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Preprint series, Vol. 43 (2005), 980

ON CONTINUA WITH HOMOTOPICALLY FIXED BOUNDARY

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ISSN 1318-4865

May 31, 2005

Ljubljana, May 31, 2005

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— Version: January 16th 2005—

Abstract. The paper presents two subcontinua of $\mathbb{R}^n$, one Peano-Continuum, and one cellular continuum with trivial fundamental group. Both of them have the remarkable property that neither the entire spaces nor (roughly speaking) any part of them is homotopy equivalent to a lower-dimensional space. This extends work of the last three authors and of Karimov from the planar case to the higher dimensional case, but it also contains in the cellular case the first example with all these properties in dimension two.

1. Introduction

It is a natural property of finite simplicial complexes in $\mathbb{R}^n$ that, by systematically collapsing n-dimensional simplices with free faces, the entire complex can be homotopy equivalently transformed into an at most $(n-1)$-dimensional complex that is usually called "spine". Also, open subsets of $\mathbb{R}^n$ have this property of being homotopy equivalent to an at most $(n-1)$-dimensional set; apparently this has been known and used as an exercise in the Bing-School. However, we are not aware of a written version apart from [Vr]. Hence it came to some people as a kind of surprise when [Za97].A.4.13 contained an example of a planar Peano-continuum, which apparently did not offer any possibility for a collapse. In [CCZ; §5] it was then rigorously proven for an analogously constructed space that this space is definitely not homotopy equivalent to any lower-dimensional space. The examples of [Za97] and of [CCZ] had large (i.e. uncountable) fundamental groups. In [KRRZ; Expl.1] it was shown that for a planar set, in order to achieve the property of being not homotopy equivalent to any one-dimensional space, it is not necessary to have a non-trivial fundamental group. By extending the use of the term "homotopy dimension" from complexes to arbitrary topological spaces, this property was called "of homotopy dimension two" (cf. our Def.3.3(i)). [KRRZ; Def.2.1(ii)] also introduced the term "everywhere homotopically two-dimensional" as a precise mathematical term that mimics the intuitive feeling that every part of a space has homotopy dimension two. We will be also using this phrase here (cf. Def.3.3(ii)). The codiscrete subsets of the two-sphere of homotopy dimension two have meanwhile been characterized by Cannon and Conner in [CC; Thm.1.1].

[KRRZ] already contains as Expl.3, apart from the cellular planar homotopically two-dimensional continuum with trivial fundamental group that we mentioned before, an everywhere homotopically two-dimensional planar Peano-continuum. By

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1991 *Mathematics Subject Classification.* Primary: 54F15, 55M10. Secondary: 54D05.

Key words and phrases. Homotopically fixed point, homotopy dimension, cellular sets, Peano continua

This research was supported by the Slovenian-Polish research grant No. SLO-POL 24 (2002–2003) and the MESS research program No. P1-0292-0101-04.

the way, as also shown in [KRRZ; Prop.1.1(i)], no planar continuum can have all three properties: being homotopically two-dimensional, with trivial π_1, and a Peano-continnum altogether.

In this paper now we are giving two examples of subcontinua of $\mathbb{R}^n$ which both also have (and have everywhere) homotopy dimension n. One of them will be a Peano-continuum (Example B), the other a cellular continuum with trivial fundamental group (Example A). We construct these spaces by using the same ideas as in [KRRZ; Expl.1–3], however since some theorems in topology do only hold for the plane, here at some places we do have to work harder than it was necessary in [KRRZ; §2–3] (cf. 1.3 below). The idea for our Example B is using a fractal-like iterated gluing of thickened Hawaiian earrings. I.e. the construction starts with a space H that is pictured in Fig.1, and it uses the property that the accumulation point of this space is homotopically fixed (cf. Def.3.4). Then we will glue infinitely many copies of H to a dense subset of the boundary of H, in this way making all attaching points again homotopically fixed. Next we will glue a second collection of copies of H to a dense subset of the boundary of the previous set, and after infinitely many such steps we will close our construction. It will be necessary to precisely understand how this space changes during this closing-process for being able to prove that even after closing it still has the desired properties. Example A is a similar fractal-like iterated gluing of thickened triangular doublecomb-spaces. Hence for $n = 2$ our Example A is a planar homotopically and everywhere homotopically two-dimensional cellular continuum, and such an example was not contained in [KRRZ]. We will construct these examples in Sect.2 of this paper, and in Sect.3–Sect.5 we will prove that they have the following properties:

Properties 1.1. OUR EXAMPLE A *is a subcontinuum of* $\mathbb{R}^n$ *which satisfies:*

(1) *It is cellular.*
(2) *It is simply connected.*
(3) *The local dimension at each of its points is* n.
(4) *It is everywhere homotopically* n*-dimensional.*
(5) *Each of its boundary points is homotopically fixed (cf. Def.3.4).*
(6) *There does not exist a deformation retraction onto a proper subset.*

Properties 1.2. OUR EXAMPLE B *is a subcontinuum of* $\mathbb{R}^n$ *which satisfies:*

(1) *It is a Peano-continuum.*
(2) *It is simply connected for* $n > 2$.
(3) *The local dimension at each of its points is* n.
(4) *It is everywhere homotopically* n*-dimensional.*
(5) *Each of its boundary points is homotopically fixed (cf. Def.3.4).*
(6) *There does not exist a deformation retraction onto a proper subset.*

Remark 1.3. Cell-like sets can be interpreted as sets which have the shape of a point. In the planar case these sets can be characterized by their Vietoris-homology (cf. [Brsk], Ch. VII, Thm.7.1) and also by their Čech-homology (cf. [Lefs; Ch. VII, §6, Thm.26.1]). However, this easy characterization is only available in the planar case. Also, the second claim of Props.1.1(1) would in the planar case directly follow from known results, cf. [FiZa; Corl.6]. For higher dimensions this is wrong, since Griffiths' space (this is Expl.0.12 from [Za94], but with the arc C removed, cf. also [Brwn; p.315]) is a cellular continuum in $\mathbb{R}^3$ and has non-trivial fundamental group (as follows from Griffiths' original proof for his Example C

in [Gr1].[3.4], [Gr1].(Thm.4), [Gr2].(Thm.6.3) (corrections in [Gr1.5] and [MoMo]) using the elementary proof that was sketched in [Za94].(3.11(ii)/0.12) for the space $\widehat{A} \cup \widehat{B}$. Therefore the proof of Props.1.1 will require some concrete constructions.

2. Construction of our Examples

We start with **Example B**

STEP B1: Let $B_1 \supset B_2 \supset B_3 \supset \dots$ be the sequence of round n-balls in $\mathbb{R}^n$, having radii $1/k$ and centerpoints $(0, 1/k, 0, 0, \dots 0), k = 1, 2, 3, \dots$. The *thickened Hawaiian earring* is the space $H \subset \mathbb{R}^n$ defined as (see Fig. 1):

$$H := (B_2 \setminus B_3) \cup (B_4 \setminus B_5) \cup (B_6 \setminus B_7) \cup \dots .$$

As already described before stating Props.1.1, we will as a next step construct the space H_∞ which is a fractal-like iterated glueing of this space. By adding limit-points we will obtain the desired space $\tilde{H}_\infty$. Spaces H_i and $\tilde{H}_i$ will occur as approximating spaces from inside and from outside. All these spaces are already infinite gluings of copies of H. Since for some purposes we need also finite polyhedra as approximating spaces, it will also be necessary to construct the spaces H'_i and $H'_{i,j}$ in the fortcoming steps:

STEP B2: Now construct the iterated Hawaiian earring H_∞ as follows. The space H is embedded in the n-ball $C := B_1$. On the boundary ∂H choose a countable dense set of points $V := \{v_i | i \in \mathbb{N}\} \not\ni (0, 0, \dots, 0)$.

