap K = \emptyset$, thus the square $C_j$ would be unnecessary in the cover which is minimal, see Figure 3, left.

**Lemma 2.4.** *Let* $\bigcup_{i=1}^{m'_A} A'_i \cup \bigcup_{i=1}^{m'_B} B'_i$ *be a cover of* $K$ *by* $A$ *and* $B$*-squares. It has a refinement* $\bigcup_{i=1}^{m_A} A_i \cup \bigcup_{i=1}^{m_B} B_i$ *with the following properties*

$$\begin{array}{l} \textit{if } A_i \cap A_j = \emptyset \textit{ then } p(A_i) \cap p(A_j) = \emptyset \\ \textit{if } B_i \cap B_j = \emptyset \textit{ then } q(B_i) \cap q(B_j) = \emptyset \end{array} \tag{2.5}$$

$$\textit{if } A_i \cap B_j \neq \emptyset \textit{ then } A_i \cap B_j \cap K \neq \emptyset. \tag{2.6}$$

*for every* $i, j$.

*Proof.* If $A'_i \cap A'_j = \emptyset$ then by definition $A'_i \cap p^{-1}p(A'_j) \cap K = \emptyset$. By Lemma 2.2 there exists an open interval $I_{ij} \supseteq p(A'_j)$ with $A'_i \cap p^{-1}(I_{ij}) \cap K = \emptyset$, see Figure 3, middle. If $A'_i \cap A'_j \neq \emptyset$, let $I_{ij} := \emptyset$.

If $A'_i \cap B'_k \neq \emptyset$ and $A'_i \cap B'_k \cap K = \emptyset$ then $A'_i \cap q^{-1}q(B'_k) \cap K = \emptyset$. Since $K$ is compact, similarly as in Lemma 2.2, it follows that there exists an open interval $J_{ik} \supseteq q(B'_k)$ with $A'_i \cap q^{-1}(J_{ik}) \cap K = \emptyset$, see Figure 3, right. For other pairs of squares let $J_{ik} := \emptyset$.

Let

$$A_i := A'_i - \left(\bigcup_{j=1}^{m'_A} p^{-1}(I_{ij}) \cup \bigcup_{k=1}^{m'_B} q^{-1}(J_{ik})\right)$$

for all $i$. If $A_i \cap K = \emptyset$ then remove the square $A_i$ from the cover. Similarly for $B$-squares. □

By a *cover by $A$ and $B$-squares* we shall mean a minimal one with the properties from Lemma 2.4.

*Example* 2.5. Assume that the diameters of the squares covering $L$ are bounded by $\mu \leq \delta$ and that $f \in C(L)$ is $(2\delta, \epsilon)$-uniformly continuous. We define the function $g$ on $p(\mathbf{A}) = p(\bigcup A_i)$, which is the union of several intervals, and $h$ on $q(\mathbf{A})$, which is the union of two intervals, so that $||f - (g+h)|| \leq 6\epsilon$. Similarly $g, h$ may be defined on $\mathbf{B}$. This construction gives an idea of how Theorem 1.3 is proved.

i. Define $h$ as a continuous function on $q(\mathbf{A})$ so that $|h(y) - h(y')| \leq \epsilon$ for $(x, y), (x', y') \in A_i \cup A_j$ with $A_i \cap A_j \neq \emptyset$.

The points $(x_i, y_i) \in A_i$ for which the $A$-squares were constructed are in $D_A(L)$ (see (1.2)) thus $x_i \neq x_j$ for $i \neq j$.

ii. Let $g(x_i) = f(x_i, y_i) - h(y_i)$, for all $i$, and on $p(\mathbf{A})$ extend $g$ as a piecewise linear function, which is constant on the ends of the intervals forming $p(\mathbf{A})$.

Note that this construction corresponds to (2) in Example 1.3. □

To define a function $h$ fulfilling the condition (i) from the example above we define: For a cover by $A$ and $B$-squares fix a maximal family of $q$-disjoint $A$-squares $\mathcal{E}_A = \{A^1, \ldots, A^{l_A}\}$, $A^1 <_q A^2 <_q \ldots <_q A^{l_A}$ (define $C <_q C'$

if $\max q(C) < \min q(C'))$ with the property that there does not exist any $A$-square $A$ such that

$$q(A^i) \cap q(A) \neq \emptyset \quad \text{and} \quad q(A^{i+1}) \cap q(A) \neq \emptyset \tag{2.7}$$

for any $i = 1, \ldots, l_A - 1$. If a function $h$ defined on $q(\mathbf{A})$ is such that its change on the $q$-projection of each square form $\mathcal{E}_A$ is bounded by $\epsilon$ and it is constant elswhere, then it fulfills condition (i) from the example above, i.e. its change over the $q$-projection of any $A$-square is bounded by $\epsilon$.

**Definition 2.6.** The union $\mathcal{A} := \bigcup_{i\in I} A_i$ of a family of $A$-squares is said to be an *$A$-component* if $q(\bigcup_{i\in I} A_i)$ is an interval and $\{A_i\}_{i\in I}$ is a maximal family with this property.

*Example* 2.7. For the cover of $L$ fix $\mathcal{E}_A = \{A^1, \ldots, A^4\}$ and $\mathcal{E}_B = \{B^1, B^2\}$. The cover consists of two $A$-components which are above each other, and of two $B$-components, one on the right side and one on the left side formed by two squares each. □

For a cover by $A$ and $B$-squares the $A$-components are determined uniquely. It follows directly from the definition that $q(\mathcal{A}) \cap q(\mathcal{A}') = \emptyset$, while (2.5) implies $p(\mathcal{A}) \cap p(\mathcal{A}') = \emptyset$, for every two $A$-components $\mathcal{A}, \mathcal{A}'$. Also note that every $A$-component contains at least one square from $\mathcal{E}_A$.

**Lemma 2.8.** *Let $K$ be covered by $A$ and $B$-squares, whose diameters are bounded by $\mu$. Let $[a, a'] \subseteq \mathbb{R}^2$, $a' - a \geq 6\mu$ be a closed interval which is $q$-contained in some $A$-component $\mathcal{A}$. Then $[a, a']$ $q$-contains at least one square from $\mathcal{E}_A$.*

*Proof.* Let $[a, a']$ be as in the statement but let on the contrary $q(A^i) \not\subseteq [a, a']$ for every $A^i \in \mathcal{E}_A$. Then $q(A^i) \cap [a + \mu, a' - \mu] = \emptyset$, for every $A^i \in \mathcal{E}_A$. Thus if $q(A) \cap q(A^i) \neq \emptyset$ for some $A$-square $A$, then $q(A) \cap [a + 2\mu, a' - 2\mu] = \emptyset$. But the interval $[a + 2\mu, a' - 2\mu]$, whose length is at least $2\mu$, is $q$-contained in an $A$-component, thus it $q$-contains at least one $A$-square, which could have been added to $\mathcal{E}_A$. Contradiction with maximality. □

# 3 Graph

Let us have $f \in C(K)$, a cover of $K$ by $A$ and $B$-squares, $K \subseteq \bigcup_{i=1}^{m_A} A_i \cup \bigcup_{i=1}^{m_B} B_i$, whose diameters are bounded by $\mu$, and the families $\mathcal{E}_A = \{A^1, \ldots, A^{l_A}\}$ and $\mathcal{E}_B = \{B^1, \ldots, B^{l_B}\}$.

We shall construct a graph $\Gamma = \Gamma(\mu)$ with vertices $V(\Gamma)$ that are points in $K$, and two types of edges $E_A(\Gamma)$ and $E_B(\Gamma)$ called *$A$-edges* and *$B$-edges*, respectively. Values will be assigned to both the vertices and the edges.

For every $i, j$ such that $A_i \cap B_j \neq \emptyset$ choose exactly one point in the set $A_i \cap B_j \cap K$ which is nonempty according to (2.6). The resulting points are the vertices $V(\Gamma)$ of the graph. Assign the value $f(u,v)$ to each vertex $(u,v)$.

To construct $A$-edges $E_A(\Gamma)$, order the vertices $(u_i,v_i)$ lexicographically in ascending order, starting with the $y$-coordinate. This ordering shall be called the *$y$-ordering.* If two consecutive vertices $(u_i,v_i)$, $(u_{i+1},v_{i+1})$ lie in the same $A$-component, i.e. if $[v_i,v_{i+1}] \subseteq q(\mathbf{A})$, then connect them by an $A$-edge $e_i = \{(u_i,v_i),(u_{i+1},v_{i+1})\}$.

The property of $\Gamma$ that with this ordering of vertices every path consisting of $A$-edges, called *$A$-path*, connects vertices $(u_m,v_m)$, $(u_{m+1},v_{m+1})$, ..., $(u_n,v_n)$, $m \leq n$, $v_m \leq \ldots \leq v_n$ shall be called the *$y$-ordering property of $A$-edges.* A path $\gamma$ in $\Gamma$ is a connected subgraph with two vertices of degree 1 and the rest of the vertices of degree 2, or the trivial path consisting of one vertex.

Assign non-negative values $z(e_i)$ to the $A$-edges in the following way. For every edge $e_i = \{(u_i,v_i),(u_{i+1},v_{i+1})\}$ such that $v_i = v_{i+1}$ let $z(e_i) := 0$. Let $e_i \in E_A(\Gamma)$ be an edge with no value assigned to it, and let $M := \{A \in \mathcal{E}_A \mid q(\operatorname{Int} A) \cap q(e) \neq \emptyset\}$. Let $z(e_i) := \operatorname{card}(M)\,\epsilon$. For every edge $e_j$ such that $q(e_j) \cap q(\operatorname{Int} A) \neq \emptyset$ for some $A \in M$, let $z(e_j) := 0$.

The construction of $B$ edges and the assigned values $z(e)$ is analogous.

Edges $e = \{(u_i,v_i),(u_{i+1},v_{i+1})\}$ with $u_i = u_{i+1}$ or $v_i = v_{i+1}$ will be called *vertical* or *horizontal*, respectively.

Instead of $(u,v) \in V(\Gamma)$, $e \in E_A(\Gamma) \cup E_B(\Gamma)$, $e \in E_A(\gamma) \cup E_B(\gamma)$, etc. we shall sometimes write $(u,v) \in \Gamma$, $e \in \Gamma$, $e \in \gamma$, etc.

The diameters or distances of edges or paths, or their $p$ or $q$ projections are defined as diameters or distances of the the vertices of the edges or paths.

*Example* 3.1. The graph for the cover of $L$ from Figure 2 is drawn on Figure 4. The lower $A$-edge $e^A = \{(\overline{a},\overline{b}),(a,b)\}$ is $q$-intersected by two squares $A^1, A^2$ from $\mathcal{E}_A$, thus $z(e^A) = 2\epsilon$, similarly $z({e'}^A) = 2\epsilon$. Each of the two $B$-edges $e^B = \{(a,b),(c,d)\}, {e'}^B = \{(\overline{a},\overline{b}),(\overline{c},\overline{d})\}$ is $p$-intersected by exactly one square from $\mathcal{E}_B$, thus $z(e^B) = z({e'}^B) = \epsilon$. Every cover of $L$ contains four $B$-squares forming two $B$-components and some $A$-squares forming two $A$-components, see Example 2.3. Thus the structure of the graph is allways the same, only the labels may change. The labels of the two $B$-edges are allways equal to $\epsilon$ and the labels of the $A$-edges increase as the squares become smaller (Lemma 2.8, see also Lemma 4.1). □

*Remark* 3.2. Note that if $\Gamma$ contains a pair of vertices $(u,v) \in A \cap B \cap K$, $(u',v) \in A' \cap B' \cap K$, $u \neq u'$ then since $q(A) \cap q(A') \neq \emptyset$, the squares belong to the same $A$-component. Thus $(u,v)$ and $(u',v)$ are connected by an $A$-path $\gamma$. The construction is such that $\gamma$ consists of horizontal edges and so $\sum_{e\in\gamma} z(e) = 0$. Analogously for vertices $(u,v)$, $(u,v')$ with $v \neq v'$.

