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MULTIPLE KRONECKER  
COVERING GRAPHS

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# Multiple Kronecker Covering Graphs

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## Abstract

A graph may be the Kronecker cover in more than one way. In this note we explore this phenomenon. Using this approach we show that the least common cover of two graphs need not be unique.

## 1 Introduction

A graph  $\tilde{G}$  is said to be a covering graph over a graph  $G$  if there exists a surjective homomorphism (called a covering)  $f: \tilde{G} \rightarrow G$  such that for every vertex  $v$  of  $\tilde{G}$  the set of edges incident with  $v$  is mapped bijectively onto the set of edges incident with  $f(v)$ . A covering  $f$  is  $k$ -fold if the preimage of every vertex of  $G$  consists of  $k$  vertices.

To simplify the description of large graphs, the concept of *voltage graphs* and *covering graphs* is generally used, see for example [2] or [8].

In 1982 F.T. Leighton proved in [5] that any two graphs with a common universal cover have a common finite cover. It is not hard to see that any two graphs with a common cover have a unique maximal common cover: the universal cover. In this note we show that the result does not extend in the opposite direction. There are graphs with a common cover whose minimal common cover is not unique.

## 2 Graphs are not determined by their Kronecker covers

The *Kronecker cover* of  $G$  is the  $\mathbb{Z}_2$ -covering graph over  $G$  with voltages 1 on all edges (note that in this case the direction of edges is irrelevant). We will denote the Kronecker cover of  $G$  by  $KC(G)$ . Alternatively,  $KC(G)$  can be defined as the tensor product of  $G$  and  $K_2$ . See [3] for more about graph products.

It is easy to see the following properties of  $KC(G)$ .

**Proposition 1.** *Kronecker covers of graphs are bipartite. If  $G$  is bipartite, then  $KC(G)$  consists of two copies of  $G$ . If  $G$  is connected and non-bipartite then  $KC(G)$  is connected.*

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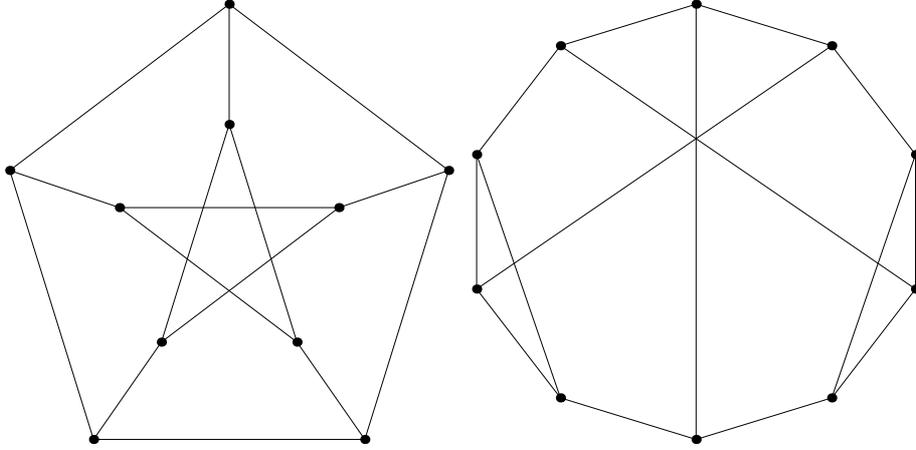


Figure 1: The Desargues graph  $G(10, 3)$  is a Kronecker cover of Petersen graph  $G(5, 2)$  and of the graph  $X$ .

*Proof.* By definition, the vertex set  $V(KC(G))$  is a union of two sets  $V(G)$  and the edges of  $KC(G)$  connect only vertices in different copies of  $V(G)$ .  $\square$

It follows from the theory of tensor products developed by Imrich et al. that  $KC(G) = KC(G')$  does not necessarily imply  $G = G'$ . Recently, Imrich et al. have determined all possibilities for a hypercube  $Q_n$  to be a Kronecker cover.

Here we open the problem for all simple graphs.

**Problem 1.** *Given a connected, simple graph  $K$ , determine all simple graphs  $G$  such that  $K = KC(G)$ .*

Clearly, not all graphs can be Kronecker covers. Here is a simple criterion for a graph  $K$  to be Kronecker cover. Let  $K$  be bipartite with bipartition  $(V_1, V_2)$  and let  $\pi \in \text{Aut } K$  be a fixed-point free involution such that  $\pi$  interchanges the bipartition:  $\pi(V_1) = V_2$ . Furthermore, we require that for any vertex  $v$  of  $K$  vertices  $v$  and  $\pi(v)$  are non-adjacent. Such an automorphism is called a (*combinatorial*) *polarity*.

**Proposition 2.** *Let  $K$  be a connected graph. Then  $K$  is a Kronecker cover of some graph  $G$  if and only if  $K$  is bipartite and there exists a polarity  $\pi \in \text{Aut } K$ .*

Let  $\Pi K \subset \text{Aut } K$  denote the set of all polarities of  $K$ . Clearly if  $\pi$  is a polarity and  $\alpha$  an arbitrary automorphism, then  $\pi^\alpha = \alpha\pi\alpha^{-1}$  is also a polarity, because every automorphism either fixes or interchanges the bipartition. Let  $\pi^{\text{Aut } K}$  denote the class  $\pi^{\text{Aut } K} = \{\pi^\alpha \mid \alpha \in \text{Aut } K\}$ . If we define an equivalence relation  $\cong$  in  $\Pi K$  so that  $\pi$  is equivalent to  $\pi'$  if and only if there exists an  $\alpha \in \text{Aut } K$  such that  $\pi' = \pi^\alpha$ , then the equivalence classes are exactly of the form  $\pi^{\text{Aut } K}$ .