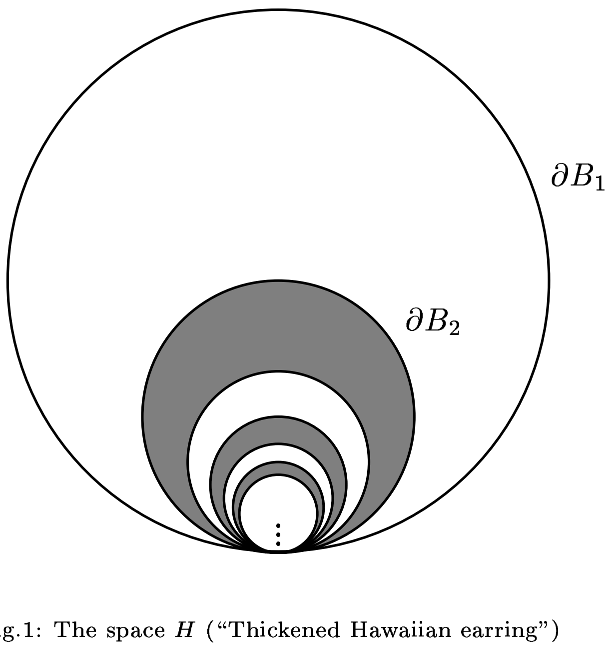


Fig.1: The space H ("Thickened Hawaiian earring")

Subdivide the set $\mathbb{N}$ of indices into a sequence of finite disjoint sets

$$\mathbb{N} = V_1 \cup V_2 \cup \dots$$

with the additional demand that none of the $\{v_\nu | \nu \in V_i\}$ takes points inside the gaps between the rings $B_{2j-1} \setminus B_{2j}$ for $j > i$. We will use the sets V_i to inductively define $H'_i, H'_{i,j}$ and U_i.

In the *zero-step* we let $H'_0 := H$, $H'_{0,j} = (B_2 \setminus B_3) \cup (B_4 \setminus B_5) \cup \ldots \cup (B_{2j-2} \setminus B_{2j-1}) \cup B_{2j}$ and $U_0 = \mathbb{R}^n$. The induction will be exclusively over the index i, the index j will in any step mean that $B_{2j} \cup \bigcup_{\nu=2}^{j-1}(B_{2\nu-2} \setminus B_{2\nu-1})$ is used as a base-space.
As *inductive hypothesis* we assume that $\forall_{l \leq k-1}\, U_l, H'_l$ and $H'_{l,j}$ have been chosen.
In the *inductive step* we first choose U_k as a regular open neighbourhood of $H'_{k-1,k}$ with the additional conditions that $\overline{U_k} \subset U_{k-1}$ and that U_k is contained in the metric neighbourhood $U(H'_{k-1,k}, (\frac{1}{2})^k)$. The understanding of this formula is that the space $H'_{k-1,k}$ itself also is part of this metric neighbourhood. For all index tuples $(i_1, \ldots, i_k) \in (V_1 \cup \ldots \cup V_k)^k$ and for all such shorter index tuples $(i_1, \ldots i_m)$ which satisfy that at least one of those indices i_l is contained in V_k we then choose similarities $S_{i_1} : B_1 \to B_1$, $S_{i_1,\ldots,i_m} : B_1 \to S_{i_1,\ldots,i_{m-1}}(B_1)$ $(1 \leq m \leq k)$ which satisfy that

1. $S_{i_1,\ldots i_m}$ maps $v := (0, 0, \ldots, 0)$ to $S_{i_1,\ldots i_{m-1}}(v_{i_m})$.
2. For all $m \leq k$ and for all permitted indices i_l all $S_{i_1,\ldots,i_m}(H)$ are disjoint apart from that $(S_{i_1,\ldots,i_m}(H) \cap S_{i_1,\ldots i_{m-1}}(H)) = \{S_{i_1,\ldots i_{m-1}}(v_{i_m})\}$.
3. For each m and all permitted indices i_l all $S_{i_1,\ldots i_m}(B_1)$ are disjoint and are contained in U_m.

Observe that such similarity maps can be inductively chosen, based on the idea of taking their images in the neighbourhoods of the attaching points $S_{i_1,\ldots i_{m-1}}(v_{i_m})$ of the previously chosen steps. The disjointness demands can be satisfied by making the ratio of dilatation of each map $S_{i_1,\ldots i_m}$ accordingly small. Based on such choices of similarities we finally let:

$$H'_k := H'_{k-1} \cup \bigcup_{i_1 \in V_k} (S_{i_1}(H)) \cup \ldots \cup \bigcup_{\substack{i_1,\ldots,i_m \in V_1 \cup \ldots V_k \\ \text{but at least one } i_l \in V_k}} S_{i_1,\ldots,i_m}(H) \cup \ldots \tag{4}$$
$$\ldots \cup \bigcup_{\substack{i_1,\ldots,i_{k-1} \in V_1 \cup \ldots V_k \\ \text{but at least one } i_l \in V_k}} S_{i_1,\ldots,i_{k-1}}(H) \cup \bigcup_{i_1,\ldots,i_k \in V_1 \cup \ldots V_k} S_{i_1,\ldots,i_k}(H).$$

Analogously $H'_{k,j} := H'_{k-1,j} \cup \ldots$, where in the remainder of (4) the basic space H is each time to be replaced by $H'_{0,j}$.
These primed spaces H'_i have been constructed as finite unions. Since the associated spaces $H'_{i,j}$ are simplicial complexes, regular neighbourhoods must exist.

Step B3: However, in the remainder of the paper we will also need the following spaces, which are based on infinite unions using the same set of similarities:

(1) $$H_0 := H \ , \ H_1 := H_0 \cup \bigcup_{i \in \mathbb{N}} S_i(H) \ , \ H_2 := H_1 \cup \bigcup_{i,j \in \mathbb{N}} S_{i,j}(H) \, , \ldots$$

(2) $$\tilde{H}_0 := B_1 \ , \ \tilde{H}_1 := H_0 \cup \bigcup_{i \in \mathbb{N}} S_i(B_1) \ , \ \tilde{H}_2 := H_1 \cup \bigcup_{i,j \in \mathbb{N}} S_{i,j}(B_1) \, , \ldots$$

(3) Put $H_\infty := H_0 \cup H_1 \cup H_2 \cup \ldots$ and $\tilde{H}_\infty := \tilde{H}_0 \cap \tilde{H}_1 \cap \tilde{H}_2 \cap \ldots$.