**Definition 3.3.** Consider $\Gamma = \Gamma(\mu)$ as an abstract graph, i.e. with no embedding of vertices. Let values $g(u)$ and $h(v)$ be given for every vertex

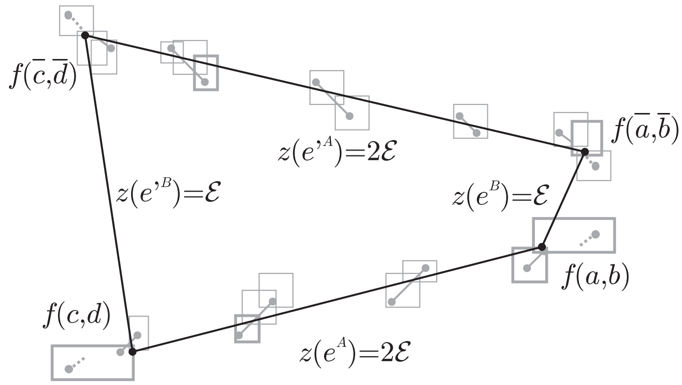


Figure 4: Graph for $L$

$(u, v)$ of $\Gamma$. Denote $|\triangle_e h| = |h(v) - h(v')|$, $|\triangle_e g| = |g(u) - g(u')|$ for edges $e = \{(u, v), (u', v')\}$. We say that the functions $g$ and $h$ are *$\alpha$-good with respect to* $\Gamma$, $\alpha \geq 0$ if

1. $|f(u, v) - g(u) - h(v)| \leq \alpha$, for every vertex $(u, v)$
2. $|\triangle_e h| \leq z(e)$ for every $A$-edge $e$
3. $|\triangle_{e'} g| \leq z(e')$ for every $B$-edge $e'$.

Because properties (2), (3) are independednt of $\alpha$, we shall sometimes say that $g$, $h$ are *good* with respect to some edge.

If $\Gamma$ is considered in the way it was constructed, that is with $V(\Gamma) \subseteq \mathbb{R}^2$, then functions $g$, $h$ which are $\alpha$-good with respect to the abstract representation are well-defined as functions on $p(V(\Gamma)), q(V(\Gamma)) \subseteq \mathbb{R}$, respectively. Namely by Remark 3.2 every pair of vertices $(u, v)$, $(u', v)$, $u \neq u'$ is connected by a 0-valued $A$-path $\gamma$. Thus since $g$ is $\alpha$-good with respect to $\Gamma$, the value of $g$ in both the vertices is the same. Similarly for $(u, v)$, $(u, v')$, $v \neq v'$

**Theorem 3.4.** *Let $f \in C(K)$. Let $\delta > 0$ be such that $f$ is $(2\delta, \epsilon)$-uniformly continuous. Let functions $g$ and $h$ be $\alpha$-good, with $\alpha = 4\epsilon$, with respect to a graph $\Gamma = \Gamma(\delta)$. Then they can be extended to continuous functions on $\mathbb{R}$, such that*

*1.* $||f - (g + h)|| \leq 100\epsilon$ *on* $K$

*2.* $\max\{||g||_K, ||h||_K\} \leq ||f|| + ||g||_\Gamma + ||h||_\Gamma$.

*Proof.* Assuming that the functions $g$ and $h$ are given on the vertices of the graph $\Gamma = \Gamma(\delta)$, we shall describe their extension to $\mathbb{R}$. We shall start by extending $h$ to $q(\mathbf{A})$ and $g$ to $p(\mathbf{B})$, then we shall extend $g$ to $p(\mathbf{A})$ and $h$ to $q(\mathbf{B})$.

*Extension of $h$ to $q(\mathbf{A})$ and $g$ to $p(\mathbf{B})$.* Temporarily denote $\overline{h} := h$ on $q(V(\Gamma))$. As the $A$-components forming $\mathbf{A}$ are $q$-disjoint, it suffices to describe the construction on one $A$-component $\mathcal{A}$.

In case $\mathcal{A}$ contains no graph vertex, let $\overline{h} := 0$ on $q(\mathcal{A})$. Let $\mathcal{A}$ contain $y$-ordered vertices $\{(u_i, v_i)\}_{i=1}^{n_1}$, as in the construction of $A$-edges of $\Gamma$. Denote $q(\mathcal{A}) := [d, d']$. On $[d, v_1]$ and $[v_{n_1}, d']$ and on every $[v_i, v_{i+1}]$ with $\overline{h}(v_i) = \overline{h}(v_{i+1})$, let $\overline{h}$ be constant.

For every $i$ such that $\overline{h}(v_{i+1}) - \overline{h}(v_i) = \beta \neq 0$ do the following. Let for example $i = 1$ and denote $e_1 := \{(u_1, v_1), (u_2, v_2)\}$. Since $\overline{h}$ is $\alpha$-good with respect to $\Gamma$, we have $|\beta| \leq z(e_1)$. From the graph construction we know that $z(e_1) = l\epsilon$, where the interval $[v_1, v_2] = q(e_1)$ is $q$-intersected by the interiors of at least $l$ squares from $\mathcal{E}_A$, for example $\{A^1, \ldots, A^l\}$, we have $A^1 <_p \ldots <_p A^l$, (see (2.7). Denote the endpoints of the $q$-projections of these squares in $[v_1, v_2]$ by $b^i := \min\{q(A^i) \cap [v_1, v_2]\}$ and $\overline{b}^i := \max\{q(A^i) \cap [v_1, v_2]\}$. Let

$$\overline{h}(b^j) := \overline{h}(v_1) + (j-1)\frac{\beta}{l} \quad \text{and} \quad \overline{h}(\overline{b}^j) := \overline{h}(b^{j+1}) = \overline{h}(v_1) + j\frac{\beta}{l}$$

for every $j$. Extend $\overline{h}$ as a piecewise linear function on $[v_1, v_2]$.

If a square $B_j$ does not intersect $\mathbf{A}$, then Lemma 2.2 implies that there exists an open neighborhood $V_j$ of $B_j$ such that $\mathbf{A} \cap K \cap q^{-1}(V_j) = \emptyset$. For squares $B_j$ with $B_j \cap \mathbf{A} \neq \emptyset$ let $V_j := \emptyset$. Denote $\mathcal{V}_B := \bigcup_{j=1}^{m_B} V_j$. Thus

$$\mathbf{A} \cap K \cap q^{-1}(\mathcal{V}_B) = \emptyset. \tag{3.8}$$

Let $h := \overline{h}$ on $q(\mathbf{A}) - \mathcal{V}_B$.

Since $h$ was extended linearly between the values given in the graph vertices we have

$$||h||_{q(\mathbf{A}) - \mathcal{V}_B} \leq ||h||_\Gamma. \tag{3.9}$$

Let us show

$$|h(y) - h(y')| \leq \epsilon, \quad (x, y), (x', y') \in A_i \cap K \tag{3.10}$$

for every $i$. First note that if $(x, y), (x', y') \in A_i \cap K$, then $y, y \notin \mathcal{V}_B$, thus the values $h(y)$ and $h(y')$ are defined. The function $h$ is nonconstant only on the $q$-projections of the squares from $\mathcal{E}_A$. On each the difference between any two values of $h$ is bounded by $\epsilon$. By definition there is at most one square $A^j \in \mathcal{E}_A$ such that $q(A^j) \cap q(A_i) \neq \emptyset$.

The following will be needed for the extension of $h$ to $q(\mathbf{B})$:

$$|h(y) - h(y')| \leq 2\epsilon, \quad y, y' \in q(A_i \cup A_j) - \mathcal{V}_B \tag{3.11}$$

for every $i, j$ with $A_i \cap A_j \neq \emptyset$. The reason is similar as before, in this case there are at most two squares from $\mathcal{E}_A$ that $q$-intersects the union $A_i \cup A_j$.

The extension of $g$ to $p(\mathbf{B})$ is analogous to the extension of $h$ to $q(\mathbf{A})$. Again the function $g$ is defined on $p(\mathbf{B}) - \mathcal{U}_A$, where $\mathcal{U}_A$ is defined similarly as $\mathcal{V}_B$, i.e. it is formed by the neighborhoods $U_i$ of the squares $A_i$ with $A_i \cap \mathbf{B} = \emptyset$, and $\mathbf{B} \cap K \cap q^{-1}(\mathcal{U}_A) = \emptyset$. Thus analogies of (3.9), (3.10), (3.11) hold.

*Extension of $g$ to $p(\mathbf{A})$ and $h$ to $q(\mathbf{B})$.* The function $g$ is already defined on $p(\mathbf{B}) - \mathcal{U}_A$, here only the definition on $p(\mathbf{A}) \cap p(\mathbf{B}) - \mathcal{U}_A$ is of interest. We shall extend it to $p(\mathbf{A})$.

Let us show first that the set $p(\mathbf{A}) \cap p(\mathbf{B}) - \mathcal{U}_A$ is formed by closed intervals containing the $p$-projections of the graph vertices. Namely since $\mathcal{U}_A$ is formed by the neighborhoods $U_i$ of $A$-squares $A_i$ with $A_i \cap \mathbf{B} = \emptyset$, we have

$$p(\mathbf{A}) \cap p(\mathbf{B}) - \mathcal{U}_A = p(\bigcup_{i \in I_B} A_i) \cap p(\mathbf{B}) - \mathcal{U}_A$$

where $I_B := \{i \mid A_i \cap \mathbf{B} \neq \emptyset\}$. Each $A_i$, $i \in I_B$ $p$-contains a graph vertex. On the other hand $\mathcal{U}_A$ contains no graph vertices, as they are all contained in $\mathbf{B} \cap K$, see (3.8). The difference $p(A) - p(A')$ is an interval for any pair of $A$-squares $A, A'$ (see 2.4). Thus $p(A) - \mathcal{U}_A$ and also $p(A) \cap p(\mathbf{B}) - \mathcal{U}_A$ is an interval. Therefore each set $p(A_i) \cap p(\mathbf{B}) - \mathcal{U}_A$, $i \in I_B$ is a closed interval containing a graph vertex.

If a square $A_{i_0}$ does not contain a graph vertex, i.e. $A_{i_0} \cap \mathbf{B} = \emptyset$, then $p(A_{i_0}) \subseteq \mathcal{U}_A$ so the function $g$ is not defined in any point of $p(A_{i_0})$. For every such square $A_{i_0}$ choose a point $(x_{i_0}, y_{i_0})$ in the set $[A_{i_0} - p^{-1}p(\bigcup_{i \neq i_0} A_i)] \cap K$, which is nonempty because the cover by $A$ and $B$-squares is minimal. Thus the $p$-projections of these points are pairwise different. Since they are contained in $\mathbf{A} \cap K$, none of them belongs to $q^{-1}(\mathcal{V}_B)$, see (3.8), so the function $h$ is defined in them. Let $g(x_{i_0}) := f(x_{i_0}, y_{i_0}) - h(y_{i_0})$ in each such point. In combination with (3.9) we have

$$|g(x_{i_0})| \leq ||f|| + ||h||_{q(\mathbf{A}) - \mathcal{V}_B} \leq ||f|| + ||h||_{\Gamma}. \tag{3.12}$$

We write $A_i \leq_p A_j$ for two $A$-squares if

$$\min p(A_i) \leq \min p(A_j) \quad \text{and} \quad \max p(A_i) \leq \max p(A_j).$$

From (2.4) it easily follows that $\leq_p$ defines a linear ordering on $A$-squares, so order them $A_1 \leq_p \ldots \leq_p A_{m_A}$. Moreover (2.5) implies that if $A_i \cap A_{i+1} = \emptyset$ then $A_i <_p A_{i+1}$.