**Proposition 3.** *Let  $K$  be a connected graph. Then  $K$  is a Kronecker cover of  $k$  simple graphs if and only if  $k = |\Pi K / \cong|$ .*

Let us consider the case presented in Figure 1. The Desargues graph  $G(10, 3)$  can be represented as a Kronecker covering graph in two distinct ways. Let us

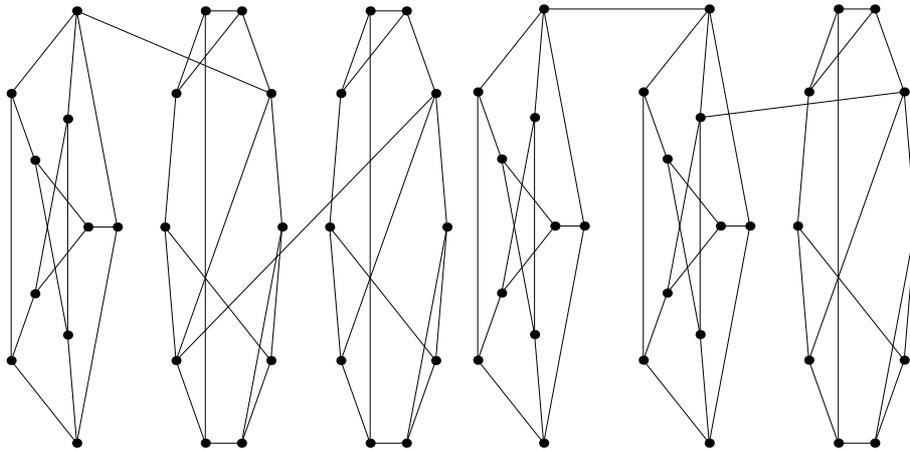


Figure 2: The graphs  $H_1$  and  $H_2$  have more than one common connected double cover.

label the vertices of Petersen graph  $G(5, 2)$  in such a way that the vertices in the outer pentagon are labeled 1,2,3,4,5 and the vertices in the inner pentagram 6,7,8,9,10 with 1 being adjacent to 6, 2 to 7, etc. The labeling of the vertices of the bipartite  $G(10, 3)$  is chosen in such a way that the vertex  $i$  of Petersen lifts to a black vertex  $i$  and a white vertex  $i'$ . Each pair  $i$  and  $i'$  of vertices is antipodal in  $G(10, 3)$ . In order to specify the second quotient, the graph  $X$  we have to define a new polarity  $\pi$  of  $G(10, 3)$  that tells which black vertex  $\pi(i)$  projects onto the vertex  $i$  of  $X$ . We do this with the aid of an involution  $\alpha$  of  $G(5, 2)$  defined by  $\alpha = (1, 8)(2, 10)(3, 5), (4)(6)(7)(9)$  with four fixed points by setting  $\pi(i) = \alpha(i)'$  and  $\pi(i') = \alpha(i)$ . If we now identify  $\pi(i')$  with  $i$  we obtain a covering projection of  $G(10, 3)$  onto the graph  $X$  labelled in the following order along the Hamilton cycle of the graph on the right side of Figure 1:  $\{10, 1, 3, 4, 9, 6, 8, 5, 2, 7\}$  where  $10 - 1 - 3$  and  $8 - 5 - 2$  are the two triangles. More generally we have the following proposition.

**Proposition 4.** *Let  $\alpha$  be an involution of a graph  $G$  that does not interchange the endpoints of an edge, then  $\pi(i) = \alpha(i)'$  and  $\pi(i') = \alpha(i)$  is a polarity of  $KC(G)$ .*

### 3 Least common covers are not unique

It is well-known that if two graphs share a common cover then they have the same universal cover, that is the largest possible connected cover of the two graphs.

Here we show that the converse problem, namely finding the least common cover may have more than one solution. Let  $G$  and  $H$  be disjoint connected graph. Let  $G \smile H$  be a graph composed from  $G$  and  $H$  adding an edge that connects some vertex of  $G$  to some other vertex of  $H$ . This operation depends on the choice of vertices, but we will indicate the choice simply by referring to the figure. Let  $G^*$  and  $H^*$  be any two double covers of  $G$  and  $H$  respectively.

Then there is a unique way to extend this to a double cover of  $G \smile H$  that we shall denote by  $G^* \asymp H^*$ .

Let us take the two familiar graphs  $G(5, 2)$  and  $X$ . Now form two graphs  $H_1 = G(5, 2) \smile X \smile X$  and  $H_2 = G(5, 2) \smile G(5, 2) \smile X$ ; see Figure 2. Let  $G_0 = G(10, 3) \asymp G(10, 3) \asymp G(10, 3)$ ,  $G_1 = G(10, 3) \asymp G(10, 3) \asymp 2X$  and  $G_2 = 2G(5, 2) \asymp G(10, 3) \asymp G(10, 3)$ .

We claim that  $G_0, G_1$ , and  $G_2$  cover  $H_1$  and  $H_2$ . Using the computer system VEGA (see [7]) we checked that  $G_0$  and  $G_1$  are nonisomorphic covers of  $H_1$  and  $H_2$ .

We may conclude by stating our finding in a more formal way.

**Theorem 5.** *There exist connected graphs  $H_1$  and  $H_2$  with a common universal cover such that their minimal common cover is not unique.*

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