In this way we have

$$H = H_0 \subset H_1 \subset H_2 \subset \ldots \subset H_\infty \subset \tilde{H}_\infty \subset \ldots \subset \tilde{H}_2 \subset \tilde{H}_1 \subset \tilde{H}_0 = B_1$$

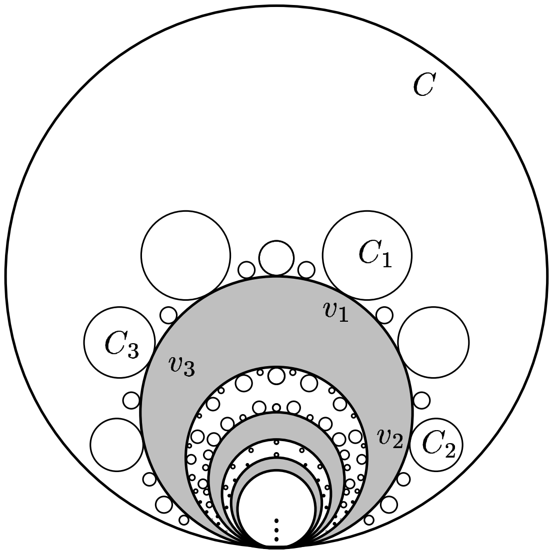


Fig 2: The construction of $\tilde{H}_\infty$ in the two-dimensional case ($C_\nu := S_\nu(B_1)$)

We will call the points of $\tilde{H}_\infty \setminus H_\infty$ "limit points"; this notion will be justified by Proposition 3.2. Note that H_∞ is by construction just an infinite gluing of copies of thickened Hawaiian earring spaces H. We will associate to each of these copies a "generation index" and a "degree". The generation index is the first index k of a space H'_k which contains this copy, while the degree of a copy $S_{i_1,\ldots i_k}(H)$ is the index k. H_0 has degree and generation index zero, and is the only space with one of these indices being zero. Every point in H_∞ has such a finite generation index and a finite degree. Both are unique apart for the points, where two different Hawaiian earrings are glued together. These points have precisely two of both types of indices. However, while for each number our space H_∞ contains only finitely many copies of H which have this generation index, apart from index zero it always contains infinitely many copies with the same finite degree. Vice versa, an earring of degree k must be glued to an Hawaiian earring of degree $k-1$, while an earring of generation index k might be glued to a Hawaiian earring of any generation index $\leq k$.

$\tilde{H}_\infty$ is our Example B, see Section 4 for proofs that it has the above claimed properties.

Example A

STEP A1: Take the point $a_1 = (1,0)$ and the following sequences of points in the plane $\mathbb{R}^2$:

$$a_0 = (0,0), a_2 = (1/2,0), a_3 = (1/3,0), \ldots a_k = (1/k,0), \ldots,$$

$$b_0 = (0,1), b_2 = (1/2,1/2), b_3 = (1/3,2/3), \ldots b_k = (1/k,1-1/k), \ldots.$$

The *triangular comb space* is the following union of line segments:

$$T' := \overline{a_0\ a_1} \cup \overline{a_0\ b_0} \cup \bigcup_{k=2}^{\infty} \overline{a_k\ b_k}$$

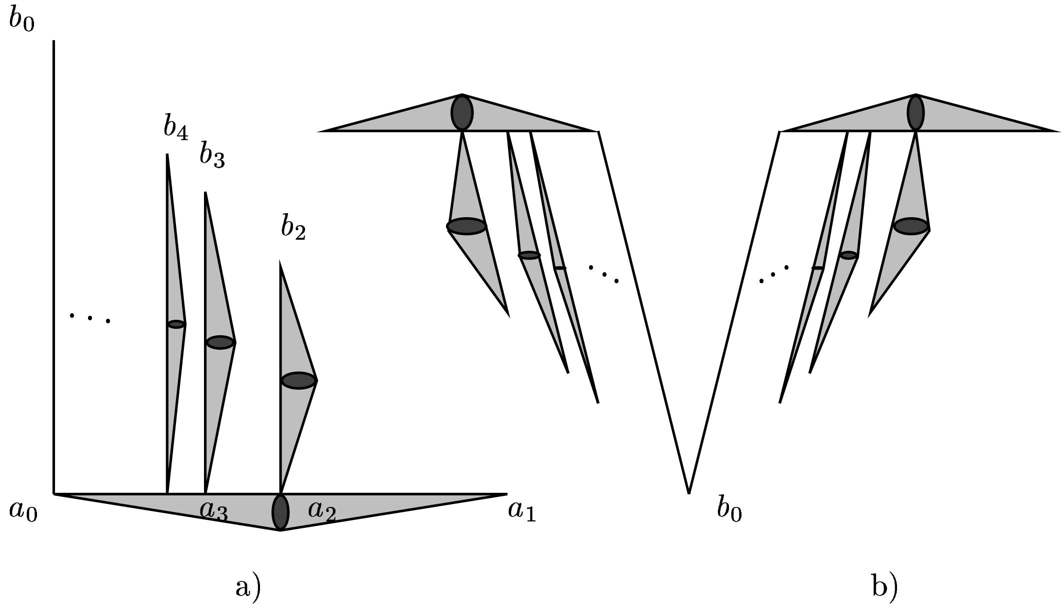


Fig 3: The spaces T (Fig.3a) and D=T∨T (Fig.3b) in case of three dimensions

To construct the thickened triangular comb space $T \subset \mathbb{R}^n$ proceed as follows. Stick a sufficiently small polygonal $(n-1)$-disk orthogonally to each line segment except of the segment $\overline{a_0\ b_0}$ ("accumulation line") , e.g. at its midpoint, and then construct the suspensions of that disks, with the endpoints of the segments as suspension points (Fig. 3). Those disks should be so small chosen that the suspensions along different segments $\overline{a_i\ b_i}$ ("teeth") are disjoint and that a_i is the only intersection point of the suspensions along $\overline{a_i\ b_i}$ and along $\overline{a_0\ a_1}$ ("stem"). Apart from this, each thickened tooth is contained in the convex hull of each predecessor and of the accumulation line.

The *thickened double comb space* $D \subset \mathbb{R}^n$ is a wedge of two copies of T.

In principle the forthcoming steps are analogous as for our above described Example B, with the comb-space D taking over the role of the Hawaiian earrings.

Step A2: Embed D into the interior of a circular n-cone C with $v := b_0$ going to the vertex of the cone. Analogously as for Example B we choose a countable dense set $V = \{v_i | i \in \mathbb{N}\} \subset \partial D$, $V \cap \{b_0, a_i \mid i \geq 2\} = \emptyset$, then choose pairwise disjoint circular n-cones C_i $(i \geq 1)$ with $\partial D \cap C_i = \{v_i\}$ and let $S_i : C \to C$ with $\mathrm{Im}(S_i) = C_i$ be similarity maps. $S_i(v) = v_i$.

These S_i and similaries $S_{i_1,\ldots,i_k}$ are in an analogous inductive process chosen as for Step B2, where instead of B_1 we are now using the cone C, instead of B_2 we are now using the wedge of two topological balls each of which is defined as convex hull of each of our two thickend comb-spaces T of D, and instead of $B_{2\cdot j} \cup \bigcup_{\nu=2}^{j-1}(B_{2\cdot j-2} \setminus B_{2\cdot j-1})$ we are using a space which is obtained by applying such a convex-hull operation only to the $(j+1)^{\mathrm{st}}$, $(j+2)^{\mathrm{nd}}$, $(j+3)^{\mathrm{rd}}, \ldots$ teeth of each of the two subspaces T of D, i.e. to those teeth which originally have been defined as suspensions over the lines $\overline{a_\nu, b_\nu}$ with $\nu > j$ in the definition of T. As in Step B2 we obtain that such spaces are finite simplicial complexes, but in difference to Step B2 this time these complexes all are collapsible. Therefore the regular neighbourhoods U_i that we define in this case will have all the topological type of a tame cell. This property will have to be essentially used in the proof of

the cellularity claim of Prop.1.1(1).
With the above described substitutions, with subdividing our set of indexes for V into a countable number of finite sets V_j such that V_j does not have attaching points in gaps between those teeth of D that are filled if the convex-hull operation is applied to all teeth with a number $\geq j$, and by using v_i, V_i and $S_{i_1,\dots,i_m}$ now in the new meaning, we will define spaces D'_k and $D'_{k,j}$ according to the scheme as described in Step B2. Note that this in practice means that in B2(4) only the letter "H" has to be substituted by "D".