The set $p(\mathbf{A})$ may consist of several intervals of the form $p(A_i \cup A_{i+1} \cup \ldots \cup A_{i+j})$ with $A_{i-1} <_p A_i$ and $A_{i+j} <_p A_{i+j+1}$. Each of them contains a subset, consisting of one or more of the chosen points $(x_{i_0}, y_{i_0})$ or intervals forming $p(\mathbf{B}) - \mathcal{U}_A$, on which $g$ is defined. On each such interval $[a, a']$ extend $g$ as a piecewise linear function, which is constant on both, possibly trivial, segments between the left-most given value of $g$ in $[a, a']$ and $a$, and the right-most given value of $g$ in $[a, a']$ and $a'$.

Thus in combination with the analogy of (3.9) for $g$ on $p(\mathbf{B})$ and (3.12) we have

$$||g||_{p(\mathbf{A}\cup\mathbf{B})-\mathcal{U}_A} \leq ||f|| + ||g||_\Gamma + ||h||_\Gamma \tag{3.13}$$

Let us show

$$|g(x) - g(x')| \leq 13\epsilon + 6\alpha, \quad (x,y),(x',y') \in A_i \cap K \tag{3.14}$$

for all $i$.

Order the set of graph vertices and the newly chosen points $(x_{i_0}, y_{i_0})$ lexicographically, in ascending order starting with the $x$-coordinate and denote the resulting sequence $\{(x_i, y_i)\}_i$. Every $A$-square contains one or more of the points $(x_i, y_i)$.

Consider some $(x_i, y_i)$, $(x_{i+1}, y_{i+1})$ that lie in two intersecting, or the same $A$-square. For simplicity let $(x_i, y_i) \in A_i$, $(x_{i+1}, y_{i+1}) \in A_{i+1}$. We shall compute an upper bound on $|g(x) - g(x_i)|$ for $x \in [x_i, x_{i+1}]$. Let for example $g(x_i) \leq g(x_{i+1})$.

Because the diameters of the squares are bounded by $\delta$, we have $|(x_i, y_i) - (x_{i+1}, y_{i+1})| \leq 2\delta$, therefore by the assumption that $f$ is $(2\delta, \epsilon)$-uniformly continuous $|f(x_i, y_i) - f(x_{i+1}, y_{i+1})| \leq \epsilon$. Since the functions $g$, $h$ are $\alpha$-good with respect to $\Gamma$ and $f = g + h$ in the chosen points which are not graph vertices, we have $|f(x_k, y_k) - g(x_k) - h(y_k)| \leq \alpha$, for all $k$. By (3.11) we have $|h(y_i) - h(y_{i+1})| \leq 2\epsilon$. Thus

$$|g(x_i) - g(x_{i+1})| \leq 3\epsilon + 2\alpha. \tag{3.15}$$

If both $(x_i, y_i)$ and $(x_{i+1}, y_{i+1})$ are graph vertices then $[x_i, x_{i+1}] \cap p(\mathbf{B}) - \mathcal{V}_B$ is formed by two intervals $[x_i, x_i']$, $[x_{i+1}', x_{i+1}]$, $x_i' \leq x_{i+1}'$ (we have $x_i' = x_{i+1}'$ in case $[x_i, x_{i+1}] \subseteq p(\mathbf{B}) - \mathcal{V}_B$). By (2.4), if a $B$-square $p$-intersects the union $A_i \cup A_{i+1}$, then it $p$-contains the minimum or the maximum of $p(A_i \cup A_{i+1})$. Thus at most two of all $B$-squares $p$-intersecting $[x_i, x_{i+1}] \subseteq p(A_i \cup A_{i+1})$ are in $\mathcal{E}_B$, so the difference between two values of $g$ in $[x_i, x_i']$ or in $[x_{i+1}', x_{i+1}]$ is bounded by $2\epsilon$. On the interval $[x_i', x_{i+1}']$, if it is non-trivial, the extension is linear. Thus $g(x) \in [g(x_i) - 2\epsilon, g(x_{i+1}) + 2\epsilon]$ and in particular $|g(x) - g(x_i)| \leq |g(x_i) - g(x_{i+1})| + 2\epsilon$ for $x \in [x_i, x_{i+1}]$. If only one or none of the points $(x_i, y_i)$, $(x_{i+1}, y_{i+1})$ is a graph vertex, the situation is even simpler and the same inequalities hold. So in combination with (3.15):

$$|g(x) - g(x_i)| \leq 5\epsilon + 2\alpha, \quad x \in [x_i, x_{i+1}]. \tag{3.16}$$

To prove (3.14) let $(x,y),(x',y') \in A_i \cap K$, $x < x'$. Let $x \in [x_j, x_{j+1}]$ and $x' \in [x_{j+k}, x_{j+k+1}]$, $k \geq 1$. The points $(x_{j+1}, y_{j+1})$ and $(x_{j+k}, y_{j+k})$ may be identical or they lie in the same $A$-square. Thus by (3.15) we have $|g(x_{j+1}) - g(x_{j+k})| \leq 3\epsilon + 2\alpha$. If the two points $x_j, x_{j+1}$ (likewise for $x_{j+k}, x_{j+k+1}$) lie in two $A$-squares $A_{i-1}$, $A_i$ that do not intersect, i.e. $A_{i-1} <_p A_i$, then on the intervals $[x_j, \max p(A_{i-1})]$ and $[\min p(A_i), x_{j+1}]$ the function $g$ was extended constantly from $p(\mathbf{A}) \cap p(\mathbf{B}) - \mathcal{U}_A$ to $p(\mathbf{A})$. Thus similarly as before $|g(x) - g(x_j)| \leq 2\epsilon$. Otherwise we have (3.16). Therefore (3.14) holds.

The extension of $h$ to $q(\mathbf{B})$ is similar to the extension of $g$ to $p(\mathbf{A})$, so analogies of (3.13) and (3.14) hold.

*Extension of functions $g$ and $h$ to $\mathbb{R}$.* The function $g$ is defined on the closed set $(p(\mathbf{A}) \cup p(\mathbf{B})) - \mathcal{U}_A$. Since $K \subseteq (p^{-1}p(\mathbf{A}) \cup p^{-1}p(\mathbf{B})) - \mathcal{U}_A$, by extending $g$ to $\mathbb{R}$ its values on $p(K)$ do not change. We extend the function $h$, too.

The functions $g$, $h$ are such that $|g(x) - g(x')| \leq 13\epsilon + 6\alpha$ and $|h(y) - h(y')| \leq 13\epsilon + 6\alpha$ for all $(x, y), (x', y') \in C_i \cap K$, $C \in \{A, B\}$, for all $i$, see (3.10), (3.14). Because the diameters of the squares are bounded by $\delta$ and $f$ is $(2\delta, \epsilon)$-uniformly continuous, we have $|f(x, y) - f(x', y')| \leq \epsilon$, for all such $(x, y)$, $(x', y')$. Moreover in each $C_i \cap K$, there is a point $(x, y)$, which is a graph vertex or one of the points chosen for $A$ and $B$-squares, with $|f(x, y) - g(x) - h(y)| \leq \alpha$. Thus (1) of the theorem holds. And (3.9), (3.13) imply point (2) of the theorem. □

# 4 Adjusted graph

**Lemma 4.1.** *Let $\gamma$ be an $A$-path in $\Gamma = \Gamma(\mu)$ with $\sum_{e \in \gamma} z(e) < k\epsilon$ for some $k \in \mathbb{N}$. Then*

$$\operatorname{diam} q(\gamma) < 6k\mu \qquad (4.17)$$

*Proof.* Assume the $y$-ordering of the vertices $\{(u_i, v_i)\}_i$ of $\Gamma$. If $\gamma = (u_i, v_i)$, $(u_{i+1}, v_{i+1}), \ldots, (u_j, v_j)$, $i < j$ is an $A$-path, then by definition $v_i \leq \ldots \leq v_j$ and all its vertices lie in one $A$-component. If $\operatorname{diam} q(\gamma) = v_{i+j} - v_i \geq 6k\mu$ then by Lemma 2.8 the interval $[v_i, v_{i+j}]$ contains at least $k$ squares from $\mathcal{E}_A$, so $\sum_{e \in \gamma} z(e) \geq k\epsilon$. □

The following example shows that functions $g$, $h$ that are 0-good with respect to a given graph $\Gamma(\mu)$ do not allways exist. It also shows the solution for the particular set $L$.

*Example* 4.2. Consider the graph for the set $L$ from Figure 4 and let $f \in C(L)$ be such that $f(a, b) = f(\overline{c}, \overline{d}) = 0$, $f(\overline{a}, \overline{b}) = f(c, d) = 10$. Let $\epsilon = 1$. Functions $g, h$ that are 0-good with respect to this graph do not exist. The obstacle for their definition are combinations of pairs of adjecent edges of different $A$ and $B$-type, whose labels are smaller that the difference of the function $f$ between their ends: $z(e) < |\triangle_e f|$; here it is any pair of adjecent edges.

We consider a sequence $\Gamma(1/i)$, $i \in \mathbb{N}$ of graphs for the set $L$. The structure of every $\Gamma(1/i)$ is the same (see Example 3.1): the endpoints of the $A$-edges $e_i^A = \{(a_i, b_i), (\overline{a}_i, \overline{b}_i)\}$ (similarly for ${e'}_i^A$) in the graphs $\Gamma(1/i)$ converge to the pair $(1,-1), (-1,-3) \in L$. Thus we may assume

$$\operatorname{diam} q(e_i^A) \geq 1, \qquad \text{for all } i.$$

So by Lemma 4.1 the labels $z(e_i^A)$ are increasing:

$$z(e_i^A) \to \infty, \text{ as } \ i \to \infty.$$

On other hand the endpoints of the $B$-edges $e_i^B = \{(a_i, b_i), (c_i, d_i)\}$ (similarly ${e'}_i^B$) converge to the pair $(1,-1), (1,1) \in L$ so the diameters of their $p$-projections converge to zero:

$$\operatorname{diam} p(e_i^B) \to 0, \text{ as } \ i \to \infty.$$

And the labels remain the same

$$z(e_i^B) = \epsilon, \qquad \text{for all } i.$$

Thus for some $i$ adjecent edges in $\Gamma(1/i)$ with $z(e) < |\triangle_e f|$ dissapear, as on Figure 5. Hence it is possible to define 0-good functions $g$, $h$ by $g := 0$, $h := f$ in the graph vertices. Note that this definition is similar to that in Example 1.3, where $g$ and $h$ on $L$ were defined using the sets $D_A(L)$, $D_B(L)$. □