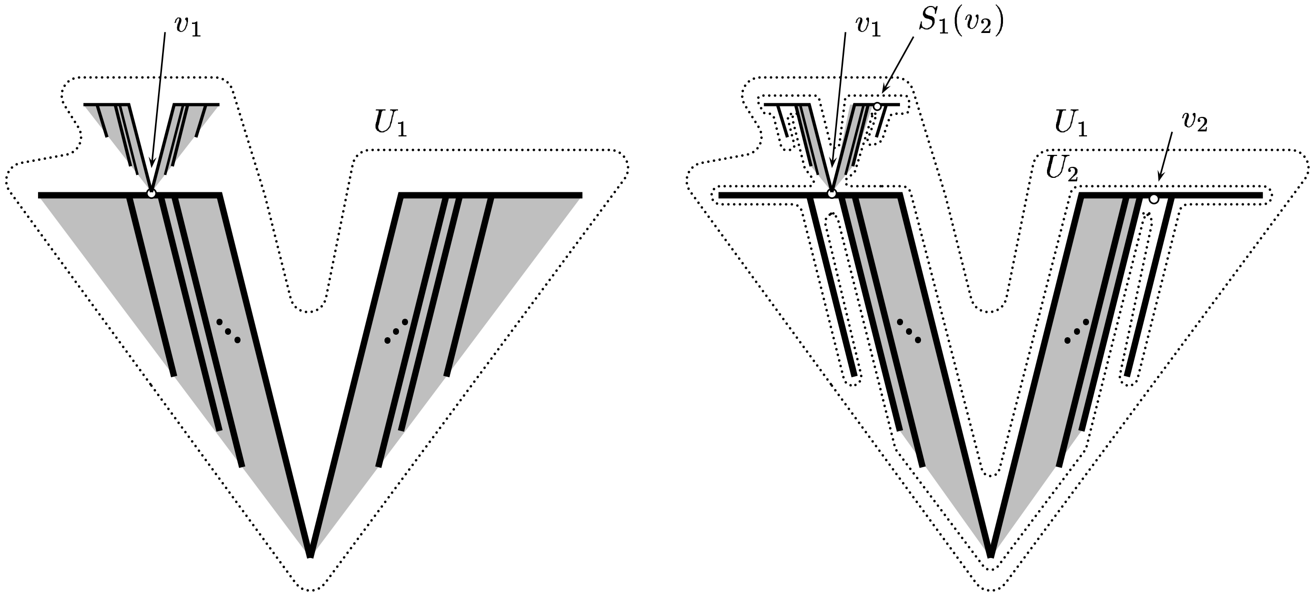
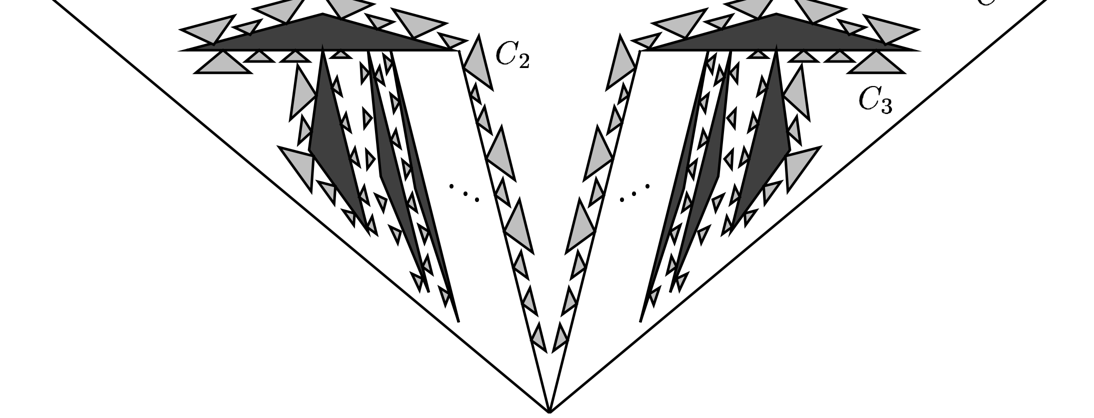


Fig.4: The two figures above show the first two steps of the construction of $\tilde{D}_\infty$. The top left subfigure is D'_1, provided we assume that $V_1 = \{v_1\}$. If we further assume that $V_2 = \{v_2\}$, the top right subfigure shows D'_1as a subspace of D'_2: In order to obtain the full D'_2 it would be necessary to attach further combs at the points $v_2, S_1(v_2), S_1(v_1), S_2(v_1)$ and $S_2(v_2)$, but these additional combs could not have been drawn for reasons of scale.For similar reasons we also refrained from drawing the space D in a thickend way. The subfigure below shows a more advanced state in the construction of $\tilde{D}_\infty$. Here all triangles in light grey replace analogous iterated glueings of comb spaces as is the entire triangle C.



Step A3: With symbols v, V, V_i and $S_{i_1,\dots,i_k}$ in its new meanings, by substituting B_1 by C and by replacing the letter "H" by "D" in formulae (1), (2) and (3) from

Step B3, we obtain the following chain of sets:

$$D = D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_\infty \subset \tilde{D}_\infty \subset \ldots \subset \tilde{D}_2 \subset \tilde{D}_1 \subset \tilde{D}_0 = C$$

Analogously as for Example B we regard the points of $\tilde{D}_\infty \setminus D_\infty$ as "limit points" and introduce the generation index and the degree on D_∞ by calling k the degree of the subspace $S_{i_1,\ldots,i_k}(D)$, and by calling such a subspace of "generation index m", if m is the lowest index of a space D'_m that already contains this copy of D.

Below in Section 3 we will prove that $\tilde{D}_\infty$, our Example A, has the properties that we claimed at the end of Section 1.

3. Proofs of Properties of Example A

Proposition 3.1. *The ratio of dilatation of the unique similarity map that maps D to some comb-space of the degree k is at most $(\frac{1}{2})^k$.*

Proof: This is an immediate consequence of the fact that according to its definition in Step A1 D has a height bigger than one, and that by B2(3) our comb of degree k must fit into a neighbourhood of some diameter that is smaller than $(\frac{1}{2})^k$.

Proposition 3.2. $\overline{D_\infty} = \tilde{D}_\infty$.

Proof: By the fixed chosen embedding of Step A2 a maximum is defined with which any point of C can lie apart from the nearest point of D_0. With the geometric regression from Prop.3.1, it follows that any point from $\tilde{D}_k$ can lie at most

$$(\frac{1}{2})^k \cdot \text{that distance}$$

apart from the nearest point of D_k. Since $\frac{1}{2} < 1$, in the limit $\tilde{D}_\infty$ can be at most the closure of D_∞. Since, on the other hand, $\tilde{D}_\infty$ is closed as an infinite intersection of closed sets, it is the closure.

We shall need some preliminary definitions.

Definition 3.3. (cf. [KRRZ; Def.2.1])

(i) A space X is said to be *homotopically n-dimensional* (or to have *homotopy dimension n*), if it is homotopy equivalent to some n-dimensional space and is not homotopy equivalent to any $(n-1)$-dimensional space. Here by dimension of space we mean covering dimension.

(ii) We say that a space X is *everywhere homotopically n-dimensional* if for every open subset U of X and every id-homotopy $h : X \times I \to X$ which is *stable* on $X \setminus U$, i.e. $h(x,t) = x$ for every $x \in X \setminus U$ and for every t, the intersection $U \cap h(X,1)$ is n-dimensional. By an *id-homotopy* we mean any mapping $h : X \times I \to X$ such that $h(x,0) = x$ for all $x \in X$.

Definition 3.4. A point x_0 of a space X is called a *homotopically fixed point (hf-point,* cf. [KRRZ; Def.2.2]), if for any homotopy $h: X \times I \to X$ starting from the identity mapping and for every $t \in I$ the point x_0 is a fixed point of the mapping $h(\cdot,\ t)$, i.e. $h(x_0,\ t) = x_0$ for all $t \in I$.

Proof of Property 1.1(3): It needs to be proven that every neighbourhood of every point of $\tilde{D}_\infty$ contains an open part of a Euclidean n-dimensional ball. This property holds for all points of the double-comb D (including the accumulation lines that have not been thickened, since these segments are accumulation points of the thickened teeth, cf. §2, Step A2). Therefore it holds for all points of D_∞, since this is just an iterated gluing of copies of D. Finally, the remaining points of $\tilde{D}_\infty$ are by Prop.3.2 proven to be limit points, so that they also satisfy this property.

Proof of Property 1.1(5): Again, we prove this property first for the boundary points of D_∞, because then the desired property necessarily extends to the points of $\tilde{D}_\infty$, since for such an arrangement of spaces in $\mathbb{R}^n$ (not really intersecting but only touching in a zero-dimensional set of points) a limit-point of such sets is automatically also a limit point of its boundary points. Note that all arguments that in Lemma 2.3 of [KRRZ] were given to prove that the wedge point of the ordinary not-thickened double comb is a homotopical fixed point, are valid for this thickened version as well. Then analogously with the arguments from the list of Remarks 2.4 of [KRRZ] it follows also in our case that all boundary points for which we attach such double points (and thus their limit points also) remain homotopically fixed points also in the space that is composed by these attaching processes. That way we see that all boundary points of D_∞ are homotopically fixed, which, as stressed in the first sentence of this proof, suffices to obtain the desired result.

Proof of Property 1.1(6): This is an immediate consequence of 1.1(5): There is no chance to deformation retract a cell in $\mathbb{R}^n$ to a proper subset, if the boundary has to be fixed. The same arguments as given in Remarks 2.4(i) and (ii) of [KRRZ] work, of course, using here n-dimensional cohomology instead of two-dimensional. Now please observe, that all connected components of inner points of our comb-spaces are just cells.

Proof of Property 1.1(4): This is an immediate consequence of 1.1(6), since any homotopy that qualifies a space for being not everywhere homotopically n-dimensional is by Definition 3.3(ii) a deformation retraction with the additional demand of keeping the exterior of some neighbourhood fixed.

The last two of our Properties 1.1, namely 1.1(1) and (2), will be proven in the following two propositions:

3.5 Proposition. *Example A is cellular.*

The proof of this fact requires the construction of approximating n-dimensional cells in the spirit of Figure 4b of [KRRZ]. However, the construction here is somehow more complicated as it was in [KRRZ; Proof 3.2], since we do here not only have two degrees of spaces which need to be respected by the positions of our approximating disks. However the construction of our approximating cells has been prepared by choosing ball-neighbourhoods U_i in Step A2, and all that needs to be shown is that $\bigcap_{i=1}^\infty U_i = \tilde{D}_\infty$. By construction D_∞ consists of copies $S_{i_1,\dots,i_k}(D)$, and in B2(3) each such copy has been placed in some neighbourhood U_m, where $m > k$ is the generation index of the copy $S_{i_1,\dots,i_k}(D)$. Therefore it is contained in all neighbourhoods U_i with $i < m$, and by construction as part of the space $D'_m \subset D'_i$ it lies also in all U_i with $i > m$. Hence $D_\infty \subset \bigcap_{i=1}^\infty U_i$. Since we have $\overline{U_m} \subset U_{m-1}$, we also have that $\bigcap_{i=1}^\infty U_i = \bigcap_{i=1}^\infty \overline{U_i}$ and hence is closed, so that we have $\tilde{D}_\infty = \overline{D_\infty} \subset \bigcap_{i=1}^\infty U_i$. By closedness of $\tilde{D}_\infty$, any point that is not belonging to $\tilde{D}_\infty$ would lie at positive distance ε apart. Hence it cannot be contained in a metric neighbourhood of radius $(\frac{1}{2})^m$ of $\tilde{D}_\infty$ with sufficiently high index m, and therefore also not in the neighbourhood U_m which is by construction contained in the metric neighbourhood of the same diameter of the smaller space D'_m.

Proposition 3.6. *Example A is simply connected.*

Proof: Let $u : [0,1] \to \tilde{D}_\infty$ be an arbitrary closed path in our space. Since it is just a homotopic process to shrink all constant domains of such a path to points, we can and will in the forthcoming proof assume that u is free of constant segments.

In order to show that $\pi_1(\tilde{D}_\infty) = 1$, we need to construct a nullhomotopy for such a path u. According to a proposal of Kenyon ([Keny]), for that purpose we look at the closed unit disk $\mathbb{B}^2$, and we will use in the forthcoming constructions the metric that can be put on such a disk to give the Poincaré Disk model for the hyperbolic (or "Lobachevskian") space $\mathbb{H}^2$ (cf. [Moi]; §9.2). Accordingly we will talk of "hyperbolic lines" (which are Euclidean semicircles that are perpendicular to the boundary) and similar things. We use $\partial\mathbb{B}^2$ as a parameter domain for our path u, and hence by using the parametrization for u we get a map $f : \partial\mathbb{B}^2 \to \tilde{D}_\infty$. If we manage to extend this map f to a continuous map $F : \mathbb{B}^2 \to \tilde{D}_\infty$, then we succeeded in proving that u was nullhomotopic.

Let us, without loss of generality, assume that u passes through D_0, i.e. through the only comb-space of degree zero (if it does not, then the comb-space of lowest degree through which u passes takes this role in the forthcoming proof). We treat it as evident that also the thickened double-comb space D has trivial fundamental group (a rigorous proof could be given by precisely the same method as this proof when treating the various n-dimensional cells and line-segments of which D comprises as we treat comb-spaces in this proof). Our path might spend various different segments inside D_0. According to the tree-like gluing structure of D_∞, if u leaves D_0 and goes into the combs of higher degrees, it has to come back to D_0 through the same boundary point. Accordingly we can on $\partial\mathbb{B}^2$ mark those (closed) segments where our path is inside D_0, and those (open) segments where it is outside D_0, and we will find that f takes on the two endpoints of every interval where u is outside D_0 the same boundary point of D_0. We connect the start- with the endpoint of each such interval (in the worst case there can be countable many of such intervals which are separated by a Cantor set) by a hyperbolic line segment through $\mathbb{B}^2$, and, as a first step of extending f to F, we hereby agree that the restriction of F to such a line segment shall be the same value that is by f taken on both endpoints of such a hyperbolic line. This construction gives a system of non-nested hyperbolic lines which seal off various semi-disks near the boundary of $\partial\mathbb{B}^2$ from a "central region". F has already been completely defined on the boundary of the central region, and the restriction of F to the boundary of the central region can be interpreted as a path which entirely remains in D_0. Since $\pi_1(D_0) = 1$, this path is nullhomotopic in D_0, and we can use this nullhomotopy to assign to F now also a continuous definition inside the entire central region. Hence we reduced the problem of defining F now to the remaining semidisks near $\partial\mathbb{B}^2$.

The above construction might be treated as the zero-step of an inductive process, the inductive step will consist of filling in some parts of the missing definition of F in one of the outer semidisks. Let $S(v)$ be the value that f takes on the endpoints of the outside boundary u of this semidisk, where $S = S_{i_1,\ldots,i_k}$. By assumption, $f(u)$ is, apart from the endpoints of u, contained in $S(C)$. The subset of $\partial\mathbb{B}$, where $f(u)$ is not contained in $S(D)$, is open, and can in the worst case consist of countably many disjoint open subintervals. Here f will map to combs of degree $k+1$ or higher. At the beginning and at the endpoint of each of those subintervals f will take on the same value. As in the zero-step of this construction, connect each pair of endpoints of those subintervals by a hyperbolic line, and use the f-value from the endpoints of these lines as constant value for F on each line. These hyperbolic lines, together with the inner boundary of our semidisk and the appropriate segments of $\partial\mathbb{B}$ in between form a closed Jordan curve c, on which F already is defined and only takes on values in $S(D)$. Since $\pi_1(D) = 1$, a nullhomotopy for $F(c)$ exists inside $S(D)$ an can be used to define F in the interior region of c. In addition, the geometric shape of D_0 allows to choose this nullhomotopy in such a way, that its diameter does not exceed the diameter of the loop to be contracted. Hence we have extended the definition of F to some parts of our outer semidisk, and by iterating this construction we will continue to extend it to parts of the still missing smaller semidisks.