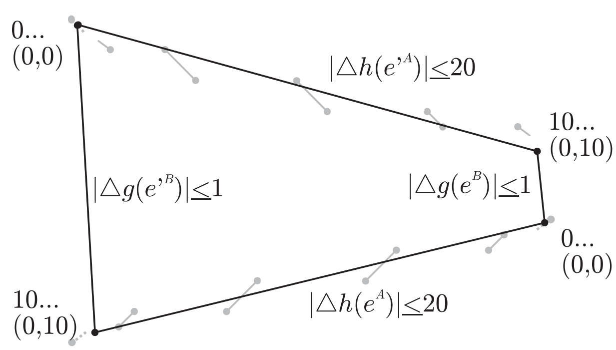


Figure 5: Definition of $g$, $h$

The essential idea from the example is this. Let $\Gamma(1/i)$ be a sequence of graphs for some compactum $K$ with $E(K) = \emptyset$. Let $e_i^A, e_i^B \subseteq \Gamma(1/i)$ be two sequences of $A$ and $B$-edges with the following properties:

$$z(e_i^A), z(e_i^B) \leq \kappa \tag{4.18}$$

$$\operatorname{diam} p(e_i^A) \geq \omega > 0 \quad \text{and} \quad \operatorname{diam} q(e_i^B) \geq \omega > 0. \tag{4.19}$$

Then their ends converge to the endpoints of a non-trivial vertical segment $e^A = [(a_1, b_1), (a_2, b_1)]$, $|a_1 - a_2| \geq \omega > 0$, and a non-trivial horizontal segment $e^B = [(c_1, d_1)(c_1, d_2)]$, $|d_1 - d_2| \geq \omega > 0$. Since $E(K) \neq \emptyset$ we have $c_1 \neq a_1, a_2$ and $b_1 \neq d_1, d_2$. In other words $\operatorname{dist}(\{b_1\}, \{d_1, d_2\}) \geq \beta > 0$, $\operatorname{dist}(\{c_1\}, \{a_1, a_2\}) \geq \beta > 0$, for some $\beta > 0$. Therefore we may assume

$$\operatorname{dist}(p(e_i^A), p(e_i^B)) \geq \beta/2 \quad \text{and} \quad \operatorname{dist}(q(e_i^A), q(e_i^B)) \geq \beta/2$$

for all $i$.

Thus the obstacle for defining $g$ and $h$ which can not be eliminated even with decreasing square size are $A$-edges $e_i^A \subseteq \Gamma(1/i)$ (or analogously $B$-edges) with small label

$$z(e_i^A) < |\triangle_{e_i^A} f| \leq 2||f|| = \kappa$$

and such that the diameter of their $p$-projection is not bounded from bellow

$$\operatorname{diam} p(e_i^A) \to 0, \quad i \to \infty.$$

Because their labels are bounded by $2||f||$, Lemma 4.1 implies that the diamater of their $q$-projection is approaching zero: $\operatorname{diam} q(e_i^A) \to 0, \quad i \to \infty$. Hence their diameter is arbitrarily small: $\operatorname{diam} e_i^A \leq \delta$, for $i > i_0$. Since we are assuming that $f$ is $(3\delta, \epsilon/2)$-uniformly continuous its change over one such edge is thus bounded by $\epsilon$: $|\triangle_{e_i^A} f| \leq \epsilon$, $i > i_0$. In Theorem 3.4 functions $g$, $h$ should be only $\alpha = 4\epsilon$-good with respect to $\Gamma$. This allows us to define them also when such small edges are present, as in the example bellow.

*Example* 4.3. Let $M$ be the closure of the union of a sequence $\{L_i\}_{i=0}^{\infty}$, $L_0 = L$ of smaller and smaller images of the set $L$ converging to the two points $(\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$, as on Figure 6.

Let $M'$ be formed by two points $(-1, 5)$ and $(1, 7)$ and segments similar to those in the set $L$ that converge to them and lie outside $p^{-1}(L)$:

$$[(1 - \frac{3}{12i+1}, 7 - \frac{1}{i+1}), (1 - \frac{3}{12i+2}, 7 - \frac{1}{i+2})]$$

$$[(-1 + \frac{3}{12i+4}, 5 + \frac{1}{i+1}), (-1 + \frac{3}{12i+5}, 5 + \frac{1}{i+2})].$$

Let the set $N$ consist of the sets $M$ and $M'$ and the closure of copies of $M'$ which converge to the point $(\frac{1}{2}, \frac{13}{2})$.

The set $D_B(N)$ is thus formed by vertical triples of points, represented by filled balls on Figure 6. In any cover, each triple of points from $D_B(N)$ is covered by three $p$-intersecting $B$-squares forming a $B$-component, thus each $B$-component contains exactly one square from $\mathcal{E}_B$. Therefore the $B$-edges in a graph for any cover of $N$ form paths of length 2, with edges labeled 0, $\epsilon$. Figure 7 shows two graphs, the right one is for smaller square size, it is

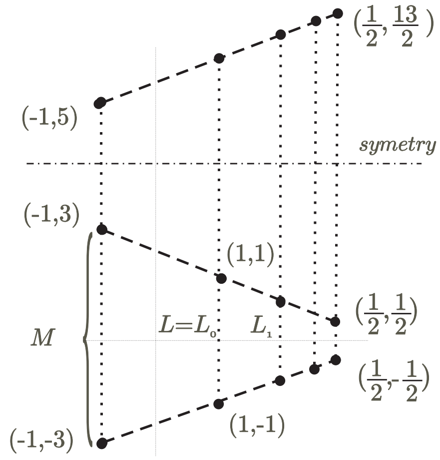


Figure 6: Set $N$

'finer'. The obstacle for defining 0-good functions $g, h$ are for example the cycles $E, F, I, H$ in the left graph on Figure 7, and the cycle $E', F', H', I'$ on the right one. Similar cycles may appear with any size of the squares.

Let for example $\epsilon = 1$ and $f(E) = -1, f(F) = 0, f(I) = 10, f(H) = 11$. Functions $g, h$ which are $2\epsilon$-good with respect to the graph may be defined in the vertices of the cycle $E, F, I, H$ in the following way, see Figure 8:

$$\begin{aligned}
&g(F) + h(F) = 0 + 10 = f(F) \\
&g(I) + h(I) = g(F) + h(I) = 0 + 0 = f(I) \\
&g(H) + h(H) = g(H) + h(I) = -1 + 0 = f(H) \\
&g(E) + h(E) = g(H) + h(F) = -1 + 10 = f(E) - 2
\end{aligned}$$

□

To encode the possibility of $g+h$ being $\alpha$-different from $f$, we shall apply certain local transformations to $\Gamma$ and obtain the *adjusted graph* $\Gamma^{BA}$, with $V(\Gamma^{BA}) \subseteq V(\Gamma)$. We shall define $g, h$ in the vertices of the adjusted graph so that $f = g + h$ (Theorem 6.2) and then extend this definition to the vertices of $\Gamma$ so that $||f - (g+h)|| \leq 4\epsilon$ (Theorem 4.5). The transformations are done in two steps. First we construct the graph $\Gamma^B$ by contracting certain well chosen zero-valued $B$-edges whose $q$-projection is bounded by $\delta$, increasing the values of the $A$-edges whose endpoints have thus moved, and adding certian $A$-edges. Secondly we construct the graph $\Gamma^{BA}$ from $\Gamma^B$ by contracting some short zero-valued $A$-edges, increasing the values of the $B$-edges whose endpoints have moved and adding some $B$-edges, in a similar manner. Let us illustrate the idea on the previous example.

*Example* 4.4. Assume that $EF$ is the only zero-edge in the left graph on Figure 7, whose diameter is bounded by $\delta$. We contract it (see left graph in

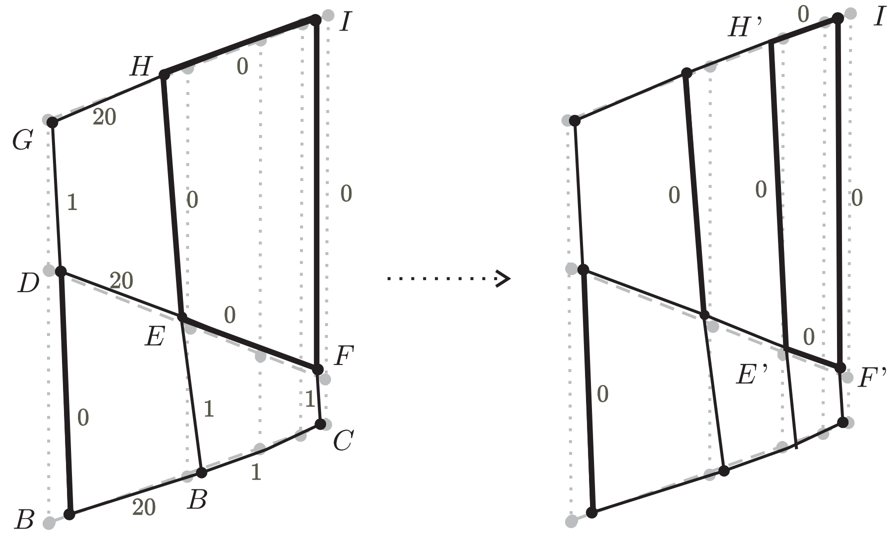


Figure 7: The graphs for $N$

Figure 9): The value of the *new* $A$-edge $DF$ obtained from the $A$-edge $DE$ is $z(DF) = z(DE)$. The $B$-edges $HE$, $BE$ are replaced by *new* $B$-edges $HF$, $BF$ with values $z(HF) = z(HE) + \epsilon$, $z(BF) = z(BE) + \epsilon$. In this process the information that the difference of the values of $g$ in $H$ and $B$ is bounded by $z(HE) + z(BE)$ is lost. Therefore we add an *added* edge $BH$ with the value $z(BH) = z(HE) + z(BE)$.

Finally the right graph on Figure 9 shows a 0-good assignment of $g, h$ in the vertices of the adjusted graph. This assignment may be extended to the one from Figure 8. Details are in Theorem 4.5.

□

Let us give the construction of the adjusted graph $\Gamma^{BA} = \Gamma^{BA}(\mu)$ obtained from $\Gamma^B = \Gamma^B(\mu)$ which is obtained from $\Gamma = \Gamma(\mu)$. To construct the graph $\Gamma^B$ from $\Gamma$ and show its properties, we need the following properties of edges of $\Gamma$, which we know are true:

1. the $x$-ordering property of the 0-valued $B$-edges
2. the $y$-ordering property of $A$-edges
3. if $\gamma$ is a 0-valued $B$-path then $\operatorname{diam} p(\gamma) < 6\mu$ (i.e. Lemma 4.1 for 0-valued $B$-edges)
4. Lemma 4.1 for $A$-edges

To construct $\Gamma^{BA}(\mu)$ and show its properties we shall need the same properties of edges of $\Gamma^B$ with $A$, $B$ and $p$, $q$ replaced.

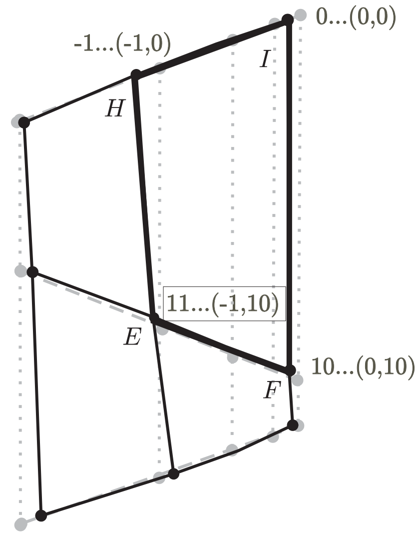


Figure 8: The graph for $N$ and assignement of values

Assume the $x$-ordering of the endpoints $\{(x_i, y_i)\}_{i\geq 1}$ of the 0-valued $B$-edges of $\Gamma$, see property (1). To avoid confusion, vertices in general will be denoted by $\{(u_i, v_i)\}_{i\geq 1}$, while the vertices which are the endpoints of the 0-valued $B$-edges (or $A$-edges) will be denoted by $\{(x_i, y_i)\}_{i\geq 1} \subseteq \{(u_i, v_i)\}_{i\geq 1}$, as here.