In Section 5 we will complete this proof by showing

(1) that this definition fills the entire interior of $\mathbb{B}$,
(2) that the gluing structure of the various subdisks on which F is independently defined is finite in the interior and hence the resulting map is continous, and
(3) that this definition on $\mathrm{Int}(\mathbb{B})$ continously extends to the map that by the original path u is defined on $\partial\mathbb{B}$.

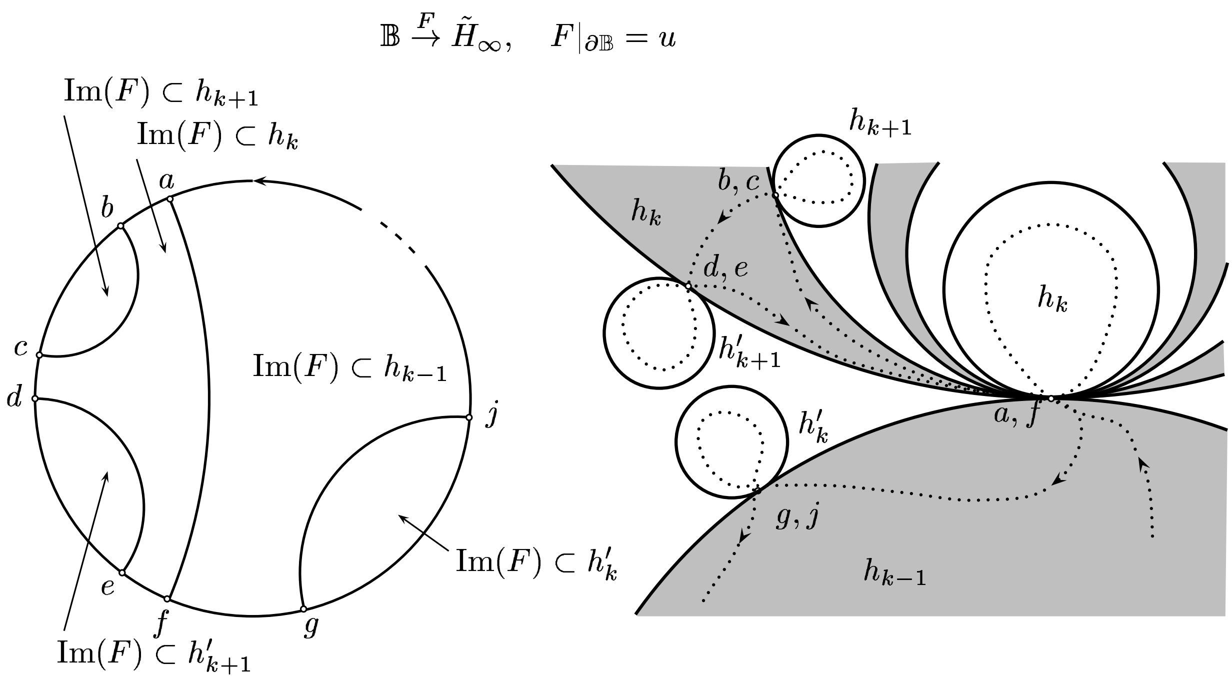


Fig.5: The figure corresponds to the proof of the analogous property 1.2(2) of Example B. The subfigure on the right shows a part of $\tilde{H}_\infty$, drawn essentially as in Fig.6. In dotting we also put a segment of a sample path u into this figure and indicated the parameter values of when this path passes through the attaching points. These values then are used in the left subfigure where the corresponding steps of constructing a Kenyon-diagram are indicated.

4. Proofs of Properties of Example B

Due to the analogous way of how the spaces $\tilde{B}_\infty$ and $\tilde{H}_\infty$ are built, the proofs of Properties 1.2(2)–(6) are analogous to those to Props.1.1(2)–(6) that we exposed in Sect.3. Essentially just each thickened double-comb has to be replaced by a thickened Hawaiian earring. Hence we restrict ourselves in this section to proving Property 1.2(1). The following propositions prepare this proof, which essentially has to show the local pathwise connectivity of $\tilde{H}_\infty$.

Proposition 4.1. *Each limit point of $\tilde{H}_\infty$ is the limit point of one and only one sequence $(h_0, h_1, h_2, \dots)$, which is constructed by one sequence of naturals $i_0, i_1, i_2, \dots$ according to*

$$h_0 = H_0,\ h_m = S_{i_1, i_2, \dots i_m}(H_0) \text{ for } m > 0.$$

Proof: The fact that each limit point only in one way can be approximated like this, follows directly from the construction of $\tilde{H}_\infty$. Since our point as a limit point lies in $\tilde{H}_\infty$ but not in H_∞, it also lies in $\tilde{H}_k$, but not in H_k for all $k \in \mathbb{N}$. However,

$$\tilde{H}_k \setminus H_k \subset \bigcup_{(i_1, \dots, i_k)} (S_{i_1, \dots, i_k})(B_1 \setminus H_0).$$

In particular $\tilde{H}_k$ and H_k do not differ in any part that has a degree k or smaller. Since this argument holds for any k, and since all the attached Hawaiian earrings of any fixed degree are disjoint, we get a unique sequence as claimed in this proposition.

Proposition 4.2. *$\tilde{H}_\infty$ is pathwise connected.*

Proof: By the iterative gluing construction, it is evident that H_∞ is pathwise connected. Hence the remaining question is, whether the limit points of $\tilde{H}_\infty$ can somehow by paths be connected to the main body H_∞. Here is a method for how to construct such a path: Choose a limit point P and associate according to Proposition 4.1 a sequence $(h_0, h_1, h_2, \dots)$ to this point. Recall that these sequences could be uniquely constructed so that each h_k denotes Hawaiian earrings of degree k. Choose a path $u : [0, 1] \to \tilde{H}_0$ which between the parameter values 0 and $\frac{1}{2}$ runs from its start point in h_0 to some point in h_1, between parameter values $\frac{1}{2}$ and $\frac{3}{4}$ from this point in h_1 via the unique attaching point to some point in h_2, and so on. Finally define artificially that $u(1) = P$. Since by Prop.3.1 the geometric regression $(\frac{1}{2})^k \cdot \pi \cdot (1 + \frac{1}{2}) \cdot \mathrm{diam}(H_0)$ is an upper bound for the length of the above constructed path between it parameter values $\frac{1}{k}$ and $\frac{1}{k+1}$, the above construction gives a continuous path which hence connects P to the main body H_∞.