For $(x_i, y_i) \in V(\Gamma)$ let $\gamma^{Cb}(x_i, y_i) = (x_i, y_i)\dots(x_j, y_j)$, $i \leq j$ be the maximal, possibly trivial, 0-valued $B$-path with

$$\operatorname{diam} q(\gamma^{Cb}(x_i, y_i)) \leq \delta. \tag{4.20}$$

By property (3) we have

$$\operatorname{diam} p(\gamma^{Cb}(x_i, y_i)) = x_j - x_i < 6\mu. \tag{4.21}$$

Let $\mathrm{H}^B(x_i, y_i)$ be the subgraph of $\Gamma$ generated by the vertices of $\gamma^{Cb}(x_i, y_i)$. Define a subset $W^B = \{(x_{i_k}, y_{i_k})\}_{k\geq 1}$ of $V(\Gamma)$ in the following way. Let $i_0 = 0$, $j_0 = 0$. For every $k > 0$ let $i_k$ the smallest number such that $i_k > i_{k-1}$ and the path $\gamma^{Cb}(x_{i_k}, y_{i_k}) = (x_{i_k}, y_{i_k})\dots(x_{j_k}, y_{j_k})$ is nontrivial.

The construction of $\Gamma^B$ is as follows. Delete all edges of $\mathrm{H}^B(u, v)$ and map all vertices of $\mathrm{H}^B(u, v)$ to $(u, v)$, for all $(u, v) \in W^B$.

Replace every edge $e = \{(u', v'), (u'', v'')\}$ of $\Gamma$ with $(u'', v'') \in V(\mathrm{H}^B(\overline{u}, \overline{v}))$ for some $(\overline{u}, \overline{v}) \in W^B$, and $(u', v')$ not in any $V(\mathrm{H}^B(u, v))$, $(u, v) \in W^B$ by the edge $e^{new} = \{(u', v'), (\overline{u}, \overline{v})\}$, called *new*, of the same $A$ or $B$-type. If $e^{new}$ is an $A$-edge, let

$$z(e^{new}) := z(e) + \epsilon$$

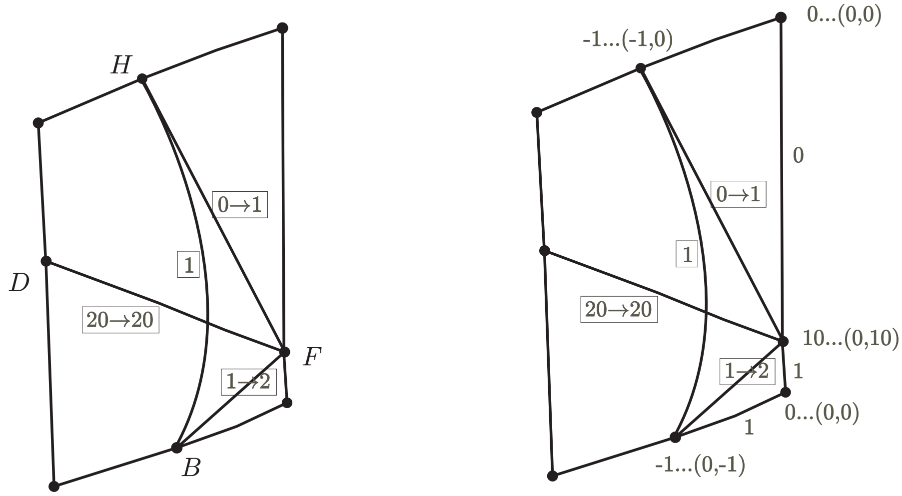


Figure 9: Adjusted graph for $N$

if $e^{new}$ is a $B$-edge, let

$$z(e^{new}) := z(e).$$

Replace every edge $e = \{(u',v'),(u'',v'')\}$ of $\Gamma$ with $(u',v') \in V(\mathrm{H}^B(u,v))$ and $(u'',v'') \in V(\mathrm{H}^B(\overline{u},\overline{v}))$ for some $(u,v),(\overline{u},\overline{v}) \in W^B$ by the edge $e^{new} = \{(u,v),(\overline{u},\overline{v})\}$, likewise called *new*, of the same $A$ or $B$-type. If $e^{new}$ is an $A$-edge, let

$$z(e^{new}) := z(e) + 2\epsilon,$$

if $e^{new}$ is a $B$-edge, let

$$z(e^{new}) := z(e).$$

Other edges of $\Gamma$ are kept unchanged.

Assume the $y$-ordering of the vertices of $\Gamma$, see property (2). For every, possibly trivial, $A$-path $\gamma = (u_m,v_m)\dots(u_n,v_n)$, $v_m \leq \dots \leq v_n$ in $\Gamma$ which is maximal with respect to the property that each vertex $(u_i,v_i)$ is moved to a vertex $(u'_i,v'_i)$, $m \leq i \leq n$ add the following $A$-edges, called *added* to $\Gamma^B$.

For every pair $(u'_i,v'_i) \neq (u'_j,v'_j)$, $j-i \geq 2$ add the $A$-edge

$$e^{add} = \{({u_i}',{v_i}'),({u_j}',{v_j}')\}$$

with the value

$$z(e^{add}) = z(e_i) + z(e_{i+1}) + \ldots + z(e_{j-1}) + 2\epsilon.$$

Since the vertex $(u_{m-1},v_{m-1})$ is not in the path $\gamma$, either $e_{m-1} = \{(u_{m-1},v_{m-1}),(u_n,v_n)\}$ is not an $A$-edge of $\Gamma$, or if it is then the vertex

$(u_{m-1}, v_{m-1})$ was not moved. In the latter case, for every vertex $(u'_j, v'_j)$, $m \leq j \leq n$ with $(u_{m-1}, v_{m-1}) \neq (u'_j, v'_j)$ add the $A$-edge

$$e^{add} = \{(u_{m-1}, v_{m-1}), (u_j', v_j')\}$$

with the value

$$z(e^{add}) = z(e_{m-1}) + z(e_m) + \ldots + z(e_{j-1}) + \epsilon.$$

Likewise for the vertex $(u_{n+1}, v_{n+1})$.

In addition, if both $e_{m-1}$, $e_n$ are $A$-edges of $\Gamma$ then add the $A$-edge

$$e^{add} = \{(u_{m-1}, v_{m-1}), (u_{n+1}, v_{n+1})\}$$

with the value

$$z(e^{add}) = z(e_{m-1}) + z(e_m) + \ldots + z(e_n).$$

If more $A$-edges that we have constructed should connect two vertices, keep only the one with the smallest value $z(e)$.

Summarizing, for every $A$-edge $e = \{(u_m', v_m'), (u_n', v_n')\}$ in $\Gamma^B$ there exist two, possibly trivial, 0-valued $B$-paths $\gamma_1^B(e) = (u_m, v_m) \ldots (u'_m, v'_m)$, $\gamma_2^B(e) = (u_n, v_n) \ldots (u_n', v_n')$ in $\Gamma$, $\gamma_1^B(e) \subseteq \mathrm{H}^B(u_m, v_m)$, $\gamma_2^B(e) \subseteq \mathrm{H}^B(u_n, v_n)$, and a nontrivial $A$-path $\gamma^A(e) = (u_m, v_m) \ldots (u_n, v_n)$ in $\Gamma$. We shall write

$$e = \gamma_1^B(e) \cup \gamma^A(e) \cup \gamma_2^B(e).$$

If there is no reason for confusion, we shall simply write $e = \gamma_1^B \cup \gamma^A \cup \gamma_2^B$. Of course none of the edges of $\gamma_i^B$ are in $\Gamma^B$. If $e$ is a new or an added edge then none of the edges of $\gamma^A$ are in $\Gamma^B$ either.

If both paths $\gamma_i^B$ are trivial, then $e$ is either an edge of $\Gamma \cap \Gamma^B$ or it is an added edge. Otherwise it is an added or a new edge. In case $e$ is a new edge, the path $\gamma^A$ consists of a single edge, the one from which $e$ was obtained.

If we let $\omega := 0$ in case both the paths $\gamma_i^B$ are trivial, $\omega := 1$ in case exactly one of them is trivial, and $\omega := 2$ in case both are nontrivial, then

$$z(e) = \sum_{e' \in \gamma^A(e)} z(e') + \omega\epsilon. \tag{4.22}$$

In particular the edge $e$ is 0-valued if and only if both the subpaths $\gamma_i^B$ are trivial and $\sum_{e' \in \gamma^A(e)} z(e') = 0$. Hence evidently properties (1), (3) hold for $A$-edges in $\Gamma^B$.

Clearly $V(\Gamma^B) \subseteq V(\Gamma)$.

Let us show that the properties (2), (4) hold for $B$-edges of $\Gamma^B$.

Assume the $x$-ordering of the vertices of $\Gamma$. Note, that this does not correspond to any of the properties (1) $-$ (4) of $\Gamma$, thus an analogy of this argument will hold only for the 0-values $A$-edges of $\Gamma^{BA}$, see property (1). For every subgraph $\mathrm{H}^B(u_i,v_i)$, $(u_i,v_i)\in W^B$, generated by the the 0-valued $B$-path $\gamma^{Cb}(u_i,v_i)=(u_i,v_i)\dots(u_j,v_j)$, $i<j$ there exist at most two $B$-edges not in this path with one or both endpoints in $\mathrm{H}^B(u_i,v_i)$, namely $e_{i-1}=\{(u_{i-1},v_{i-1}),(u_i,v_i)\}$ and $e_j=\{(u_j,v_j),(u_{j+1},v_{j+1})\}$, and we have $(u_{i-1},v_{i-1}),(u_{j+1},v_{j+1})\notin \mathrm{H}^B(u_i,v_i)$.

Hence for every $B$-edge $e=\{(u_i,v_i),(u_n,v_n)\}$, $i<n$ of $\Gamma^B$ there exists one, possibly trivial, 0-valued $B$-path $\beta^B(e) = (u_i,v_i)\dots(u_{n-1},v_{n-1})$ in $\Gamma$, $\beta^B(e)\subseteq H^B(u_i,v_i)$, and the $B$-edge $e'=\{(u_{n-1},v_{n-1}),(u_n,v_n)\}$ in $\Gamma$ from which $e$ was obtained. We shall write

$$e=\beta^B(e)\cup e'$$

or simply $e=\beta^B\cup e'$. By definition $e$ is a new $B$-edge in $\Gamma^B$ if and only if $\beta^B(e)$ is nontrivial. Otherwise $e$ is an edge of $\Gamma\cap\Gamma^B$ and $e=e'$. Thus properties (2), (4) are satisfied for $B$-edges of $\Gamma^B$.

Since we have shown that the properties of edges in $\Gamma^B$ corresponding to properties (1)–(4) are fulfilled, the graph $\Gamma^{BA}$ may be constructed from $\Gamma^B$ analogously as $\Gamma^B$ was constructed form $\Gamma$.

As we have already mentioned, the substantial difference concerns the last argument. That is only the 0-valued $A$-edges $e$ of $\Gamma^{BA}$ are of the form $e=\beta^A(e)\cup e'$, $A$-edges $e$ of $\Gamma^{BA}$ in general are of the form

$$e=\beta_1^A(e)\cup e'\cup\beta_2^A(e).$$

Obviously if $\Gamma^B$ contains two vertices $(u,v)$, $(u,v')$, $v\neq v'$ then they are connected by a vertical, therefore 0-valued $B$-path in $\Gamma$ (see Remark 3.2), which is also a 0-valued path in $\Gamma^B$, because of property (1). If $\Gamma^B$ contians vertices $(u,v)$, $(u',v)$, $u\neq u'$ then they are connected by a 0-valued horizontal $A$-path $\gamma$ in $\Gamma$. Assume all vertices of $\gamma$ were moved, then there is a 0-valued added edge $\{(u,v),(u,v')\}$. If not all vertices of $\gamma$ were moved, consider the subpaths of $\gamma$ (of course they are horizontal) with all vertices moved. Therefore Remark 3.2 applies also for $\Gamma^B$. Similarly for $\Gamma^{BA}$.

**Theorem 4.5.** *Let $f\in C(K)$ be $(\delta,\epsilon)$-uniformly continuous. There exists $\mu>0$ such that if $\Gamma^{BA}=\Gamma^{BA}(\mu)$ (or $\Gamma^B=\Gamma^B(\mu)$) is the graph obtained from $\Gamma^B=\Gamma^B(\mu)$ ($\Gamma=\Gamma(\mu)$), then functions $g$ and $h$ that are $\alpha$-good with respect to $\Gamma^{BA}$ ($\Gamma^B$) can be extended to functions that are $\alpha+2\epsilon$-good with respect to $\Gamma^B$ ($\Gamma$) and*

$$||g||_\Gamma^B=||g||_{\Gamma^{BA}}+\epsilon \text{ and } ||h||_\Gamma^B\le||h||_{\Gamma^{BA}}$$

$$(\ ||g||_\Gamma\le||g||_{\Gamma^B} \text{ and } ||h||_\Gamma=||h||_{\Gamma^B}+\epsilon).$$

*Proof.* Let us show the part of the statement about the extension from $V(\Gamma^B)$ to $V(\Gamma)$. The transition from $V(\Gamma^{BA})$ to $V(\Gamma^B)$ is analogous. The requierement on $\mu$ will be stated bellow.

For every graph $\mathrm{H}^B(u,v)$, $(u,v) \in W^B$ let $g(u') := g(u)$ for all $(u',v') \in \mathrm{H}^B(u,v)$. This assignment is thus good with respect to all $B$-edges of $\Gamma$.

The function $h$ will be defined in such a way that it is good with respect to the $A$-edges of $\Gamma$ and

$$|h(v') - h(v)| \leq \epsilon \tag{4.23}$$

for all $(u',v') \in \mathrm{H}^B(u,v)$, for all $(u,v) \in W^B$.

Such assignment is $\alpha + 2\epsilon$-good with respect to $\Gamma$. Namely by (4.20) and (4.21) we have $\operatorname{diam} q(\mathrm{H}^B(u,v)) \leq \delta$ and $\operatorname{diam} p(\mathrm{H}^B(u,v)) < 6\mu$. Thus $(u',v') \in \mathrm{H}^B(u,v)$ implies $|(u',v')-(u,v)| \leq \delta$, provided that $6\mu \leq \delta$. Since $f$ is $(2\delta,\epsilon)$-uniformly continuous it implies $|f(u,v) - f(u',v')| \leq \epsilon$. So

$$\begin{aligned}
|f(u',v') - g(u') - h(v')| &= \\
|f(u',v') - g(u) - h(v')| &\leq \\
|f(u',v') - g(u) - h(v)| + \epsilon &\leq \\
|f(u,v) - g(u) - h(v)| + 2\epsilon &\leq 2\epsilon + \alpha.
\end{aligned}$$

So let us describe the extension of the function $h$ from $V(\Gamma^B)$ to $V(\Gamma)$. Assume the $y$-ordering of the vertices of $\Gamma$. Consider an $A$-path $\gamma = (u_m,v_m)\ldots(u_n,v_n)$, $m \leq n$, in $\Gamma$ which is maximal with respect to the property that each vertex $(u_i,v_i)$ is moved to a vertex $({u'}_i,{v'}_i)$, $m \leq i \leq n$, as in the construction of the added edges. Let both $e_{m-1}$, $e_n$ be $A$-edges of $\Gamma$, the cases when one or both are not $A$-edges of $\Gamma$ is simpler. The function $h$ is defined in the vertices $(u_{m-1},v_{m-1}),({u'}_m,{v'}_m),\ldots({u'}_n,{v'}_n),(u_{n+1},v_{n+1}) \subseteq V(\Gamma^B) \subseteq V(\Gamma)$. We shall define it in the vertices $(u_m,v_m),\ldots,(u_n,v_n)$.

Formally denote $({u'}_j,{v'}_j) = (u_j,v_j)$ and let $h(v_j') := h(v_j)$, for $j = m-1, n+1$. Let $\alpha_i = 1$ if $i = m,\ldots,n$ and $\alpha_i = 0$ if $i = m-1, n+1$.

The required (4.23) holds if and only if $h(v_i)$ belongs to the interval

$$I_i := [h(v_i') - \alpha_i\epsilon, h(v_i') + \alpha_i\epsilon],$$

$i = m-1,\ldots,n+1$.

On the other hand the values of $h(v_j)$ will be good with respect to the edges of $\gamma$ if and only if

$$|h(v_i) - h(v_{i+1})| \leq z(e_i), \tag{4.24}$$

$i = m-1,\ldots,n$.

If for all $i < j$ we have

$$\operatorname{dist}(I_i,I_j) \leq z(e_i) + z(e_{i+1}) + \ldots + z(e_{j-1}) \tag{4.25}$$

then there exist values $h(v_i) \in I_i$ such that (4.24) holds.

So take $I_i$, $I_j$, $i < j$ and assume without loss of generality that $\max I_i \leq \min I_j$. Thus (4.25) holds if and only if

$$\min I_j - \max I_i = h(v_j') - \alpha_j\epsilon - h(v_i') - \alpha_i\epsilon \leq z(e_i) + z(e_{i+1}) + \ldots + z(e_{j-1}).$$

The vertices $(u'_i, v'_i)$, $(u'_j, v'_j)$ are connected by an added $A$-edge thus

$$h(v'_j) - h(v'_i) \leq z(e_i) + z(e_{i+1}) + \ldots + z(e_{j-1}) + \alpha_i\epsilon + \alpha_j\epsilon,$$

which yields (4.25).

Obviously $||g||_\Gamma = ||g||_{\Gamma^B}$ and $||h||_\Gamma \leq ||h||_{\Gamma^B} + \epsilon$. □

# 5 Properties of the adjusted graph.

In the process of construction however, we have lost the property stated in Lemma 4.1. We have only the following statement (naturally $3||f||$ can be replaced by any positive number):

**Lemma 5.1.** *For every $\alpha > 0$ there exists $\mu > 0$ such that*

$$\textit{if} \quad z(e) < 3||f|| \quad \textit{then either} \quad \operatorname{diam} p(e) < \alpha \quad \textit{or} \quad \operatorname{diam} q(e) < \alpha$$

*for every edge $e$ in $\Gamma^{BA}(\mu)$.*

*Proof.* Let us start by showing this property for $A$-edges of $\Gamma^B$.

Let $e = \gamma_1^B(e) \cup \gamma^A(e) \cup \gamma_2^B(e)$ be an $A$-edge of $\Gamma^B(\mu)$ with $z(e) < 3||f||$. We have $\sum_{e' \in \gamma_A(e)} z(e') \leq z(e) < 3||f||$. Let $k = \lceil 3||f||/\epsilon \rceil$. By property (4) we have

$$\operatorname{diam} q(\gamma^A(e)) < 6k\mu.$$

Since the paths $\gamma_i^B(e)$ are 0-valued, by property (3) we have

$$\operatorname{diam} p(\gamma_i^B(e)) < 6\mu,$$

$i = 1, 2$.

Thus with decreasing $\mu$, the paths $\gamma_i^B(e)$ converge to vertical segments, and the paths $\gamma^A(e)$ converge to horizontal segments, for $A$-edges $e$ in $\Gamma^B(\mu)$ with $z(e) < 3||f||$. If in contrast with the statement there exists $\alpha_0$ such that for every $\mu$ there exists an $A$-edge $e_\mu \in \Gamma^B(\mu)$ with $z(e_\mu) < 3||f||$ and $\operatorname{diam} p(e_\mu) \geq \alpha_0$ and $\operatorname{diam} q(e_\mu) \geq \alpha_0$ then the endpoints of the paths corresponding to the edges $e_\mu$ converge to an array of length at least 2 in $K$.

Since we only used properties (1)–(4), analogous arguments apply for $B$-edges of $\Gamma^{BA}(\mu)$ with $z(e) < 3||f||$.

Let $e = \beta_1^A \cup e' \cup \beta_2^A$ be an $A$-edge of $\Gamma^{BA}(\mu)$ which does not belong to $\Gamma^{BA}(\mu) \cap \Gamma^B(\mu)$. Since $e'$ is an $A$-edge in $\Gamma^B$ with $z(e') \leq z(e) < 3||f||$, by the above, one of the projections of $e'$ is arbitrarily small with decreasing $\mu$. By (3) for $\mathrm{H}^A$, we have $\operatorname{diam} q(\beta_i^A) < 6\mu$, $i = 1, 2$. Therefore the statement of the lemma for $A$-edges of $\Gamma^{BA}$ is obtained by the same argument as before, showing that if the contrary holds, then there is an array of length 2 in $K$.
□

The functions $g$ and $h$ can be defined consistently on $\Gamma^{BA}$. In order to do so, we distinguish three types of edges.

**Definition 5.2.** Define the following subgraphs of $\Gamma^{BA}(\mu)$

- $\Gamma^N$ consists of edges with $z(e) \geq |\triangle_e f| + \epsilon/2$, called *neutral*
- $\Gamma^{Ha}$ consists of $A$-edges with $z(e) < |\triangle_e f| + \epsilon/2$, called *horizontal*
- $\Gamma^{Vb}$ consists of $B$-edges with $z(e) < |\triangle_e f| + \epsilon/2$, called *vertical*

**Proposition 5.3.** *Let $f$ be $(3\delta, \epsilon/2)$-uniformly continuous. There exists $\mu > 0$ such that if $e^H \in \Gamma^{Ha} \subseteq \Gamma^{BA}(\mu)$ is a horizontal edge then*

$$\operatorname{diam} p(e^H) > \delta$$

*and if $e^V \in \Gamma^{Vb} \subseteq \Gamma^{BA}(\mu)$ is a vertical edge then there exists a point $(u, v) \in K$ such that*

$$u \in p(e^V) \quad \textit{and} \quad \operatorname{diam} q(e^V) \cup \{v\} > \delta, \tag{5.26}$$

*where by $p(e^V)$ we mean the closed interval given by the p-projections of the endpoints of $e_V$. Moreover for all $\alpha > 0$ there exists $\mu > 0$ such that*

$$\operatorname{diam} q(e^H) < \alpha \quad \textit{and} \quad \operatorname{diam} p(e^V) < \alpha. \tag{5.27}$$

*Proof.* Let $e = e^H$ be horizontal edge of $\Gamma^{BA} = \Gamma^{BA}(\mu)$, i.e. it is an $A$-edge with

$$z(e) < |\triangle_e f| + \epsilon/2. \tag{5.28}$$

Let $e = \beta_1^A \cup e' \cup \beta_2^A$, $e' = \gamma_1^B \cup \gamma^A \cup \gamma_2^B$. Let us show

$$\operatorname{diam} q(\beta_1^A) + \operatorname{diam} q(\gamma_1^B) + \operatorname{diam} q(\gamma^A) + \operatorname{diam} q(\gamma_2^B) + \operatorname{diam} q(\beta_2^A) \leq 3\delta \tag{5.29}$$

for small enough $\mu$.