Proposition 4.3. *$\tilde{H}_\infty$ is locally pathwise connected.*

Sketch of Proof: Thanks to all the self-symmetries of the fractal-like iteration in building H_∞, the essential step has been done with having proven Proposition 4.2 already. We have to construct an arbitrary small pathwise connected neighbourhood for each point in $\tilde{H}_\infty$. Therefore we distinguish cases according to the various types of points of $\tilde{H}_\infty$

(1) *Interior points of one of the domains $B_{2i} \setminus B_{2i+1}$ (in H_0 or in its similarity images):* As interior points of a open domain these points clearly have appropriate neighbourhoods.
(2) *Attaching points:* The set of all attaching points is $\tilde{V} = \{S_{i_1, \dots, i_k}(v_j) \mid k \in \mathbb{N}_0,\ j, i_1, \dots, i_k \in \mathbb{N}\}$. Observe that each $w \in \tilde{V}$ has precisely two representations of that type according to $w = S_{i_1, \dots, i_k}(v_j) = S_{i_1, \dots, i_k, j}(v)$. By our construction in Step B2 of §2, for such a w the domain $S_{i_1, \dots, i_k, j}(B_1)$ gives a ball in $\mathbb{R}^n$ such that $\partial(S_{i_1, \dots, i_k, j}(B_1)) \cap \tilde{H}_\infty = \{w\}$. The part of $\tilde{H}_\infty$

that sits inside this ball we call the "associated subspace of w". This space is analogously built as $\tilde{H}_\infty$.

A path-connected neighbourhood of w with a diameter smaller than ε can be constructed according to

$$\Big(U(w,\delta) \cap \Big(S_{i_1,\dots,i_k}(H) \cup S_{i_1,\dots,i_k,j}((B_2 \setminus B_3) \cup \dots \cup (B_{2l-2} \setminus B_{2l-1}))\Big)\Big) \cup \\ \cup\, S_{i_1,\dots,i_k,j}(H \cap B_{2l}) \ \cup \text{ all subspaces that are associated} \\ \text{to points of the boundary of the above described space.} \qquad (*)$$

By Prop.4.2 the path-connectedness of such a neighbourhood is clear. It will be contained in $U(w,\varepsilon)$, provided that we choose the two variables l and δ as follows:

(i) First choose l sufficiently big, such that $S_{i_1,\dots,i_k,j}(B_{2l})$ fits, including all its subspaces that are associated to points of these area, into $U(w,\varepsilon)$. Since according to our construction in B2(3) every space that is associated to a ring $B_{2\nu} \setminus B_{2\nu+1}$ with sufficiently high ν has to fit into a neighbourhood of diameter $(\frac{1}{2})^\nu$, this condition will be satisfied for sufficiently high l. In addition, by 3.1, the ratio of dilatation of all $S_{i_1,\dots,i_k,j}$ is < 1.

(ii) Let μ be the index so that v_j sits at $\partial(B_{2\mu} \setminus B_{2\mu+1})$.

(iii) Now choose $\delta' < \varepsilon/2$ and smaller than the width of $S_{i_1,\dots,i_k}(B_{2\mu} \setminus B_{2\mu+1})$ at w.

(iv) Analyse how many of the balls that are associated to points of $U(w,\delta') \cap (S_{i_1,\dots,i_k}(H) \cup S_{i_1,\dots,i_k,j}((B_2 \setminus B_3) \cup \dots \cup (B_{2l-2} \setminus B_{2l-1})))$ are still sticking out of $U(w,\varepsilon)$. Since each of them must have a diameter of at least $\varepsilon/2$, there can be only finitely many of them. Now choose $\delta \le \delta'$ such that each of the attaching points of these sticking out balls lies more than δ apart from w.

(3) *Boundary points of a ring h_k which are no attaching points:* The case is analogous, but just simpler as the case before: The construction in Formula $(*)$ just corresponds to

$$(U(w,\delta) \cap S_{i_1,\dots,i_k}(H)) \cup \text{all subspaces that are associated} \\ \text{to points of the boundary of this space}$$

and corresponding to the missing terms we have accordingly fewer steps in constructing this set.

(4) *Limit points:* Let P be a limit point and $(h_0, h_1, h_2, \dots)$ be the associated sequence in the sense of Proposition 4.1. The basic idea is to construct a neighbourhood of P, if $h_k = S_{i_1,\dots,i_k}(H_0)$, as $S_{i_1,\dots,i_k}(B_1)$. By choosing the index k big, such neighbourhoods which are path-connected by Proposition 4.1 can be made arbitrarily small. The only trouble comes from the fact that such a neighbourhood fails to be open at one point, namely at the attaching point of h_k to h_{k-1}. However, we can make it open by attaching to our h_k-neighbourhood an appropriate semidisk-neighbourhood in h_{k-1} of this attaching point w. Such a neighbourhood can be analogously as in the preceding steps (2) and (3) constructed by taking $U(w,\delta) \cap S_{i_1,\dots,i_{k-1}}(H)$ for a suitable δ and by uniting to it all spaces that are associated to points on the boundary of it.

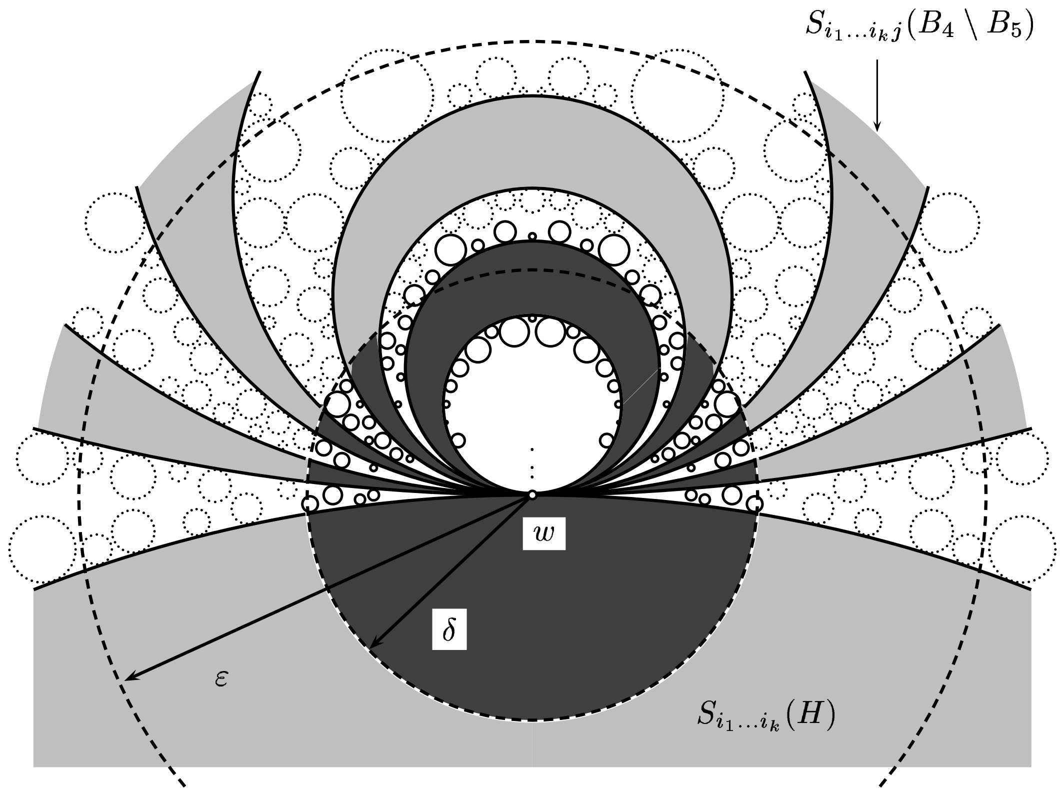


Fig.6: This figure tries to give a geometric impression for how to construct a path-connected neighbourhood W inside an ε-neighbourhood: The areas in grey are solidly filled parts of $\tilde{H}_\infty$, but only those in dark grey belong to W. Smaller copies of $\tilde{H}_\infty$ are in this figure indicated by circles, and these circles are drawn solidly/dotted, according to whetherthese copies do belong entirely or do not belong at all to W, respectively.

Since we succeeded with the construction of such neighbourhoods for all types of points in $\tilde{H}_0$, the proof of Proposition 4.3 and hence of Property 1.2(1) is complete. As stressed before, the other items from our list of Props.1.2 can be analogously proven as the corresponding claims of Props.1.1 that were proven in Section 3.