By the analogy of (4.21)

$$\operatorname{diam} q(\beta_i^A) < 6\mu, i = 1, 2.$$

By the way of construction

$$\operatorname{diam} q(\gamma_i^B) \leq \delta, \quad i = 1, 2.$$

(5.28) in particular implies

$$\sum_{e'' \in \gamma_A} z(e'') \leq z(e)' = z(e) < 3||f||.$$

Thus by the analogy of property (4) if we let $k = \lceil 3||f||/\epsilon \rceil$ then

$$\operatorname{diam} q(\gamma^A) < 6k\mu.$$

So if $12\mu + 6k\mu \leq \delta$, then (5.29) holds.

Let us show that

$$\operatorname{diam} p(e) \leq \delta \tag{5.30}$$

leads to a contradiction. Namely $\operatorname{diam} p(e) \leq \delta$ together with (5.29) implies

$$\operatorname{diam} e \leq 3\delta.$$

Thus since $f$ is $(3\delta, \epsilon/2)$-uniformly continuous, (5.28) is

$$z(e) < |\triangle_e f| + \epsilon/2 \leq \epsilon.$$

Since the values $z(e)$ are integer multiples of $\epsilon$, this implies

$$z(e') = z(e) = 0.$$

Assume the $y$-ordering of the endpoints of the 0-valued $A$-edges of $\Gamma^B$. As we have mentioned, if $e \in \Gamma^{BA}$ is a 0-valued $A$-edge, then $e = \beta^A \cup e'$ and $\beta^A \cup e' = (x_i, y_i) \ldots (x_j, y_j) \cup \{(x_j, y_j), (x_{j+1}, y_{j+1})\}$, $i \leq j$.

If $e$ is a new edge of $\Gamma^{BA}$, that is the path $\beta^A$ is non-trivial, $\beta^A \subseteq \gamma^{Ca}(x_i, y_i)$ and since the edge $e'$ was not deleted

$$e' \notin \gamma^{Ca}(x_i, y_i).$$

On the other hand $\beta^A \cup e'$ is a 0-valued $A$-path and

$$\operatorname{diam} p(\beta^A \cup e') = \operatorname{diam} p(e) \leq \delta \tag{5.31}$$

thus by definition

$$e' \in \gamma^{Ca}(x_i, y_i).$$

Contradiction.

If $e$ is an edge of $\Gamma^B \cap \Gamma^{BA}$ then the path $\beta^A$ is trivial, that is the vertex $(x_j, y_j)$ is not in any $H^A$. Thus if $k$ is the maximal index with $\gamma^{Ca}(x_k, y_k) = (x_k, y_k) \ldots (x_l, y_l)$, $(x_k, y_k) \in W^A$ and $k \leq j$, then $l < j$. But then since $z(e') = 0$ and $\operatorname{diam} p(e') \leq \operatorname{diam} p(e) \leq \delta$ the construction is such that $(x_j, y_j)$ should have been the next vertex in $W^A$. Contradiction.

(5.27) for the edges of $\Gamma^{Ha}$ thus follows from Lemma 5.1.

Let $e = e^V$ be a vertical edge of $\Gamma^{BA}$, i.e. it is a $B$-edge with (5.28). Analogously as (5.29) it can be proved that

$$\operatorname{diam} p(e) \leq 3\delta, \tag{5.32}$$

for small enough $\mu$.

Let us show that

$$\operatorname{diam} q(e) \leq \delta \tag{5.33}$$

implies the existence of the point $(u, v) \in K$ with (5.26).

Let $e = \gamma_1^A \cup \gamma^B \cup \gamma_2^A$. By (5.33) and (5.32) we have $\operatorname{diam} e \leq 3\delta$ which as before implies

$$z(e) = 0.$$

Therefore in particular both the paths $\gamma_i^A$ are trivial and

$$\sum_{e'' \in \gamma_B} z(e'') = z(e) = 0$$

so

$$z(e'') = 0$$

for each $e'' \in \gamma_B$.

With the $x$-ordering of the vertices of the 0-valued $B$-edges of $\Gamma$ (property (1)) we have $\gamma^B = (x_i, y_i) \dots (x_j, y_j)$, $i < j$, $x_i \leq \dots \leq x_j$, $e = \{(x_i, y_i), (x_j, y_j)\}$.

Similarly as (5.30) leads to a contradiction, here $\operatorname{diam} q(e_k) = |y_k - y_{k+1}| \leq \delta$ leads to a contradiction so

$$\operatorname{diam} q(e_k) > \delta$$

for all $k = i, \dots, j-1$, $e_k = \{(x_k, y_k), (x_{k+1}, y_{k+1})\} \in \gamma^B$.

Thus in particular

$$\operatorname{diam} q(e) \cup \{y_{i+1}\} \geq |y_i - y_{i+1}| > \delta$$

and $x_{i+1} \in p(e) = [x_i, x_j]$. Hence $(u, v) = (x_{i+1}, y_{i+1}) \in K$ is the desired point (recall that all vertices are points in $K$).

It can be easily seen that (5.26) holds for the edges of $\Gamma^{Vb}$. Namely for edges $e = e^V \in \Gamma^{Vb}$ with $\operatorname{diam} q(e) > \delta$ the statement follows from Lemma 5.1. As we have seen, for edges $e = e^V = \gamma_1^A \cup \gamma^B \cup \gamma_2^A \in \Gamma^{Vb}$ with $\operatorname{diam} q(e) \leq \delta$ we have $z(e) = 0$ so the paths $\gamma_i^A$ are trivial, and the path $\gamma^B$ is 0-valued, so by property (4) for $B$-edges in $\Gamma^B$ we have

$$\operatorname{diam} p(e) = \operatorname{diam} p(\gamma^B) < 6\mu.$$

□

**Proposition 5.4.** *There exists $\mu > 0$ such that for the subgraphs $\Gamma^{Vb}, \Gamma^{Ha}, \Gamma^N \subseteq \Gamma^{BA}(\mu)$ we have*

$$\begin{aligned} p(V(\Gamma^{Vb})) \cap p(V(\Gamma^{Ha})) = \emptyset, \\ q(V(\Gamma^{Vb})) \cap q(V(\Gamma^{Ha})) = \emptyset \end{aligned} \tag{5.34}$$

*and if a path $\gamma$ in $\Gamma^N$ connects a vertex in $\Gamma^{Vb} \cap \Gamma^N$ to a vertex in $\Gamma^{Ha} \cap \Gamma^N$ then*

$$\sum_{e \in \gamma} z(e) \geq \sum_{e \in \gamma} |\triangle_e f| + ||f||. \tag{5.35}$$

*Proof.* Let us show first that there exist $\beta > 0$, $\mu > 0$ such that

$$\operatorname{dist}(p(e^V), p(e^H)) \geq \beta \quad \text{and} \quad \operatorname{dist}(q(e^V), q(e^H)) \geq \beta \tag{5.36}$$

for any pair of edges $e^V \in \Gamma^{Vb}$ and $e^H \in \Gamma^{Ha}$, thus in particular (5.34) is true.

For each $\alpha = 1/n$ choose $\mu_n$ from Proposition 5.3. If (5.36) does not hold, then for each $\beta = 1/n$, $n \in \mathbb{N}$ there exist edges $e_n^H \in \Gamma^{Ha}(\mu_n), e_n^V \Gamma^{Vb}(\mu_n)$, $e_n^H = \{(a_n, b_n), (a_n', b_n')\}$, $e_n^V = \{(c_n, d_n), (c_n', d_n')\}$ and a point $(x_n, y_n) \in K$ such that

$$\begin{array}{l} x_n \in p(e_n^V) \\ |a_n - a_n'| > \delta \quad \text{and} \quad |b_n - b_n'| < 1/n \quad \text{and} \\ |c_n - c_n'| < 1/n \quad \text{and} \quad |d_n - y_n| > \delta \quad \text{and} \\ (\operatorname{dist}(p(e_n^H), p(e_n^V)) < 1/n \quad \text{or} \quad \operatorname{dist}(q(e_n^H), q(e_n^V)) < 1/n). \end{array}$$

Thus with $n \to \infty$ these points converge to an array of length 2 in $K$.

Fix the $\beta$ and $\mu$ we have found. Let $\gamma$ be a path as in the statement. By Definition 5.2 we have $z(e) \geq |\triangle_e f| + \epsilon/2$ for edges $e$ from $\Gamma^N$. Thus

$$\sum_{e \in \gamma} z(e) \geq \sum_{e \in \gamma} |\triangle f_e| + k\,\epsilon/2 \tag{5.37}$$

where $k$ is the number of edges that $\gamma$ consists of.

If $\gamma$ contains an edge $e$ with $z(e) \geq 3||f||$ and we obtain (5.35) so let $\gamma$ consist of edges with $z(e) < 3||f||$. By Lemma 5.1 the diameter of one of their projections converges to 0 with decreasing $\mu$. On the other hand, (5.36) implies that $\operatorname{diam} p(\gamma) \geq \beta$ and $\operatorname{diam} q(\gamma) \geq \beta$. Thus if with decreasing $\mu$ the number $k$ of edges on such paths $\gamma$ did not increase, then $\gamma$ would converge to an array of length at least 2 in $K$. Since $k \to \infty$ as $\mu \to 0$, the statement follows from (5.37), since it suffices to have $k\epsilon/2 \geq ||f||$. □

# 6 Construction of functions $g$ and $h$ on the adjusted graph.

*Remark* 6.1. Any of the assignments

$$g(u) := f(u, v) - k \quad \text{and} \quad h(v) := k$$

for all $(u, v) \in \Gamma^N$ or

$$g(u) := k \quad \text{and} \quad h(v) := f(u, v) - k$$

for all $(u, v) \in \Gamma^N$, with any $k \in \mathbb{R}$ is 0-good with respect to $\Gamma^N$.

**Theorem 6.2.** *Let $\mu$ be as before, and let $f \in C(K)$ be $(3\delta, \epsilon/2)$-uniformly continuous. There exist functions $g$ and $h$ that are 0-good with respect to the abstract representation of a graph $\Gamma^{BA} = \Gamma^{BA}(\mu)$ and with $||g|| \leq ||f||$, $||h|| \leq ||f||$.*

*Proof.* The representation of the graph $\Gamma^{BA}$ is abstract, thus the indices of vertices do not suggest any ordering.

Recall that $\Gamma^{BA} = \Gamma^{Vb} \cup \Gamma^{N} \cup \Gamma^{Ha}$ (Definition 5.2), and Proposition 5.4 implies $\Gamma^{Vb} \cap \Gamma^{Ha} = \emptyset$.

If $\Gamma^{BA} = \Gamma^{N}$ let $g := 0$, $h := f$ on $V(\Gamma^{BA})$, this assignment is 0-good with respect to $\Gamma^{BA} = \Gamma^{N}$, see Remark 6.1. The norms are as required.

Assume $\Gamma^{BA} \neq \Gamma^{N}$. Let

$$g := 0,\ h := f \text{ on } V(\Gamma^{Vb}).$$

Since $\Gamma^{Vb}$ consists of $B$-edges, this assignment is 0-good with respect to $\Gamma^{Vb}$. This assignment is also good with respect to the edges of $\Gamma^{N}$ connecting vertices that belong to the intersection $\Gamma^{N} \cap \Gamma^{Vb}$ (Remark 6.1). We shall define an assignment of values $h(v)$ for vertices $(u, v)$ in $\Gamma^{N} - \Gamma^{Vb}$ with

$$\begin{array}{lll} h(v) \in [0, f(u,v)] & \text{if} & f(u,v) \geq 0 \\ h(v) \in [f(u,v), 0] & \text{if} & f(u,v) < 0. \end{array} \tag{6.38}$$

Without mentioning it explicitly, if we define $h(v)$ in some vertex $(u, v)$, we let $g(u) := f(u,v) - h(v)$, which yields $g(u) \in [0, f(u,v)]$ if $f(u,v) \geq 0$ and $g(u) \in [f(u,v), 0]$ if $f(u,v) < 0$. In particular such assignment is 0-good with respect to the vertices of $\Gamma^{N}$.

On $\Gamma^{Ha}$ we wish to have $g = f$, $h = 0$. Since $\Gamma^{Ha}$ consists of $A$-edges, such assignment is 0-good with respect to $\Gamma^{Ha}$.

Let $\Gamma^{N}_{+}$ and $\Gamma^{N}_{-}$ be the subgraphs of $\Gamma^{N}$ induced by vertices $(u, v)$ with $f(u,v) \geq 0$ and $f(u,v) < 0$, respectively. We shall describe the assignment of values on $\Gamma^{N}_{+}$. The assignment on $\Gamma^{N}_{-}$ is analogous.

The subgraph $\Gamma^{N}_{+}$ may consist of several connected components.

For every component $H$ let

$$h := 0,\ g := f \text{ on } H, \text{ if } H \cap \Gamma^{Vb} = \emptyset. \tag{6.39}$$

By Remark 6.1 this assignment is 0-good with respect to $H$.

Let $H$ be a component with $H \cap \Gamma^{Vb} \neq \emptyset$. For every path $\gamma = (u_1, v_1) \ldots (u_m, v_m)$ in $H$ with $(u_1, v_1) \in H \cap \Gamma^{Vb}$ and $(u_m, v_m) \in H - \Gamma^{Vb}$ let

$$h_\gamma := f(u_1, v_1) + \sum_{i \in I_B} \triangle_i f - \sum_{i=1}^{m} z(e_i) \tag{6.40}$$

where $e_i = \{(u_i, v_i), (u_{i+1}, v_{i+1})\}$, $\triangle_i f = f(u_{i+1}, v_{i+1}) - f(u_i, v_i)$ and the edges $e_i$, $i \in I_B$ are the $B$-edges of $\gamma$. From all such paths $\gamma$ leading to a vertex $(u,v)=(u_m,v_m)$ let $\gamma^{max}(u,v)$ be one of those with

$$h_{\gamma^{max}(u,v)} = \max_{\gamma} h_\gamma.$$

Let $H^{max}$ be the subgraph of $H$ generated by these chosen paths, we have $V(H^{max}) = V(H)$. Evidently we may assume that for every vertex $(u,v) \in V(H) = V(H^{max})$ there is exactly one path in $H^{max}$ leading from $V(H \cap \Gamma^{Vb})$ to $(u,v)$. In other words if the vertices of $V(H \cap \Gamma^{Vb}) = V(H^{max} \cap \Gamma^{Vb})$ are identified to one, then $H^{max}$ is a tree.

With $\gamma^{max}(u_m,v_m) = (u_1,v_1)\dots(u_m,v_m)$, $(u_1,v_1) \in H \cap \Gamma^{Vb}$ the unique path in $H^{max}$ leading to a vertex $(u_m,v_m)$ let

$$h(v_m) := \max\{h_{\gamma^{max}(u_m,v_m)}, 0\}. \tag{6.41}$$

Let us show that this assignment is good with respect to the edges of $H^{max}$. Consider an edge $e_{m-1} = \{(u_{m-1},v_{m-1}),(u_m,v_m)\}$ of $H^{max}$, that is it is an edge on the path $\gamma^{max}(u_m,v_m)$. Assume that both values $h(v_{m-1})$ and $h(v_m)$ are nonzero.

If $e_{m-1}$ is a $B$-edge then

$$h(v_{m-1}) - h(v_m) = z(e_{m-1}).$$

Moreover if $h(v_{m-1}) \le f(u_{m-1},v_{m-1})$ then since $e_{m-1}$ is a neutral edge we have $|z(e_{m-1})| \ge |\triangle_{m-1} f| + \epsilon/2$ (Definition 5.2) thus

$$\begin{aligned} h(v_m) &= h(v_{m-1}) - z(e_{m-1}) \le \\ &\le f(u_{m-1},v_{m-1}) - f(u_m,v_m) + f(u_m,v_m) - z(e_{m-1}) \le \\ &\le f(u_m,v_m) \end{aligned} \tag{6.42}$$

If $e_{m-1}$ is an $A$-edge then

$$h(v_{m-1}) - h(v_m) = -\triangle_{m-1} f + z(e_{m-1})$$

$$g(u_{m-1}) - g(u_m) = f(u_{m-1},v_{m-1}) - f(u_m,v_m) - h(v_{m-1}) + h(v_m) = -z(e_{m-1}).$$

Here $h(v_{m-1}) \le f(u_{m-1},v_{m-1})$ also implies

$$\begin{aligned} h(v_m) &= h(v_{m-1}) + f(u_m,v_m) - f(u_{m-1},v_{m-1}) - z(e_{m-1}) \le \\ &\le f(u_m,v_m) - z(e_{m-1}) \le f(u_m,v_m). \end{aligned} \tag{6.43}$$

If one or both of $h(v_m)$, $h(v_{m-1})$ are equal to 0, the computations are similar. Thus these values are good with respect to the edge $e_{m-1}$.

Since each path $\gamma^{max}$ starts with a vertex $(u_1,v_1) \in V(\Gamma^N)$ we have $h(v_1) = f(u_1,v_1)$ so $h(v_1) \le f(u_1,v_1)$ and thus (6.43) and (6.42) imply $h(v) \in [0, f(u,v)]$ for all $(u,v) \in V(H)$, see (6.38).

It is easy to see that such values are good with respect to the rest of the edges of $H$ too. Namely let $e = \{(u',v'),(u,v)\} \notin H^{max}$, let it be for example an $A$-edge, if it is a $B$-edge, the computations are analogous. We have $h(v) = h_\gamma$, $\gamma = \gamma^{max}(u,v)$ and $h(v') = h_{\gamma'}$, $\gamma' = \gamma^{max}(u',v')$ and $e \notin \gamma, \gamma' \subseteq H^{max}$. The path $\gamma' \cup e$ leads to $(u,v)$ so

$$h(v) \geq h_{\gamma' \cup e} = h(v') - z(e).$$

Similarly

$$h(v') \geq h(v) - z(e).$$

Thus $|h(v) - h(v')| \leq z(e)$.

In this way we define $h$ on all $\Gamma^N_+$.

With this definition

$$h = 0 \text{ on } \Gamma^N_+ \cap \Gamma^{Ha}.$$

Namely if $(u,v)$ is a vertex in $\Gamma^N_+ \cap \Gamma^{Ha}$ and it lies in a subgraph $H$ with $H \cap \Gamma^{Vb} = \emptyset$ then by definition, i.e. (6.39), we have $h(v) = 0$.

If $(u,v)$ lies in a subgraph $H$ with $H \cap \Gamma^{Vb} \neq \emptyset$ then (see (6.40),(6.41)) $h(v) = \max\{h_{\gamma^{max}(u,v)}, 0\}$. We have

$$h_{\gamma^{max}(u,v)} \leq ||f|| + \sum_{e \in \gamma^{max}(u,v)} |\triangle_e f| - \sum_{e \in \gamma} z(e_n),$$

therefore since $\gamma^{max}(u,v)$ is a path in $\Gamma^N$ connecting a vertex $\Gamma^{Vb}$ and from $\Gamma^{Ha}$, by Proposition 5.4 we have

$$\sum_{e \in \gamma} z(e_n) \geq ||f|| + \sum_{e \in \gamma^{max}(u,v)} |\triangle_e f|.$$

Hence

$$\gamma^{max}(u,v) \leq 0.$$

Similarly we define $g$ and $h$ that are 0-good with respect to $\Gamma^N_-$ and with $h = 0$ on $\Gamma^N_- \cap \Gamma^{Ha}$

Trivially $g$ and $h$ are good with respect to the rest of the edges of $\Gamma^N$, i.e. those connecting $\Gamma^N_+$ and $\Gamma^N_-$. Namely if $e = \{(u,v),(u',v')\}$ in $\Gamma^N$ is an edge with $(u,v) \in \Gamma^N_+$ and $(u',v') \in \Gamma^N_-$, i.e. $f(u,v) \geq 0$ and $f(u',v') < 0$ then by (6.38)

$$0 \leq g(u) - g(u') \leq f(u,v) - f(u',v'),$$

$$0 \leq h(v) - h(v') \leq f(u,v) - f(u',v').$$

Since $e$ is a neutral edge, $z(e) \geq |\triangle_e f| + \epsilon$ which shows that these values are good with respect to $e$ whether it is an $A$-edge or a $B$-edge.

Let $g := f$, $h := 0$ on $\Gamma^{Ha} - \Gamma^N$. As we have already mentioned, since $\Gamma^{Ha}$ consists of $A$-edges, this assignment is 0-good with respect to $\Gamma^{Ha}$.

Obviously $||h|| \leq ||f||$ and $||g|| \leq ||f||$. □

## 7 Proof of Theorem 1.2.

*Proof. (of Theorem 1.2)* Let $K \subseteq \mathbb{R}^2$ with $E(K) = \emptyset$ be a compact set, let $f \in C(K)$ be given. To show that $f$ can be expressed as $f = g + h$, it suffices to find functions $g_\epsilon$ and $h_\epsilon$ that are $\epsilon$-close to $f$ and their norms are bounded by $4||f||$, for every $\epsilon$ (Proposition 1.4).

Let $\delta$ be such that $f$ is $(3\delta, \epsilon/2)$-uniformly continuous.

Let $\mu$ be as in Theorem 6.2 and also $\mu \leq \delta$. Construct $\Gamma(\mu)$ and $\Gamma^{BA}(\mu)$. Using Theorem 6.2 construct functions $g, h$ defined on $V(\Gamma^{BA})$ that are 0-good with respect to $\Gamma^{BA}$ and

$$\max\{||g||_{\Gamma^{BA}}, ||h||_{\Gamma^{BA}}\} \leq ||f||.$$

Using Theorem 4.5 extend them to functions defined on $V(\Gamma)$ that are $4\epsilon$-good with respect to $\Gamma$ and

$$\max\{||g||_{\Gamma}, ||h||_{\Gamma}\} \leq ||f|| + \epsilon.$$

Using Theorem 3.4, with $\alpha = 4\epsilon$ (we have $\mu \leq \delta$), extend them to $\mathbb{R}$ in such a way that

$$||f - (g + h)|| < 27\epsilon + 11 \cdot 4\epsilon = 100\epsilon$$

and

$$\max\{||g||_{\mathbb{R}}, ||h||_{\mathbb{R}}\} \leq ||f|| + 2||f|| + 2\epsilon \leq 4||f||.$$

□

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