5. Completion of the discussion of Example A

Observe that simple connectivity is often understood in such a way that it implies pathwise connectivity. However the pathwise connectivity of $\tilde{D}_\infty$ can be shown according to the scheme of Prop.4.2, since $\tilde{D}_\infty$ and $\tilde{H}_\infty$ have the same glueing structure. Hence we are left with the task to complete the proof of Prop.3.6 that its fundamental group is trivial, i.e. to show the items 3.6(1)–3.6(3). This will be done later in the paragraph. Before that we need one observation:

Proposition 5.1. *The set of limit points for our Examples A and B is pathwise totally disconnected.*

Proof: The proof of both cases is analogous, and is essentially only based on the analogous gluing structures on these spaces. We present it in the case of our Example A — having to replace all building blocks that are similar to D by building blocks that are similar to H in the other case: As in Proposition 4.1 for each limit point construct a unique sequence $(D_0{=}d_0\ ,\ d_1\ ,\ d_2\ ,\ \ldots)$, such that each d_k is a comb-space of degree k and is glued to its predecessor. Since these sequences are unique, the corresponding sequences for two different limit points have to differ at least from some finite index on. If this happens, then the place from which on these sequences differ determines different similarity maps which map C to disjoint regions in $\mathbb{R}^n$. And these disjoint regions contain our limit points and can be used to construct disjoint open neigbourhoods for our points as well, which do not have any limit points on their boundaries (cf. analogous constructions in Proposition 4.3). Hence our points have not been contained in the same connected component of the subset of limit points of $\tilde{D}_\infty$.

Proof of 3.6(1): Assume that our construction mechanism fails to associate to some point $P \in \mathrm{Int}(\mathbb{B})$ a definition. This implies that P does not belong to the central region that we considered in the proof of 3.6, but has to lie inside one of the outer semidisks. And it has to lie inside one of the outer subsemidisks, when in a forthcoming step of this proof we extended our definition of F into this semidisk. And so for each of the infinitely many forthcoming steps of the construction. Since P had a fixed distance to $\partial\mathbb{B}$, the hyperbolic geometry does also not allow the endpoints of the inner boundaries of these semidisks to converge to each other in $\partial\mathbb{B}$. Denote these endpoints by $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ and denote by $[\alpha_i, \beta_i]$ the appropriate segment on $\partial\mathbb{B}$, i.e. the appropriate segment on the outer boundary of the i^{th} semidisk. According to our construction we know that u left at the parameter value α_i a comb of degree $(i-1)$, and did not return to this comb before the parameter value β_i. The sequences $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ are monotone sequences in opposite direction on $\partial\mathbb{B} \approx S^1$ and therefore converging. Hence $[\lim_i \alpha_i, \lim_i \beta_i]$ is an interval on which u can only take on limit points, and hence by Prop.5.1 it can only be constant. However this has been ruled out at the beginning of our proof-construction in 3.6.

Proof of 3.6(2): If the segmentation by semidisks, according to our construction in 3.6 should not be finite in the interior, the geometry of non-intersecting hyperbolic straight lines would force that we have to obtain essentially the same picture as in the preceding proof to have some accumulation in the interior of $\mathbb{B}$. Hence we get the same type of contradiction to this assumption.

Proof of 3.6(3): By our construction in 3.6, only something needs to be proven, if for some point P in $\partial\mathbb{B}$ we have infinitely many semidisks in its neighbourhood. In the forthcoming proof we need to distingiush the cases depending on whether $u(P)$ is a limit point or an element of D_∞.

(1) Let $u(P)$ be a limit point lying in some ε-neighbourhood $U(u(P), \varepsilon)$. The neighbourhoods that we constructed in Prop.4.3(4) in the analogous case for Example B are, after removing the semidisk-part, small homeomorphic copies of our space $\tilde{D}_\infty$. Hence such a copy can be described as $S(\tilde{D}_\infty)$, where S is an appropriate plugging of our similarity-mappings $S_{i_1,\dots,i_k}$ and of their inverses for accordingly different choices of indices. By our gluing structure, the part $S(\tilde{D}_\infty)$ is connected to the rest of our space $\tilde{D}_\infty$ only through the point $S(v) = S_{i_1,\dots,i_k}(v)$. By Prop.4.1 such subspaces have for sufficient high index k a diameter that is proportional to $\sum_{\nu=k}^{\infty}(\frac{1}{2})^\nu = (\frac{1}{2})^{k-1}$. Hence they will be contained in $U(u(P), \varepsilon)$. Since our path u can have entered and left $S(\tilde{D}_\infty)$ only through the point $S(v)$, the corresponding parameter values on $\partial\mathbb{B}$ define a δ-neighbourhood in which we know that u

takes on only values inside $S(\tilde{D}_\infty)$. By construction in 3.6, these points on $\partial\mathbb{B}$ have been connected by a semidisk, and whatever has been defined as map F there, did also only take on points inside $S(\tilde{D}_\infty)$. Hence this semidisk is precisely that type of neighbourhood that needed to be constructed in an elementary proof of continuity for F at the point P.

(2) Now let $P \in \partial\mathbb{B}$ be a point where $u(P)$ is some point in D_∞. Hence $u(P)$ lies in a copy of D with finite degree, and therefore only k semidisks of $\mathbb{B}$ can be nested at the point P. Hence, if P is an accumulation point of semi-disks, this can only be due to side-by-side situated disks. Of course, we cannot rule out that inside some of these semidisks there are sitting finitely or infinitely many other semi-disks, but there always do also exist semidisks of degree k or $k+1$ which are accumulating from aside to our point P. For our path this means that $u(P)$ will be on the boundary of a comb d_{k-1} which might or might not be the attaching point to some point d_k. An accumulation of semidisks near P means that u in the neighbourhood of the parameter P either infinitely many times changes between d_k and d_{k-1}, or in this neighbourhood enters a lot of smaller comb-spaces that are in this region attached either to d_k or to d_{k-1}. However the latter phenomenon does not provide a potential for non-continuity, since analogously as in 4.3(2)–(3) an arbitrary small neighbourhood for $u(P)$ can be constructed in such a way that it contains appropriate parts of d_k and d_{k-1}, but apart from this all other copies of D either entirely or not at all. In particular, for the boundary points of d_k and d_{k-1} that belong to our neighbourhood, it contains the entire spaces that are associated to those points. On the other hand, in 3.6 F was constructed in such a way that each path was only contracted inside the comb where it was passing through, and therefore those parts of F cannot leave our neighbourhood. Also the excursions of u into d_{k-1} do not provide a potential for non-continuity, since in 3.6 the contraction of all those excursions has been done in one step by finding the definition in the interior of one curve c that touches $\partial\mathbb{B}$ in several disjoint intervals. Since this means that in the neighbourhood of P this contributes just one (and not infinitely many independently constructed) definitions of segments of F, this is not a potential for non-continuity. The situation is slightly different for the excursion of u into d_k, since each such excursion in the neighbourhood of P causes one semidisk, and here the definition of F inside this semidisk is in each step chosen independently. Since at the end of our construction in 3.6 we decided to use only homotopies which had the same diameter as the paths to be contracted, also the infinite plugging of such homotopies must give a continuous function F, also in the neighbourhood of these points.

Since by the above arguments we have seen that the construction of 3.4 gave a continuous map that can be interpreted as a homotopy of our arbitrary path u, it has been proven to be nullhomotopic, and $\tilde{D}_\infty$ to have trivial fundamental group, only.

The same line of conclusions also works for our Example B, as long as the “rings” of the Hawaiian earrings are thickened higher-dimensional spheres S^n and the case “S^1” is excluded.

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